MODELLING OF SECULAR VARIATIONS IN GRAVITY AND IN GEOIDAL UNDULATIONS

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Abstract

The known time variation of the absolute gravity and that of the height above sea level enables us to compute the vertical shift of the equipotential surfaces i.e. that of the mean sea level and by this way to determine the true vertical displacement of the earth's surface (i.e the true recent crustal movement). By a further development of Barta's ideas and numeric results, a model has been computed for the time variation of gravity which served to compute the vertical shift of the geoid. Model computations with different density of known gravity variations (i.e. absolute gravity stations) have shown that a world network of 36 stations seems to be sufficient to represent the characteristic features of the global distribution of the variation of reoidal undulations, but determining the magnitude of the variations needs 62 stations at least ($30° \times 30°$ density) in the case of the model used by the authors. Also the numeric value of the gatio $c = \delta g/\delta N$ between gravity variation and change in the geoidal undulation has been investigated and a rather varied global distribution between $-\infty$ and $+\infty$ has been stated. All these experiences are valid even if the numeric values (i.e. the magnitudes) of the variations computed by the model were overestimated.

1. Introduction

One of the main goals of the geodynamic researches is the investigation on the recent crustal (or plate) movements of the earth. The determination of these is based — according to the recommendation No 11 of the *International Association of Geodesy*, Hamburg 1983 — on the observed time variation of the absolute gravity and that of the height above sea level in stations favourably distributed around the globe.

The *Special Study Group 3.87 IAG* has received the task to develop such a new world absolute gravity network.

Until the repeated observations of this, are available it may be useful to have an appropriate model of the time variation of the earth gravity field which could serve for further model computations leading to estimations on the geodetic effects of the secular variation in gravity and on the needed number (or density) and distribution of the absolute gravity stations.

2. The true vertical surface displacement

The usual observation technique to determine recent vertical crustal movements is the repeated geodetic levelling which leads to observed variation δH in height above see level. However, in the earth gravity field, varying with time, also its equipotential surfaces such as the geoid (or the vertical datum) is subject to time variation. If the study of the *true vertical displacement* δr of the earth surface is the aim, changes δN in geoidal undulations (i.e. in the vertical datum) must be taken in account as

$$\delta r = \delta H + \delta N \tag{1}$$

Bruns' formula generalised for time variation [3. Biró 1983, (244.14)]

$$\delta N = \frac{\delta W}{g} \tag{2}$$

connects changes in the equipotential surfaces with the time variation δW of the gravity potential.

Latter can be connected with the gravity variation δg and change δH in height at the earth surface by the differential equation

$$\frac{\partial}{\partial h}\,\delta W - \frac{1}{g}\frac{\partial g}{\partial h}\,\delta W = -\left(\delta g - \frac{\partial g}{\partial h}\,\delta H\right) \tag{3}$$

wherein the derivatives are the vertical gradient of the gravity and that of the variation in gravity potential [Biró 1983, (245.6)].

If the terms on the right of (3) are considered to be numerical values known from observations, this linear differential equation is suitable to serve as a boundary condition for the determination of the harmonic function δW.

As a zero order solution of the third boundary value problem for the time variation (with spherical approximation) we receive

$$\delta W = \frac{R}{4\pi}\iint_{\sigma}\left(\delta g + \frac{2g}{R}\,\delta H\right) S(\psi)\,d\sigma \tag{4}$$

where R is the earth radius, σ the surface of the unit sphere and $S(\psi)$ Stokes' function.

Substituting (2) and (4) into (1) the *true vertical displacement* of the earth surface can be expressed as

$$\delta r = \delta H + \frac{R}{4\pi g}\iint_{\sigma}\left(\delta g + \frac{2g}{R}\,\delta H\right) S(\psi)\,d\sigma \tag{5}$$

Until repeated observations for δg and δH will be available, it may be useful to construct appropriate models so as to investigate the main characteristics of the assumed variation of the equipotential surfaces.

3. Models for gravity and height variation

The Hungarian geophysicist, Professor Barta, developed and published a model for the mass distribution of the earth, on the surface and at depth, that is suitable for our purposes [2. Barta 1979]. He used the filtering effect of increasing distances to eliminate effect of the disturbing mass irregularities near the earth surface. He recognised that the undulations of the equipotential surface of the earth gravity field at a height of 6000 km above sea level can be approximated very well by

$$N(r, \varphi, \lambda) = \sum_{n=2}^{8} [A_n P_n(\cos \beta_1) + B_n P_n(\cos \beta_2)] \tag{6}$$

the sum of two series of simple zonal spherical harmonics with poles P_1 and P_2, where β_1 and β_2 are polar angles of an arbitrary point P of the earth surface from poles P_1 and P_2, respectively.

