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CONSTRUCTIVE REPRESENTATION OF FOURTH-ORDER SPACE CURVES

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Abstract

There are several possibilities of constructing and examining fourth-order space curves, such as analytically for metric requirements, synthetically in position geometrical analyses, in projective or constructive geometry. Computer graphics expands possibilities and widens survey. There are several possibilities for the constructive geometrical classification of fourth-order space curves. Two main classes are first and second-kind fourth-order space curves, to be classified e. g. according to their relation to a plane in the infinity. Classification may be according to the appearance of the common polar tetrahedron, various involutions, decompositions, singular points, smoothness or bifurcation of fourth-order space curves (whether being pair or odd). Cases of constructing fourth-order space curves of the first kind are considered, answering different classifications, also concerning symmetry conditions. Practical applications mainly involve involutions of fourth-order space curves. Junction curves of shell surfaces are often decompositional, symmetrical fourth-order space curves of the first kind. All these will be directly illustrated, without aiming at completeness.

Keywords: constructive geometry, conjugated complex, first and second kind fourth-order space curves, set of conic sections, polar tetrahedron, osculating point, stationary fitting plane.

Introduction

There are several possibilities of constructing and examining space curves. If metric aspects prevail, then, in coordinate geometry, one may proceed analytically, while for position geometry problems, in projective and constructive geometry, one may proceed by synthesis. Also computer graphics may be applied, digital information delivered by the computer is made to design graphic display, using automatic plotter, or on screen.

The most general approximation is by means of projective or constructive geometry, having permanent recourse to spatial approach, to derive the space curve from the intersection of two surfaces, to examine it, then to display it in some mode of display, to be constructed as needed.

1. General on Fourth-order Space Curves

Algebraic space curves are obtained constructively, by the overall or partial intersection between algebraic surfaces. In a complete intersection, the order number of the resulting intersection curve equals the product of orders of the two intersecting surfaces: $N = nm$; in case of partial intersection, when the two surfaces have a common part of order k, depending on how they are indicated: $N = nm - k$.

A fourth-order space curve is an algebraic space curve, intersected by every plane at four points that may pair-wise coincide, or may be conjugate complexes. By way of perfect intersection, the fourth-order space curve is obtained by intersection between two second-order surfaces. An arbitrary plane cuts from both surfaces a second-order curve each, of them four common parts lie in both surfaces, hence also on the curve of intersection, causing it to be a fourth-order space curve.

Projection of a simple, finitely closed, continuous fourth-order space curve g^4, a plane curve of order four, is seen in *Fig. 1* Projection planes intersecting straight lines 1, 2, ..., 8 intersect g^4 at as many points as do the straight lines the projection curve. Namely,

1. at four imaginary points,
2. at two coincident and two imaginary points,
3. at two real and two imaginary,
4. at two coincident and two imaginary points,
5. at two by two coincident real points (in fact, two cover point pairs of the space curve),
6. at four real points,
7. at two by two coincident real points,
8. at a triple real and at another real points.

The fourth-order space curve results from partial intersection if at least one of the intersecting surfaces is of higher than second order. In this case, the surfaces have also other common parts in addition to the fourth-order space curves. Common, e. g. second-order parts of the two intersecting surfaces may be classified according to two groups:

1. Plane, either
   a) cone section, or
   b) two intersecting straight lines;
2. Spatial, either
   a) two skew lines, or
   b) two coincident straight lines.

Case 1*a*. is seen in *Fig. 2* Common cone section k^2 is crossed both by the cone with vertex M, and by surface F^3 indicated by its group of

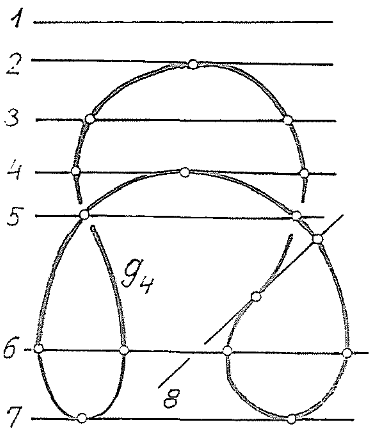

Fig. 1.

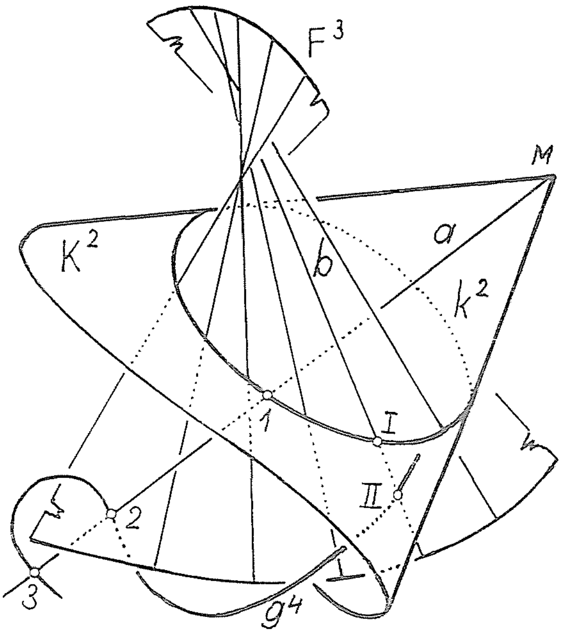

Fig. 2.

generatrices. An arbitrary generatrix a of the cone intersects surface F^3 on one hand, at point l on k^2, on the other hand, at points 2 and 3. Thus, points 2 and 3 are real intersection points lying on g^4 traced by estimation. Thus, all the cone generatrices are double secants of g^4. Since generatrices of surface F^3 intersect the cone at a point of common cone section k^2, e. g. generatrix b at point l, cone k^2 hence also g^4 may be cut but at another point, e. g. generatrix b at point II. Thus, generatrices of surface F^3 are single secants of g^4.

It is known that a space curve of order n is defined by $2n$ points, and a plane curve of order n by $\frac{n}{2}(n+3)$ points. It is self-intended that among the 8 points defining a non-degenerating space curve, more than 4 cannot lie in a plane, and more than 2 cannot be on a straight line. A surface of order n is defined by: points, of them more than $\frac{n}{2}(n+3)$ cannot lie in a plane, and more than n ones cannot be on a straight line. Thus, second-order surface F^2 is defined by 9 points. Hence, a single surface F^2 may be laid on the 8 points defining g^4 and another point outside the curve. Across the 8 points, and a single further point picked out of the three-dimensional multiplicity of points in the space, a total of ∞^3 second-order surfaces may be laid. Since, however every F^2 includes ∞^2 points, so every F^2 has been reckoned with ∞^2 times, in final account, a number of $\infty^3 - \infty^2 = \infty^1$, that is, one-times infinity of surface F^2 may be laid across the 9 points. These surfaces form a series of surfaces, with space curve g^4 as reference curve. Obviously, any surfaces F_i^2 and F_k^2 of the series of surfaces intersect at the same g^4. The g^4 derivable by this means is called a first kind fourth-order space curve.