By least squares approximation, Barta received the following figures for the spherical harmonic coefficients A_n and B_n (in meters) and for the spherical latitude φ and longitude λ of the poles P_1 and P_2 [1. Barta 1985]:

n	2	3	4	5	6	7	8
A_n	−0.62	−11.09	−0.56	−0.75	0.05	−0.07	0.05
B_n	24.36	12.28	−0.81	−0.93	0.18	0.04	0.01

	φ	λ
P_1	16°	59.5°
P_2	0°	163.5°

By reducing (6) from $r = R + 6000$ km to the sea level ($r = R$), we obtain

$$N(\varphi, \lambda) = \sum_{n=2}^{8} \left(\frac{r}{R}\right)^{n-1} [A_n P_n(\cos \beta_1) + B_n P_n(\cos \beta_2)] \tag{7}$$

where $N(\varphi, \lambda)$ is the sea level undulation of the equipotential surface of the disturbing gravity field of deep-seated mass inhomogeneities. The global distribution of (7) [2. Barta 1979, Fig. 2] shows the main characteristics of the geoidal undulations and their magnitudes amount to about 2/3 of the latter.

Authors have transformed the undulations (7) into gravity anomalies by the potential theory.

Using Bruns' formula

$$N \doteq \frac{T}{\tilde{\gamma}}$$

and

$$\tilde{\gamma} \doteq \frac{kM}{R^2}$$

for the spherical approximation of the normal gravity at sea level, (7) can be transformed into

$$T(\varphi, \lambda) \doteq \tilde{\gamma}\, N(\varphi, \lambda) = \frac{kM}{R^2} \sum_{n=2}^{8} \left(\frac{r}{R}\right)^{n-1} [A_n P_n(\cos \beta_1) + B_n P_n(\cos \beta_2)] \tag{8}$$

the spherical harmonic expansion of the disturbing potential T, where kM is the geocentric gravitational constant.

The harmonic function T of the disturbing potential can be expressed in the outer space of the sphere (replacing the geoid) with radius R, as

$$T(r, \varphi, \lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n(\varphi, \lambda) \tag{9}$$

with r being the magnitude of the geocentric radius vector of the arbitrary point of the outer space.

The disturbing potential T can be related to the gravity anomaly Δg at the same point by the linear differential equation of first order

$$\Delta g = -\frac{\partial T}{\partial r} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial r} T \tag{10}$$

[6. Heiskanen and Moritz 1967, (2–148)].

Expanding the gravity anomaly Δg into a series of spherical harmonics on the left of (10), substituting T from (9) and using the spherical approximation

$$\frac{1}{\gamma}\frac{\partial \gamma}{\partial r} \doteq -\frac{2}{r}$$

on the right of (10), we receive

$$\Delta g(r, \varphi, \lambda) = \sum_{n=0}^{\infty} \Delta g(r, \varphi, \lambda) = \frac{1}{r}\sum_{n=0}^{\infty} (n-1) \left(\frac{R}{r}\right)^{n+1} T_n(\varphi, \lambda) \tag{11}$$

for the outer space of the earth or with $r = R$

$$\Delta g(\varphi, \lambda) = \sum_{n=0}^{\infty} \Delta g_n(\varphi, \lambda) = \frac{1}{R} \sum_{n=0}^{\infty} (n-1)\, T_n(\varphi, \lambda) \tag{12}$$

for the geoid.

Comparing the terms of degree n in the series (8) and (9) we get

$$T_n(\varphi, \lambda) = \frac{kM}{R^2} \left(\frac{r}{R}\right)^{n-1} [A_n P_n(\cos \beta_1) + B_n P_n(\cos \beta_2)] \tag{13}$$

the surface spherical harmonics of the disturbing potential for $n = 2, 3, \ldots 8$.

Substituting this into (12) and considering the usual assumptions, that the origin of our system of coordinates coincides with the earth centre of mass ($T_1 \equiv 0$) and $kM = kM_{\text{earth}}$ ($T_0 \equiv 0$) we obtain

$$\Delta g(\varphi, \lambda) = \frac{kM}{R^3} \sum_{n=2}^{8} (n-1) \left(\frac{r}{R}\right)^{n-1} [A_n P_n(\cos \beta_1) + B_n P_n(\cos \beta_2)] \tag{14}$$

as the spherical harmonic expansion of the gravity anomaly at the geoid caused by the deep-seated mass irregularities wherein the zonal harmonic coefficients A_n and B_n ($n = 2, 3 \ldots 8$) are numerically given by Barta, as above.

According to Barta's ideas, assuming the angular velocity of the displacement of the deep seated mass inhomogeneities to be identical with the westward drift of the earth's magnetic field (i.e. 0.2°/a), *a model of the time variation* δg *of the gravity at sea level* can be constructed as

$$\delta g(\varphi, \lambda) = \frac{kM}{R^3} \sum_{n=2}^{8} (n-1) \left(\frac{r}{R}\right)^{n-1} \{A_n[P_n(\cos \beta_1') - P_n(\cos \beta_1)] + B_n[P_n(\cos \beta_2') - P_n(\cos \beta_2)]\} \tag{15}$$

where β_1, β_2 and β_1', β_2' are the polar angles of the arbitrary point P, referring to the actual location of the poles P_1 and P_2 at the epochs t and $t + \delta t$ [7. Thông 1985].

The global distribution of the gravity variation per year as estimated by the accepted model are demonstrated in Fig. 1. Contour intervals are 50×10^{-8} ms^{-2}/a (i.e. 50 μgals/a). Figure 1 differs (even in sign) from Fig. 4 in [2. Barta 1979], because Barta used an arbitrary approximation in his computation instead of the exact way of the theory of potential, used by us.

Even if the figures were overestimated, a tenth or a hundredth part of the computed gravity variations would suffice to be significant and the study of their global distribution could certainly help us to a favourable choice of the locations of the stations of an international absolute gravity base network.

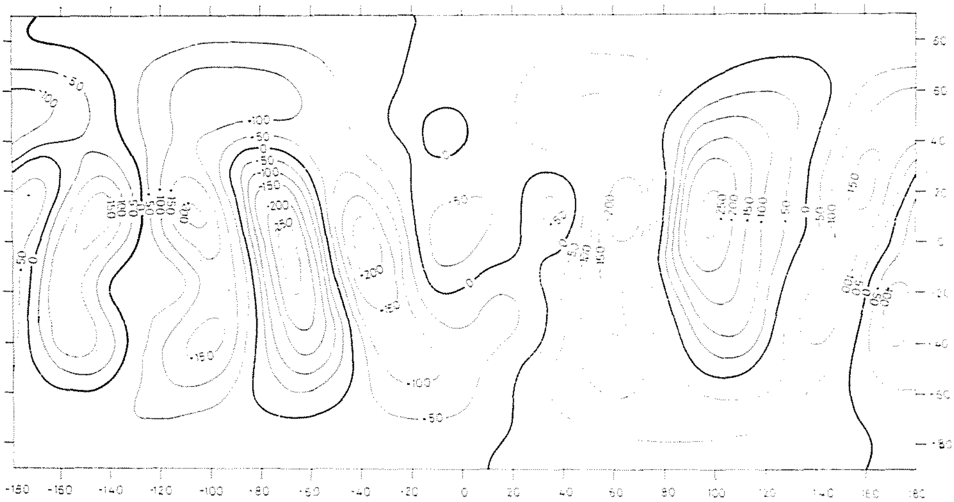

Fig. 1. The global distribution of the gravity variation δg in 10^{-8} ms^{-2}/a (i.e. $\mu gal/a$) as estimated by the accepted model

The gravity variation δg defined by (15) is the change in gravity at the displacing equipotential surface at sea level (i.e. at the geoid). Assuming that the earth body reacts against secular or long periodic variation of the earth's gravity field as an ideal liquid, the same gravity variation δg will be induced at the displacing earth's surface, too. In any other cases δg should be reduced to the topographic surface.

Gravity variations (per year) computed by (15) (as simulated observations) in a $10° \times 10°$ grid net (in 614 stations) served as one kind of input data for our further model computations.

As the other kind of input data, a model of the *variation δH in height* was needed. As a simple model $\delta H = 0$ for the oceans (a fluid-like earth body) and $\delta H = \text{const} = +10$ mm/a for all continents (as an extreme value of the Fennoscandinavian land uplift) has been accepted.

4. Variation in the equipotential surfaces

The aim of our further model computation was to gather numeric experiences in computing the true vertical displacement of the earth's surface as given by (5) and to investigate the needed number (or density) of stations to be observed to be able to determine the vertical displacement δN of the equipotential surfaces with a reliability sufficient for practical purposes.

This latter can be determined by the appropriate use of Stokes' integral ormula for time variations:

$$\delta N = \frac{R}{4\pi g} \iint_{\sigma} \left| \delta g - \frac{\partial g}{\partial H} \delta H \right| S(\psi)\, d\sigma \tag{16}$$

as showen in Section 2.

For numeric computations the integral in (16) will be replaced by a summation (numeric integration) over $m \times n$ surface elements and polar coordinates ψ, α at the surface of the unit sphere will be introduced. Further, the vertical gravity gradient will be approximated by its spherical value. This way we obtain

$$\delta N = \frac{R}{4\pi g} \sum_{i=1}^{n} F_i(\psi) \sum_{j=1}^{m} \left| \delta g_{i,j} + \frac{2g}{R} \delta H_{i,j} \right| \Delta\alpha_j \tag{17}$$

with

$$F_i(\psi) = \int_{\psi_i}^{\psi_{i+1}} S(\psi) \sin\psi \, d\psi$$

the weight function, ψ_i the angular distance of the surface element i,j of a $10° \times 10°$ grid net from the arbitrary point in which δN will be computed, $\Delta\alpha^j$ the dimension in azimuth of this surface element, $\delta g_{i,j}$ and $\delta H_{i,j}$ the mean gravity and height variation of the surface element i, j.