In the case 2*a* above, let the common pair of generatrices in deviating position of rectilinear surfaces F^2 and F^3 be *a* and *b*, as seen in *Fig. 3* All the surface F^2 — that is not represented here — is defined by *a* and *b*, as well as by straight line *c* deviating from them, while F^3 by *a* and *b*, as well as by further 15 points to be defined below, expressed by the formula above for the surface, that, however, do not fit *c*.

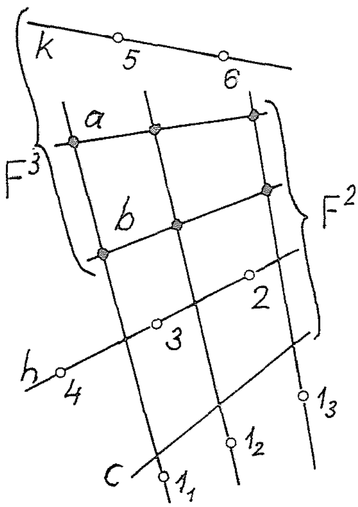

Fig. 3.

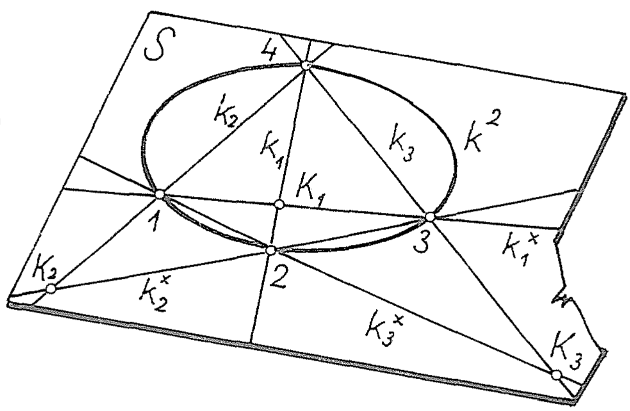

Fig. 4.

Fig. 3 shows neither these points, nor the surface itself. The other set of generatrices of surface F^2, intersecting generatrices *a*, *b* and *c*, every generatrix, are necessarily single secants of g^4, such as points l_1, l_2, l_3, since out of the three intersection points with F^3, each of generatrices *a* and *b* bears one. Generatrices of F^2 in the same series of *a*, *b* and *c*, e. g. *h* are triple secants of g^4 they being in positions deviating from *a* and *b*, and e. g. surface F^3 is cut by *h* exactly at points 2, 3 and 4 of g^4. Generatrix *k* of F^3 is twice the secant of g^4, is deviating from *a* and *b*, and necessarily intersects surface F^2 at points 5 and 6 twice in all. Thus, generatrices of surface F^3 are double secants of g^4. In this case, a single surface F^2 — just the defined one — may be laid across g^4. Hence, no surface series F_i^2 can be laid across this g^4. Namely, if several surfaces F^2 should be laid across

g^4, these would be intersected by generatrices belonging to the set a, b, c, — e. g. h — at three points, an impossibility for second-order surfaces. The fourth-order space curve derived in this way is called a second-kind fourth-order space curve.

In conclusion: On a first-kind, fourth-order space curve, a series of surfaces g_I^4, F_i^2 may be laid. Generatrices of the two derivable intersecting surfaces may be at most double secants of g_I^4. This space curve may have singular points. A single second-order distorted surface may be laid on the second-kind fourth-order space curve g_{II}^4. Generatrices of this surface in the same set as the two common generatrices of the surfaces are triple secants of g_{II}^4; generatrices of the other set intersecting the two common generatrices are single secants. Generatrices of surface F^3 deviate from a and b, so they are double secants of g_{II}^4. g_{II}^4 cannot have a singular point, since a plane passing through such a point and an arbitrary triple secant would intersect g^4 at more than four points, an impossibility. In the following, exclusively first-kind fourth-order space curves will be considered, denoted in short g^4.

2. First-kind Fourth-order Space Curves

The set of surfaces F_i^2 that can be laid on g^4 comprises several different second-order surfaces. In *Fig. 4*, plane S intersects surface series F_i^2 in a set of conic sections, of that one element is ellipse k^2. Here 1, 2, 3 and 4 are points of intersection of g^4 with plane S, basic points of the series of conic sections. In this case the line has three elements degenerating to pairs of straight lines k_1, k_1^+; k_2, k_2^+; and k_3, k_3^+, with double points K_1, K_2 and K_3. In each of the pairs of straight lines, a second-order line surface of the set of surfaces is tangential to plane S.

If the set of surfaces comprises at least two cone shells K^2 also bearing the base curve, their intersection necessarily results in space curve g^4. The set of surfaces generally comprises at most four real cones that may be pair-wise imaginary. Generatrices of these cones — intersecting their fellow cones at two points in intersection — are double secants of the intersection curve g^4. Thus, such a cone is double projection cone of the fourth-order space curve. The four cone vertices generally define a tetrahedron is space, named the common polar tetrahedron of the set of surfaces F_i^2. By finding at least two vertices of the common polar tetrahedron, construction of the curve of intersection of any two general second-order surfaces of the set of surfaces may be reduced to determine intersection between two second-order cone shells. In what follows, investigations will be extended to determine the intersection space curve g^4 between two second-order cone shells.

3. Various Fourth-order Space Curves Arising from Intersection between Two Second-order Cones

The way of constructing g^4 as intersection between two cones is known. There are several possibilities for the constructive geometrical classification of the g^4. In course of derivation it was clear that fourth-order space curves may be of the first or second kind. They are mostly classified according to their position relative to a plane in infinity. Other aspects of classification may be: appearance of the common polar tetrahedron, degeneration, decomposition of the space curve, its singular points, appearance as smooth or bifurcated, pair or odd. With respect to its position relative to a plane in the infinity, g^4 may have different appearances, depending on whether g^4 has

1. four hyperbolic,
2. two hyperbolic and one parabolic,
3. two parabolic,
4. one hyperbolic and one osculating,
5. one hyperosculating,
6. two hyperbolic and two elliptic,
7. one parabolic and two elliptic,
8. four elliptic points.

In the following, typical examples will be picked out from the possible case 8., with reference — in the respective case — to the way of definition of the two original second-order cones, to construction of the terminal tangents, to that whether the bifurcated intersection curve and its projection are pair or odd — this latter case means that an arbitrary straight line of the plane can only intersect a branch of the projection curve of the space curve at an odd number of real points. It will be illustrated how to define the two cones and their directrices, so that the intersection curve has predefined singular points.