A decomposition of (17) into two parts brings to

$$\delta N_1 = \frac{R}{4\pi g} \sum_{i=1}^{n} F_i(\psi) \sum_{j=1}^{m} \delta g_{i,j}\, \Delta\alpha_j \tag{17a}$$

and

$$\delta N_2 = \frac{1}{2\pi} \sum_{i=1}^{n} F_i(\psi) \sum_{j=1}^{m} \delta H_{i,j}\, \Delta\alpha_j \tag{17b}$$

enabling us to investigate the effect of the gravity variation on the one hand and that of the variation in height on the other, separately from each other.

For the numeric evaluation of (17a) and (17b) a simplified computer program has been developed in FORTRAN. The mean variations $\delta g_{i,j}$ and $\delta H_{i,j}$ of the surface element have been computed by linear approximation as interpolated values in the centre of the surface element using the analogy of digital terrain models, while the numeric value of the weight function $F_i(\psi)$ has been computed by a numeric integration over $1°$ elements of the circular ring i.

By using our computer program the vertical displacements δN_1 and δN_2 of the equipotential surface at sea level (i.e. the time variations of the geoid undulation) have been computed for all grid points of a $10° \times 10°$ grid.

The first experience of our model computations was that $\delta N_2 < 1$ cm/a and $\delta N_1 < 1$ m/a. This means that the effect of the height variation in δN is probably of about the same magnitude as the observed variation in the height itself. However, in the case of our model of gravity variations $\delta N_2 \ll \delta N_1$, δN_2 will thus be neglected in our further investigations.

The global distribution of the effect δN_1 of the gravity variation has been demonstrated in Fig. 2. Contour intervals are 20 cm/a. Concerning the magnitude of the computed values of δN_1 let us remember to the remark in connexion with the magnitude of the gravity variations, i.e. even if they were 10 or 100 times overestimated, the magnitude of δN_1 would be significant and could not be neglected when studying the recent vertical crustal movements.

The computed values of $\delta N_1 \doteq \delta N$ have been checked by a comparison with δN values, computed by the appropriate use of formula (7) for time variation. A root mean square deviation of ± 3 cm/a has been obtained. This means that formulae (7), (15), (17a, 17b) represent a consistent system and most probably no significant errors occurred in the computation.

It is clear that several hundred absolute gravity stations will not be observed in the near future. Therefore our computations have been repeated in several variants with less stations to prove how the reliability of the results will thus decrease.

The vertical displacement δN_1 of the equipotential surface has been computed from 146, 72, 62, 45 and 32 simulated gravity variations, too. Fictitious gravity stations have been chosen in grid nets of $20° \times 20°$, $30° \times 30°$ (in three variants), $40° \times 40°$ and $30° \times 60°$. Finally, an attempt has been made with 36

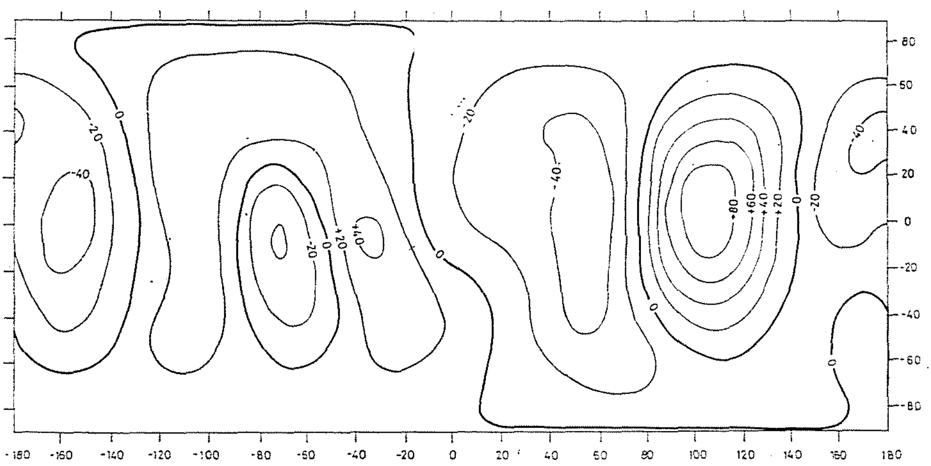

Fig. 2. The global distribution of the variation $\delta N \doteq \delta N_1$ of the geoidal undulation in cm/a as induced by the gravity variation represented in Fig. 1 (Variant 1, $10° \times 10°$ grid net)

Table I

Variant No	Number of "stations"	Grid net	Surface element	R. m. s. deviation cm/a	Demonstration
1	614	$10°\times10°$	$10°\times10°$	±3	Fig. 2
2	146	$20°\times20°$	$20°\times20°$	±7	Fig. 3
3.a	62	$30°\times30°$	$30°\times30°$	±10	Fig. 4
3.b	72	$30°\times30°$	$30°\times30°$	±12	Fig. 5
3.c	72	$30°\times30°$	$30°\times30°$	±11	Fig. 6
4	45	$40°\times40°$	$10°\times10°$	±15	—
5.a	32	$30°\times60°$	$30°\times30°$	±28	Fig. 7
5.b	36 characteristic points		$10°\times10°$	±19	Fig. 8

"stations" selected at the characteristic places of the global distribution of δg, as shown in Fig. 1. Each of the variants has been checked similarly to the first computation and root mean square deviations have been determined. The results are demonstrated in Table I and in Figs 3—8. In variant 4, a special supplement program (subroutine) has been used to derive an interpolated $10°\times10°$ net between the given "stations". Contour intervals are 20 cm/a.

By analysing our results the following *conclusions* can be drawn. The *characteristic features* of the global distribution and, generally the *order of magnitude* of the vertical displacement of the equipotential surface at sea level, can be determined by repeated observation of a world-wide net consisting of *about 36 — 45 absolute gravity stations*. In the first case the location of the stations should be selected near to the characteristic points (or lines) of the assumed gravity variation model with a constant density, as far as possible. In the second

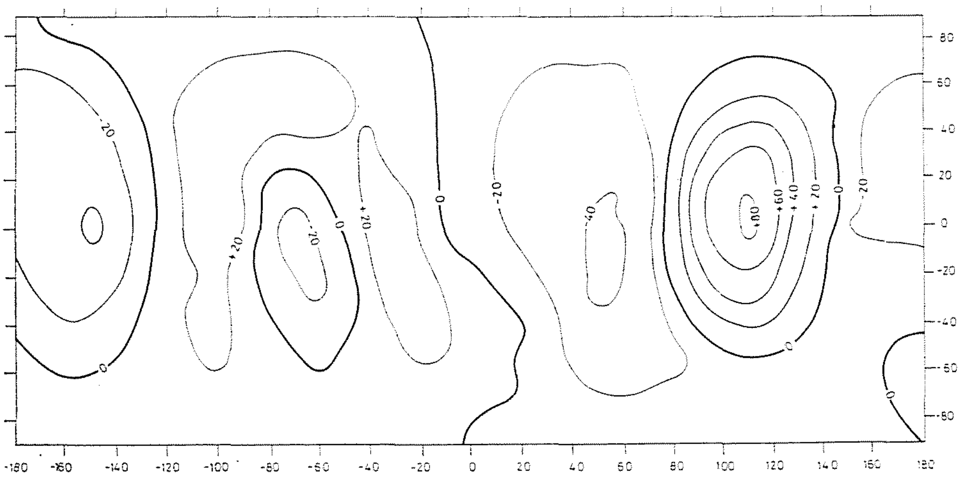

Fig. 3. The global distribution of δN in cm/a computed as Variant 2 ($20°\times20°$ grid net)

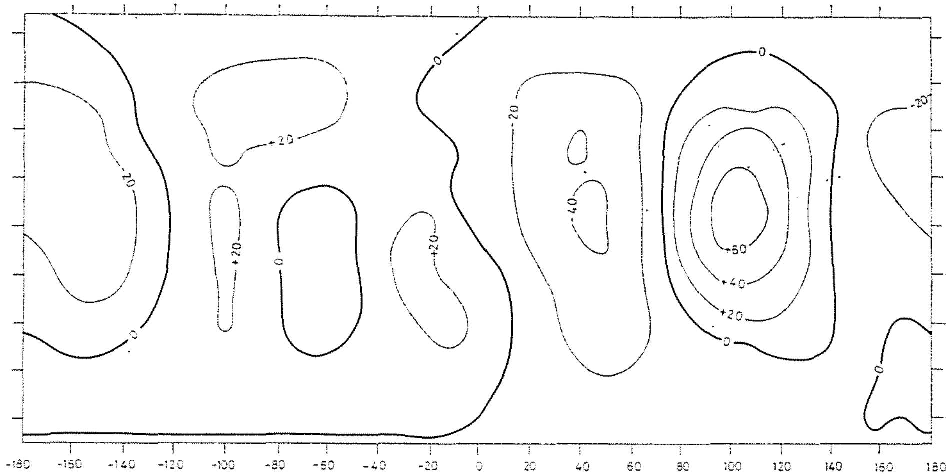

Fig. 4. The global distribution of δN in cm/*a* computed as Variant 3.a ($30° \times 30°$ grid net)