4. g^4 Having Only Real Infinitely Distant Points

4.1 g^4 with Four Hyperbolic Points

To have a curve of intersection with four hyperbolic points, the derivative pair of cones is required to have four pairs of parallel generatrices. In *Fig. 5*, a cone with vertex M_2^+ has been selected as a directional pair of cones for a cone with a given vertex M_1, having a common vertex intersecting the given cone in four generatrices. Therefore in the common base plane S, directrices of both cones intersect at real points 1_1, 2_1, 3_1 and 4_1. Shifting parallelly the cone with vertex M_2^+ to an arbitrary vertex M_2 results in co-interacting cone of the cone with vertex M_1. Vertex straight line $M_1M_2 = s$ intersects the common base plane at point S, while the four parallel pairs of generatrices intersect each other at the four hyperbolic (infinitely distant) points H_1, H_2, H_3 and H_4. At an infinitely distant point H_1 of the pair of generatrices $M_1 1_1$ and $M_2 1_2$ a terminal tangent v_1 — intersection line of two cone tangential planes tending to point H_1 — has been constructed. Because of the relative position of the cones, obviously, g^4 of intersection will have two branches, with two terminal tangents each.

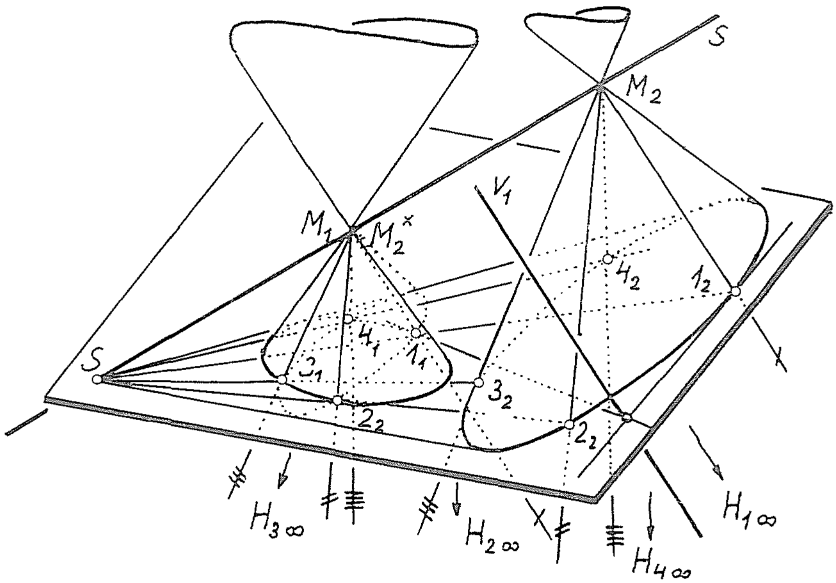

Fig. 5.

Two open, bifurcating branches, each ending in an infinitely distant pair of points, of the projection of the curve of intersection in *Fig. 5* has been traced by estimation in *Fig. 6, Fig. 7* shows the projection of g^4 with four end tangents as a one-branched, virtual double point K. Projection radius in K is double secant of the space curve. Selecting the cone with vertex M_2 as associate cone for cone with vertex M_1 by parallel shifting so that the two cones have a common tangential plane across the apical straight line, the two generatrices in the common tangential plane intersect at the double point K of the curve of intersection. Projection of this one-branched intersection curve is seen in *Fig. 8* Of course, the single- branched projection cannot be but even, with an inherited double point K. In *Fig. 9*, two of the four hyperbolic points coincide (in infinity), so projection of

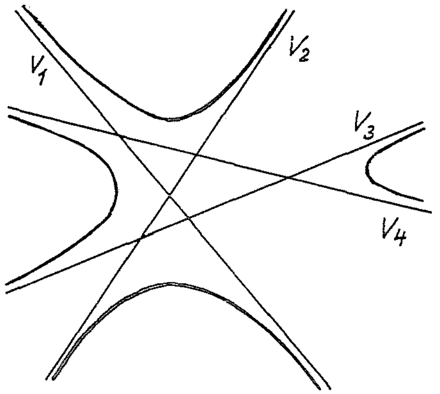

Fig. 6.

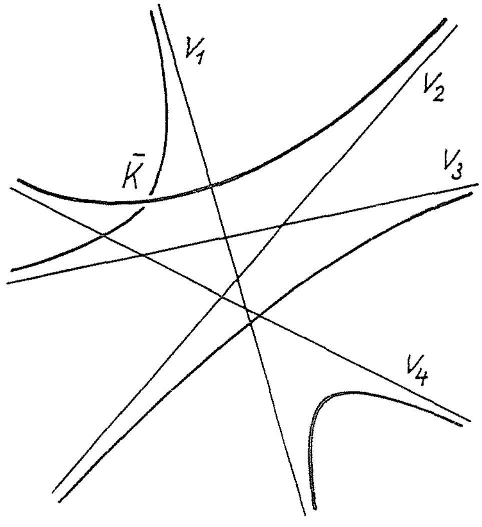

Fig. 7.

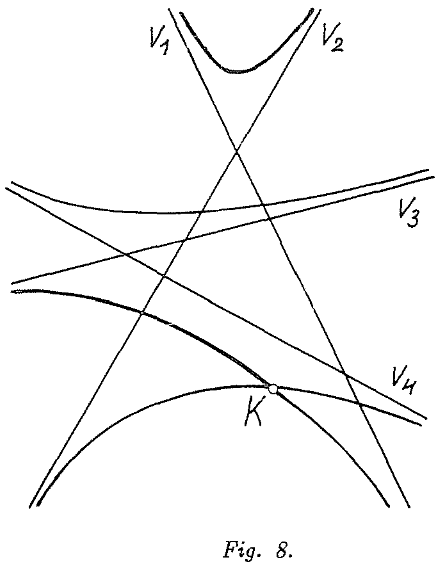

Fig. 8.

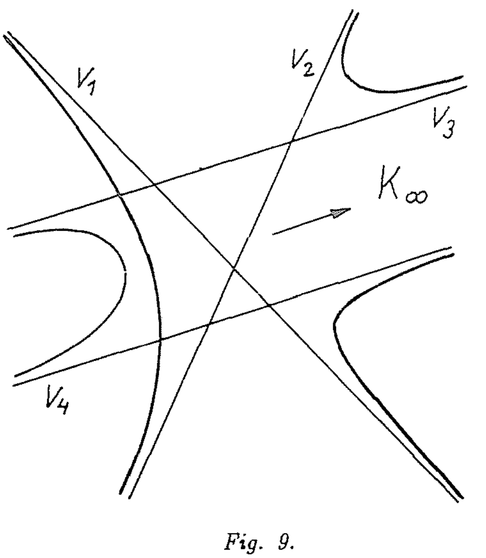

Fig. 9.

single-branch g^4 with double point K_∞ is presented. Now, two of the four terminal tangents are parallel, intersecting at K_∞.