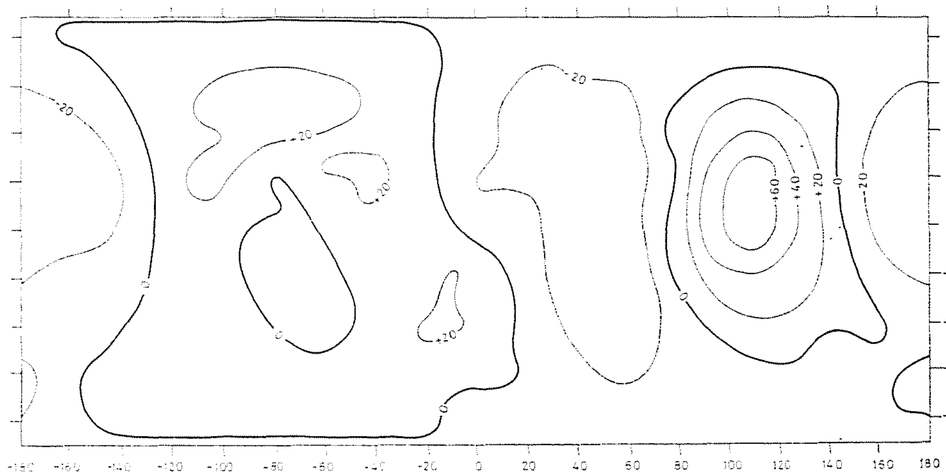

Fig. 5. The global distribution of δN in cm/*a* computed as Variant 3.b ($30° \times 30°$ grid net)

case, stations should be located at a near constant, $40° \times 40°$, distribution. Computed single δN values can be charged by an error of 50 — 100% of the true magnitude.

More reliable numeric results can be achieved by an absolute gravity net consisting of *about 62 stations* at a constant $30° \times 30°$ distribution around the globe. The error in computed single δN values can be estimated to be less than 30—50% of the true magnitude [8. Weisz 1985].

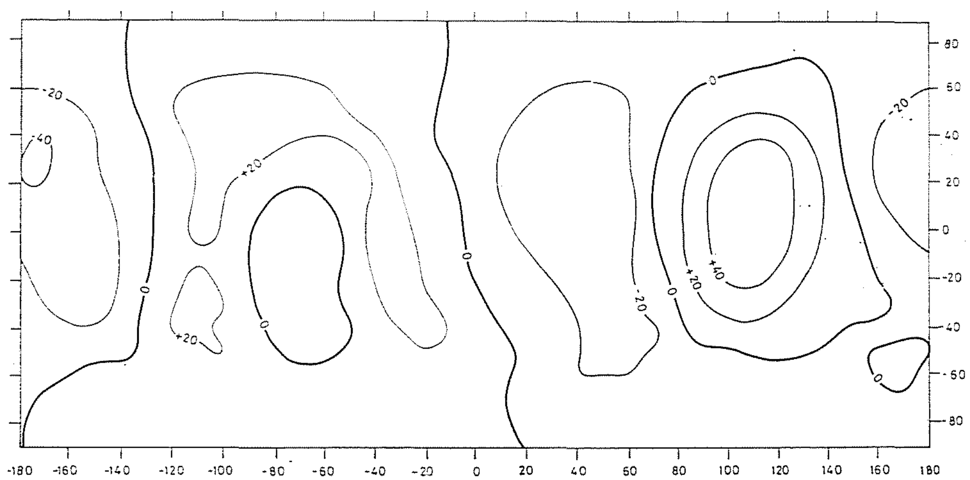

Fig. 6. The global distribution of δN in cm/a computed as Variant 3.c ($30° \times 30°$ grid net)

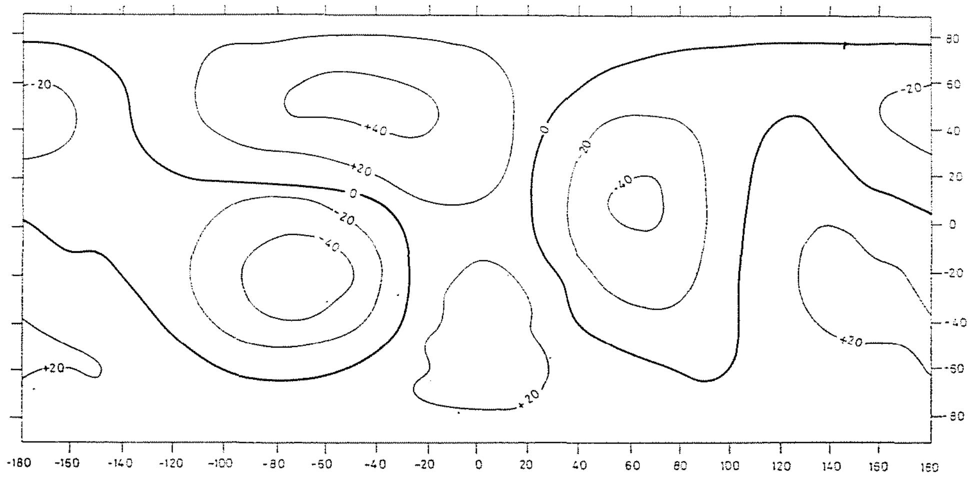

Fig. 7. The global distribution of δN in cm/a computed as Variant 5.a ($30° \times 60°$ grid net)