Taking vertex M_2 on the shell of the cone with vertex M_1 in *Fig. 5*, and if at this point, the two cones have a common tangential plane, then this cone vertex will be peak of g^4. Projection with a real peak C of the resulting single-branched g^4 is seen in *Fig. 10, Fig. 11* illustrates how to take the intersecting cone pair, if two of the four hyperbolic points coincide to become an infinitely distant hermit point of g^4. Thus, a tangential plane E common to both cones at hermit point R_∞ is needed, so that its bilateral adjacent auxiliary planes intersect but one cone in a real pair of generatrices. Furthermore, tangent generatrices $M_1 1$ and $M_2 2$ of the common tangential plane have to intersect at a point R_∞ in infinity, hence, have to be parallel. Of course, in addition, the two cones should have two pairs of parallel generatrices $M_1 3_1$, $M_2 3_2$ and $M_1 4_1$, $M_2 4_2$ defining two other hyperbolic points. As seen, in the directional pair of cones, cones with vertex M_2^+ and M_2 occur with hyperbolic directrices h^+ and h, respectively. Projection traced by estimation of g^4 constructed according to *Fig. 11* is seen in *Fig. 12*. Here two hyperbolic points coincide at hermit point R_∞. One branch bears two hyperbolic points with terminal tangents v_1 and v_2, while the other branch may be considered as shrunk to hermit point R_∞.

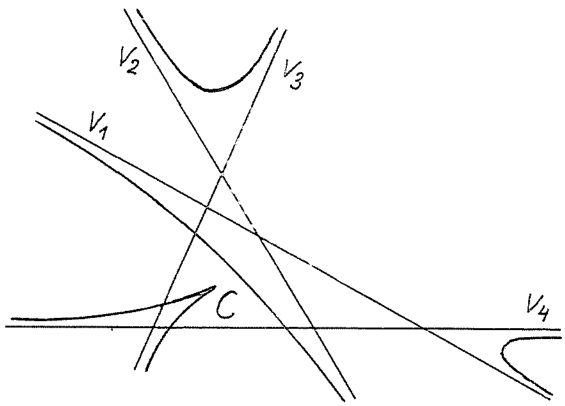

Fig. 10.

4.2 Two Hyperbolic and One Parabolic Points of g^4

In this case, generatrices of the directional cone pair have to be assumed — satisfying also the parabolic point — so that they have a common tangent at a common point, and two more real intersections. Shifting any of the cones parallelly from its position of common vertex, then — the pair of cones having a pair of parallel tangential planes — the intersecting fourth-order space curve of the shifted and the stationary cones will have a parabolic and a tangent point in the infinitely distant place. In this projection, projection branches tend unilaterally to parabolic point P_∞ — just as for the parabolic plane curve — and they close in the common tangential

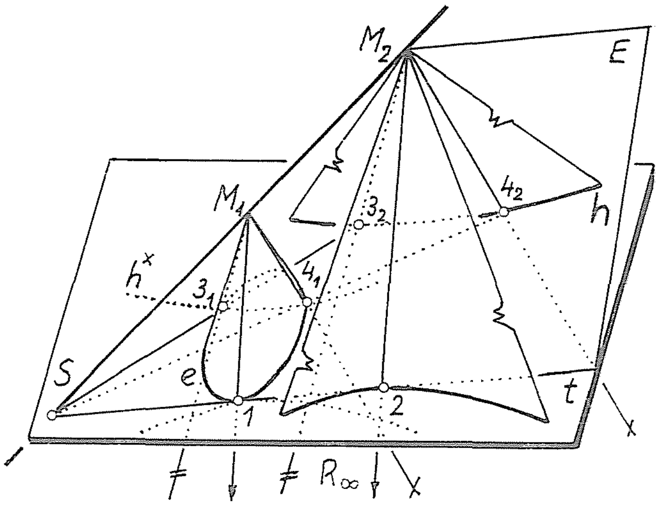

Fig. 11.

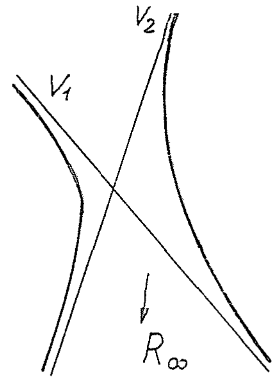

Fig. 12.

plane of the directional cone pair in infinity. This projectional curve has, of course, still two terminal tangents.

Projection of a single-branched g^4 with an apparent double point K, parabolic point P_∞ and terminal tangents v_1 and v_2, has been traced in *Fig. 13a* in conformity with those above. *Fig. 13b* shows bifurcated pairwise projection of g^4 with point P_∞, and terminal tangents v_1 and v_2. Projection of odd bifurcated g^4, with apparent double points K_1 and K_2, as well as point P_∞ and terminal tangents v_1 and v_2, is seen in *Fig. 14a*, *Fig. 14b* shows projection of a single-branched g^4 with real, and apparent double points K, and K_1, K_2, resp., point P_∞ and terminal tangents v_1

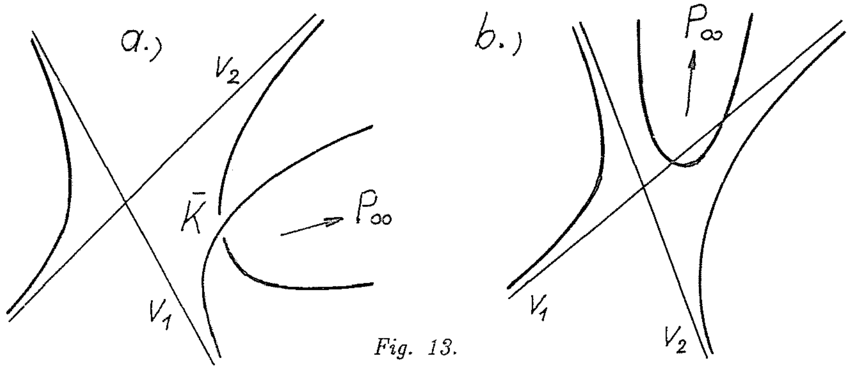

Fig. 13.

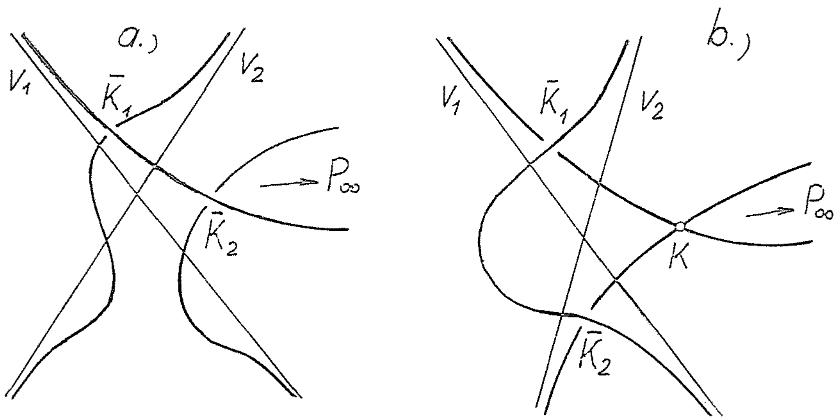

Fig. 14.

and v_2. The real double point requires the two to-be intersecting cones to have a common tangential plane.

Double point K in *Fig. 14b* is in the infinity in *Fig. 15a*, where parallel terminal tangents v_1 and v_2 coincide with two hyperbolic points, and there is also an apparent double point K_1 on the projection of g^4.

Both g^4 and its projection peak at C in *Fig. 15b*. In this case, vertex M_2 has been taken on the shell of the cone with vertex M_1 so that the two cones have a common tangential plane in it, and the cones meet heading conditions. The projection curve is bifurcated, with peak C, parabolic point P_∞, and terminal tangents v_1 and v_2. In *Fig. 16a*, point C in *Fig. 15b* is in infinity, where both hyperbolic points are coincident, and so are terminal tangents v_1 and v_2. g^4 is single, K_1 and K_2 are apparent double points of the projection. In *Fig. 16b*, g^4 is single-branched, and its projection has a single apparent double point K.

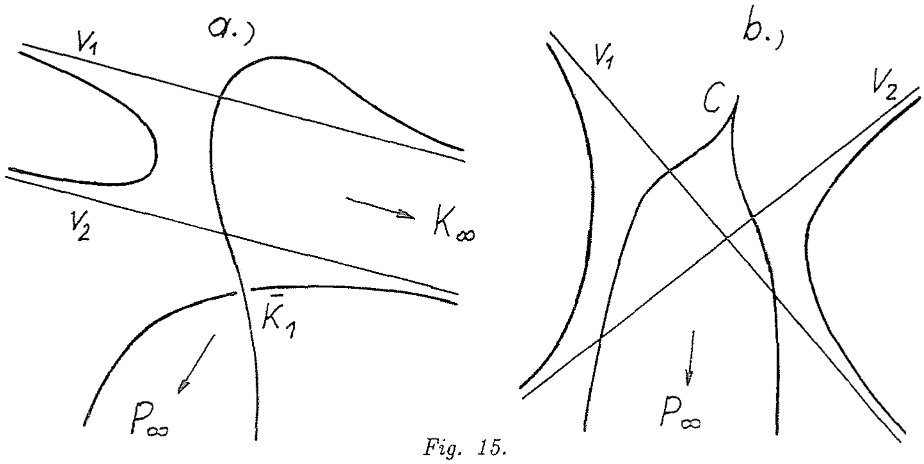

Fig. 15.

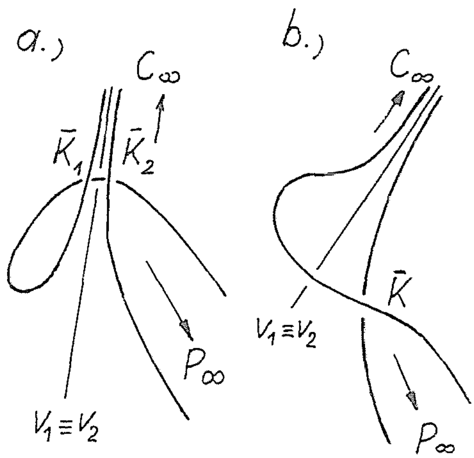

Fig. 16.

4.3 g with Two Parabolic Points

Now, the two intersecting cones must have two pairs of parallel tangential planes, of them the infinitely distant lines of intersection are tangents of a parabolic point each. Directrices of the pair of directional cones contact each other at two different points, with a common tangent each, so the pair of directional cones has two common tangential planes.

Projection of a single-branched g^4 is seen in *Fig. 17*, in all four cases with parabolic points $P_{1\infty}$ and $P_{2\infty}$, but in case

a) without a singular point,

b) with an apparent double point K,

c) with apparent double points K_1 and K_2,

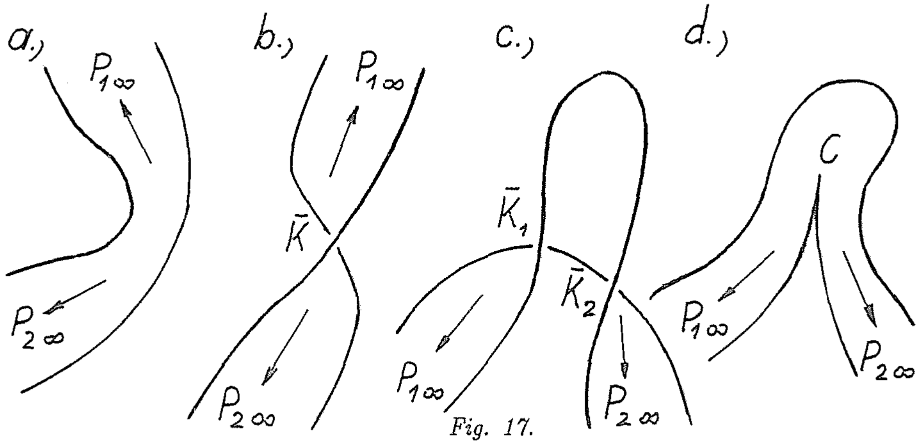

Fig. 17.

d) with an apparent peak C.

It is noteworthy that — if intersection of the vertex straight line of two intersecting cones with the plane of directrices is coincident with the intersection of two common tangents at a point common to both directrices, then the intersecting pair of cones has two common tangential planes rather than two parallel pairs of planes. Now, no parabolic points of the curve of intersection may be spoken of, but there are two real double-point intersections, where g^4 is decomposed to two cone sections. Pair-wise projection curves of a bifurcated g^4 are seen in *Fig. 18*, in case

a) with two parabolic points $P_{1\infty}$ and $P_{2\infty}$,
b) with further two apparent double points K_1 and K_2,
c) with an apparent peak C.

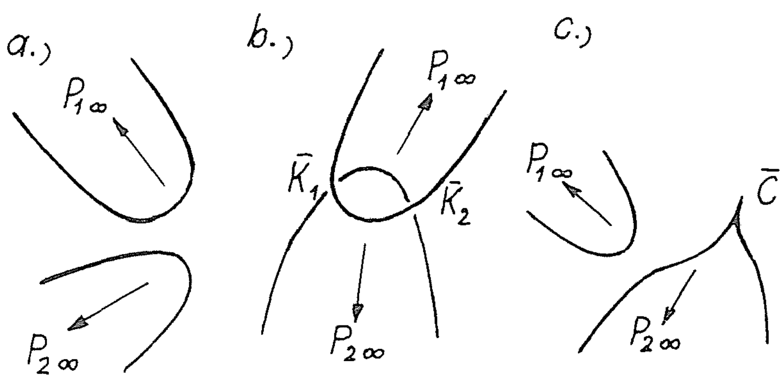

Fig. 18.