5. The ratio between the variations in gravity and in geoidal undulation

As reported in Sections 3 and 4 the time variation δg in gravity at sea level and the variation $\delta N \doteq \delta N_1$ in the geoidal undulation has been computed by using formulae (15) and (17a) in a $10° \times 10°$ grid net. These sets of data have enabled us to investigate numerically the *ratio* $c = \delta g / \delta N$.

The global distribution of the computed values of ratio c has been represented in Fig. 9. The contour lines $c = 0$ are identical with those for $\delta g = 0$

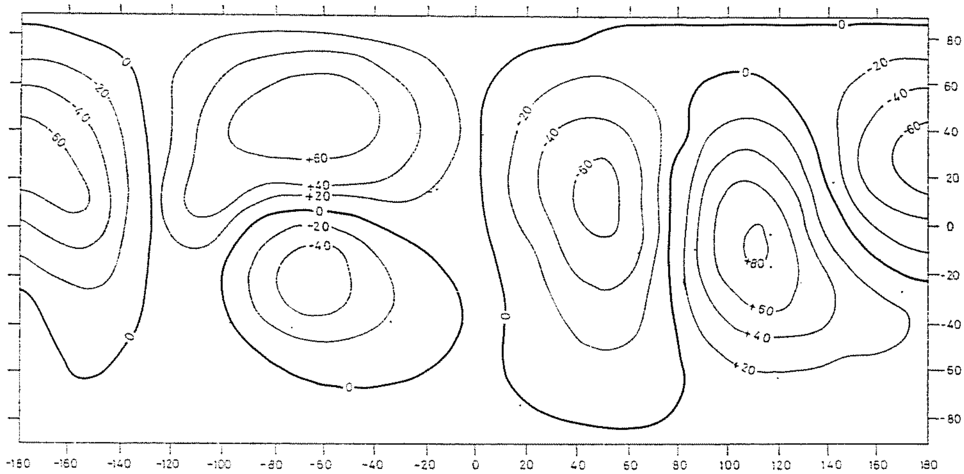

Fig. 8. The global distribution of δN in cm/a computed as Variant 5.b (characteristic points)

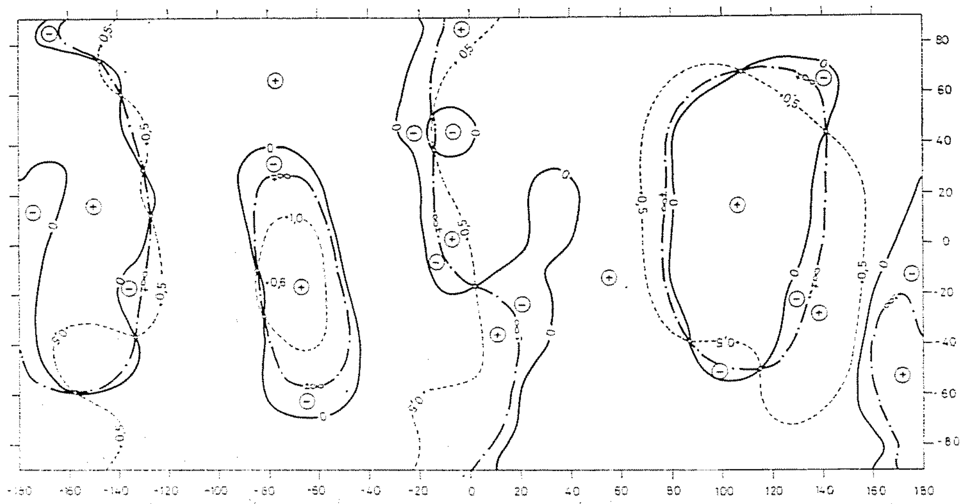

Fig. 9. The global distribution of the ratio $c = \delta g/\delta N$ in $10^{-5}\ s^{-2}$ (i.e. in mgal/m) with δ and δN represented in Figures 1 and 2

in Fig. 1. The contour lines $\delta N = 0$ in Fig. 2 represent the locations of the extreme values $c \to \pm\infty$ in Fig. 9. Between these two special contours one finds narrow strips with the values $-\infty < c < 0$, but the most extent part of the earth's surface is covered by values $0 < c < +0.5 \times 10^{-5}\ s^{-2}$ (i.e. $+0.5$ mgal/m). The mean value of ratio c can be estimated to be about $+0.3 \times 10^{-5}\ s^{-2}$ (i.e. $+0.3$ mgal/m). In the intersections of the contour lines $c = 0$ and $c \to \pm\infty$ (i.e. $\delta g = 0$ and $\delta N = 0$) the ratio $c = \delta g/\delta N$ becomes indefinite.