4.4 g^4 with a Hyperbolic Point, and an Osculating Point in Infinity

The pair of directional cones is defined in *Fig. 19*, in conformity with the heading, so that the two directrices have a triple contact at a common point X_1. Hence, the two directrices intersect at another real point 1_1. Parallelly shifting one cone of the pair of directional cones results for the intersections in a pair of cones having a pair of parallel tangential planes with the triple-valued pair of generatrices M_1X_1 and M_2X_2, intersecting at the osculating point 0_∞. Another pair of generatrices $M_1 1_1$ and $M_2 1_2$ intersect at the hyperbolic point H_∞. Thereby at all the intersections there will be a terminal tangent and curvilinear branch each tending bilaterally to infinity, closing by osculating the plane at infinity. Since in this case the plane connecting generatrices $M_1 1_1$ and $M_2 1_2$ is common tangential plane of the pair of cones, g^4 has a real singular point: double point in *Fig. 19.*

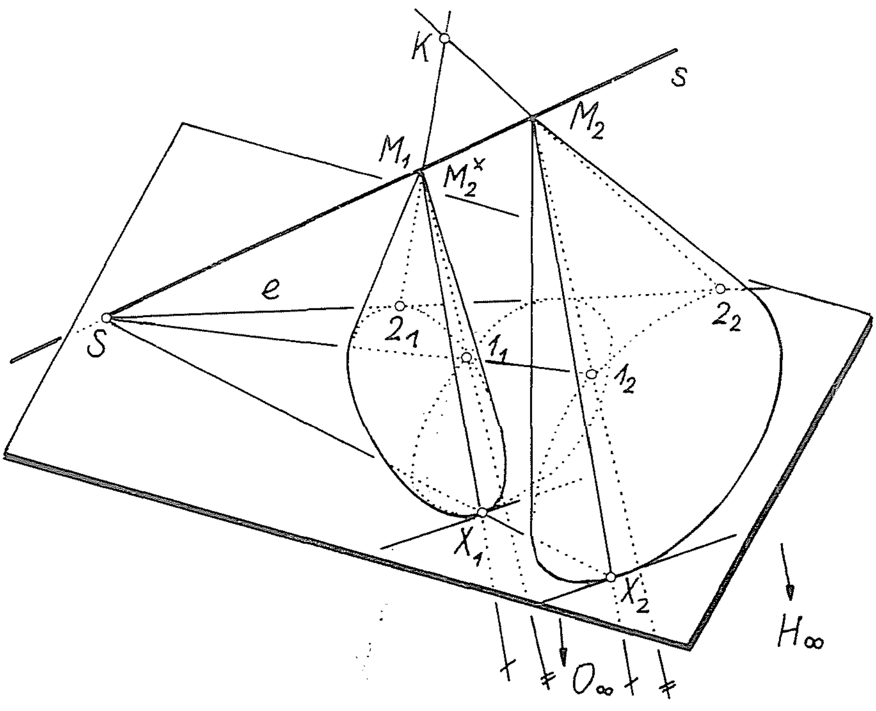

Fig. 19.

Projection of single-branched g^4 is seen in *Fig. 20,* in case

a) with osculating point 0_∞ and terminal tangent v,

b) with a further double point K,
c) with hermit point R,
d) with peak C.

Bifurcated odd projection of g^4 is seen in *Fig. 21,* in case
a) with osculating point 0_∞ and terminal tangent v;
b) with further apparent double points K_1 and K_2;
c) with an apparent double point K.

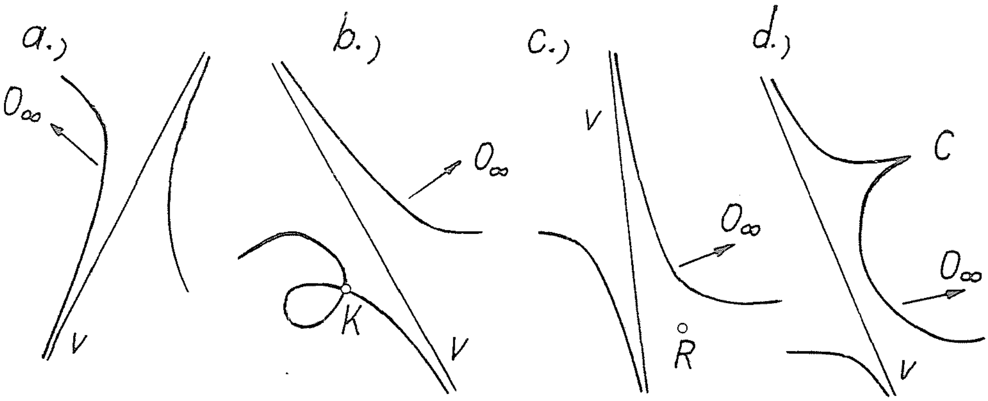

Fig. 20.

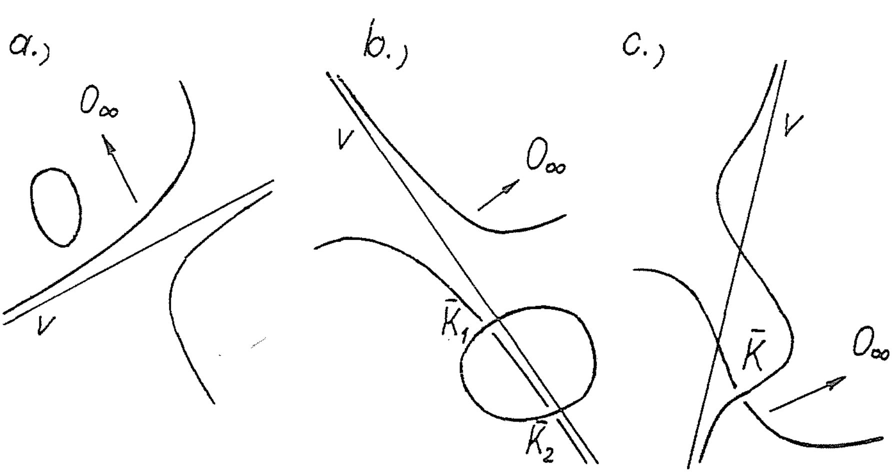

Fig. 21.

4.5 g^4 with a Hyperosculating Point at Infinity

Here the two directrices of the pair of directional cones have to be defined so that they hyperosculate at a common point, at a quadruple constant. Thus, an axial point of each of the two cone section directrices coincides while hyperosculating. Thereby the pair of directional cones have four common adjacent generatrices, along them the cones have common tangential planes. In the parallelly shifted pair of cones, at point $0_{h\infty}$ of the curve of intersection, the space curve hyperosculates the plane in infinity, thus, this plane is stationary osculatory plane of g^4. In the projection of g^4, arcs tending to, and closing at $0_{h\infty}$ open to the same side.

Projection of the single-branched g^4 meeting this condition is seen in *Fig. 22* for case

a) with hyperosculating point $0_{h\infty}$,

b) with further apparent double points K,

c) with apparent peak C.

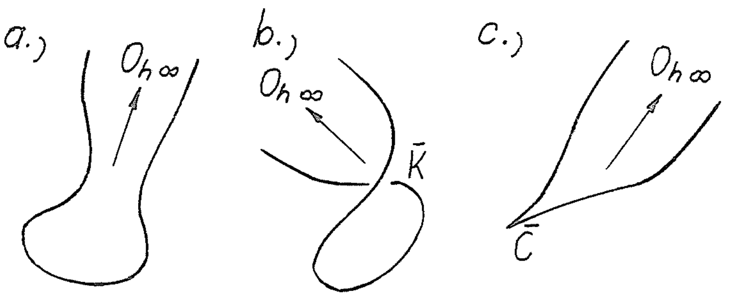

Fig. 22.

The pair of directional cones having common vertex $M_1 = M_2^+$ is seen in *Fig. 23* with two ordered views, so that directrices e and k^+ are in hyperosculating contact at point H_1 of the horizontal base plane. Vertex straight line s passes through vertex M_1 parallelly to the elevation plane. Thereafter the cone with vertex M_2^+ and a circular directrix has been parallelly shifted to the position of cone with vertex M_2, resulting in the affine cone with directrix e, conform to the heading. Tangential planes of these two cones at points H_1 and H_2 of the directrices are normal to the elevation plane, and parallel to each other, at last, they intersect at the tangent at point $0_{h\infty}$ of g^4. Because of the common plane of symmetry, elevation of g^4 is a double projection, arc sections of g^4 are in parabola p^+, with a point $0''_{h\infty}$ in infinity. In the top-view, one branch is a closed curve, the other, in the same side, closes at $0'_{h\infty}$.

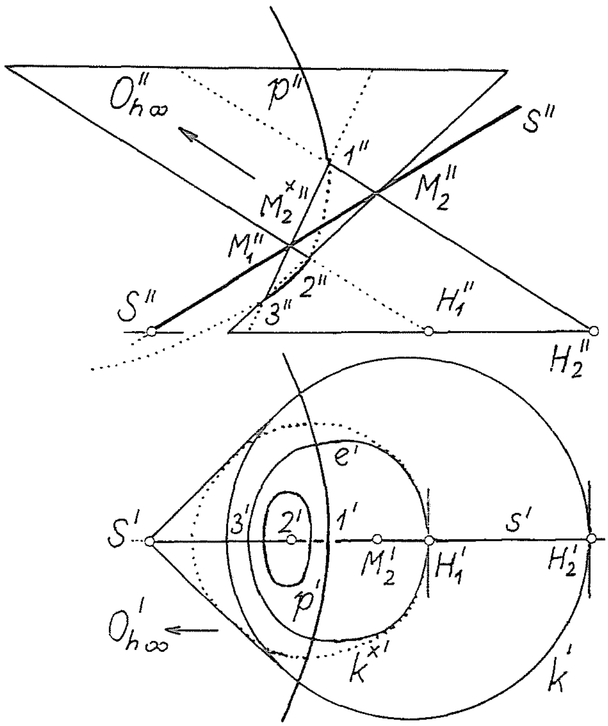

Fig. 23.

5. Space Curve g^4 with Real and Imaginary Points in Infinity

5.1 g^4 with Two Hyperbolic and Two Elliptic Points

In this case, a pair of directional cones with two real and two imaginary common generatrices has to be assumed. So directrices of the pair of directional cones intersect at two real and two imaginary points. The curve of intersection may have singular points.

All the items in *Fig. 24* show projection of a single-branch g^4 with two hyperbolic, and two conjugated complex pairs of points; in case

- *a*) without a singular point,
- *b*) with apparent double point K,
- *c*) with real double point K,
- *d*) with a real peak C,
- *e*) with a real double point K, and two apparent ones K_1 and K_1,
- *f*) for two coincident hyperbolic points with a real double point K_∞; and two apparent ones K_1 and K_2,
- *g*) with hermit point R,