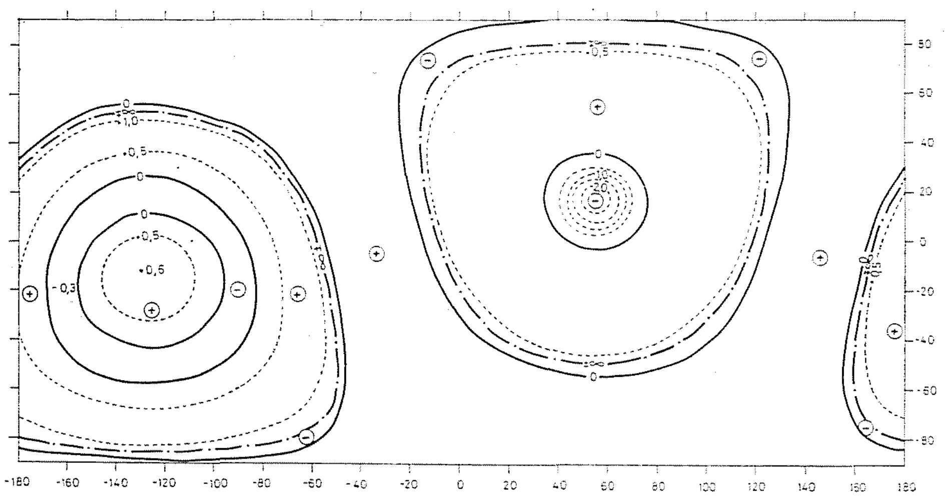

Fig. 10. The global distribution of the ratio $c = \delta g/\delta N$ in $10^{-5}\ s^{-2}$ (i.e. in mgal/m) with δg and δN computed for the simple gravity field represented by the zonal spherical harmonics with pole P_1

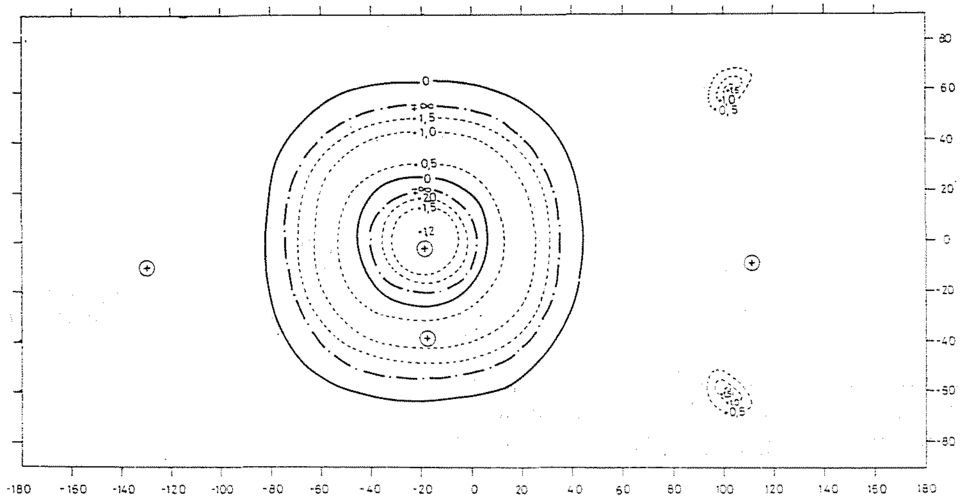

Fig. 11. The global distribution of the ratio $c = \delta g/\delta N$ in $10^{-5}\ s^{-2}$ (i.e. in mgal/m) with δg and σN computed for the simple gravity field represented by the zonal spherical harmonics with pole P_2

As shown in Section 3, the model of the time variation of the earth's gravity field has been composed by the time variations of two simple gravity fields represented by a series of zonal harmonics each with the poles P_1 and P_2, respectively.

The numeric values for $c = \delta g/\delta N$ have been computed separately also for each simple gravity field. Figs 10 and 11 represent the global distribution of the ratio c in the simple gravity fields with poles P_1 and P_2, respectively. The main characteristics are the same as before, the surroundings of the poles (or the antipoles) are much more dominant.

Our investigations have led to the conclusion that the ratio between gravity variation and the changes in the geoidal undulation shows a *rather varied distribution* between $-\infty$ and $+\infty$ around the globe even in the case of a very simple gravity field.

This numeric experience is in agreement with the consideration that the variation in the geoidal undulation is proportional with the change in gravity potential as shown by (2) but latter can be related to the gravity variation generally by the differential equation (3) of first order only. Therefore a simple proportionality in δg and δN can only be an exceptional case.

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