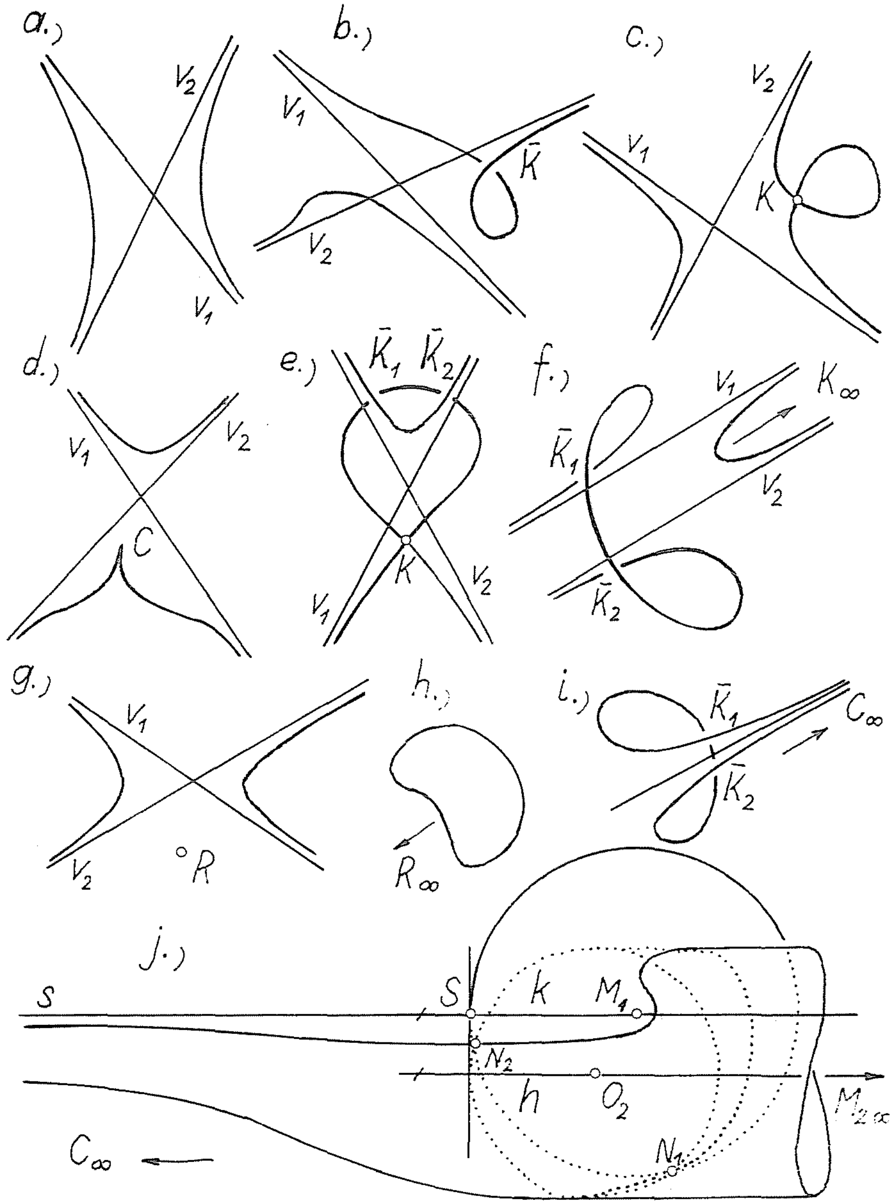

Fig. 24.

h) with two hyperbolic points coincident with hermit point R_∞,

i) with a real peak C_∞ and apparent double points K_1 and K_2,

j) with a peak C_∞ real in a single top-view, when g^4 results from the intersection between the cone with vertex M_1 and a cylinder with a circular directrix centered on O_2.

Beside parallel generatrices k and h, the two shells have a common tangent plane. Cylinder vertex $M_{2\infty}$ is on the cone. g^4 intersects the plane of directrices at points N_1 and N_2.

Bifurcated projections of g^4 are seen in *Fig. 25,* in case

a) with no singular point,

b) with apparent double points K_1 and K_2, an even one,

c) with an apparent double point K, an odd one.

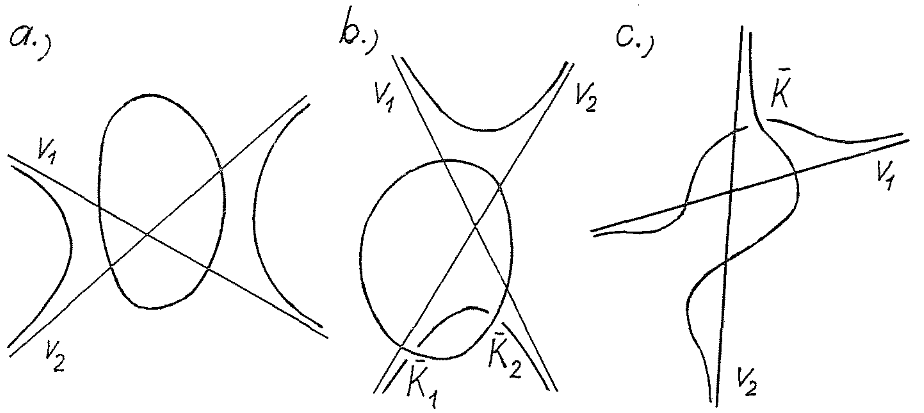

Fig. 25.

5.2 g^4 *with One Parabolic and Two Elliptic Points*

Under these conditions, the pair of cones in intersection must have a pair of parallel tangent planes. A pair of directional cones has to be assumed where the two directrix cone sections have a common tangent at a common point, while they have no other common real point. Beyond that, the pairs of cones to be intersecting may have common tangent planes, a singular point for g^4.

Single-branch projection of g^4 with one parabolic point P_∞ and two elliptic points is seen in *Fig. 26,* in case

a) with apparent double points K_1 and K_2,

b) with a real double point K, and two apparent ones K_1 and K_2,

c) with apparent peak C.

Case *a*) in *Fig. 27* shows the pair of directional cones with vertex $M_1 = M_2^+$ of them two elliptic directrices contact at point A^1. Vertex M_2 of the parallelly shifted cone has been assumed on generatrix $SM_1 = s$ of the cone with vertex M_1, therefore point $M_2 = C$ is peak of g^4. Namely,

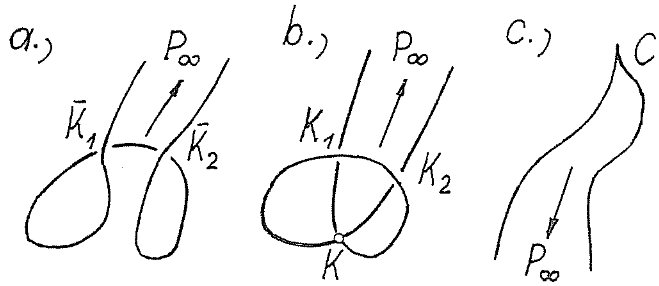

Fig. 26.

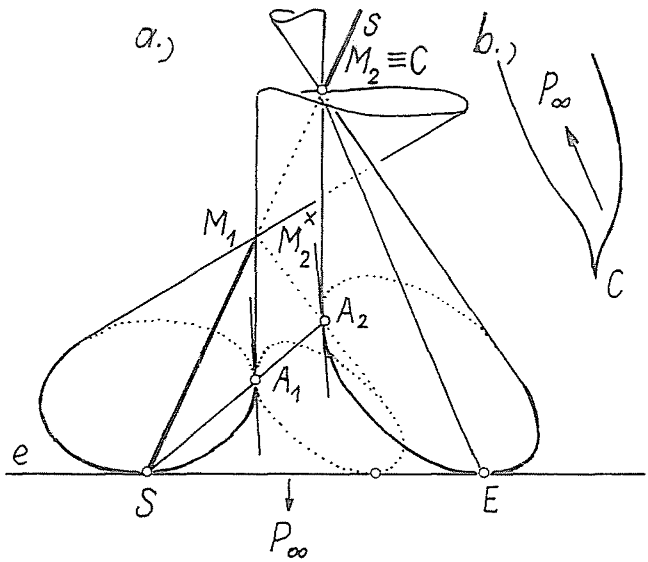

Fig. 27.

the two cones have a common tangent plane along generatrices M_1S and M_2E across tangent e. Apparently, the lower cone shells do not intersect, and the curve of intersection with peak C results from the intersection of upper shells. g^4 is single-branched, with an estimated projection seen under b).

Case a) in *Fig. 28* shows the pair of directional cones with vertex $M_1 = M_2^+$ so that elliptic and hyperbolic directrices e and h^+ have a common tangent at F. Parabolic point P_∞ lies in the direction of generatrix M_1F. M_2 is vertex of the affine cone. Hermit point R of g^4 is at the intersection of generatrices a_1 and a_2 in the common tangent plane. As seen from the position of the auxiliary plane of construction, no real intersection points adjacent to R may be constructed. Hermit point R may be consid-

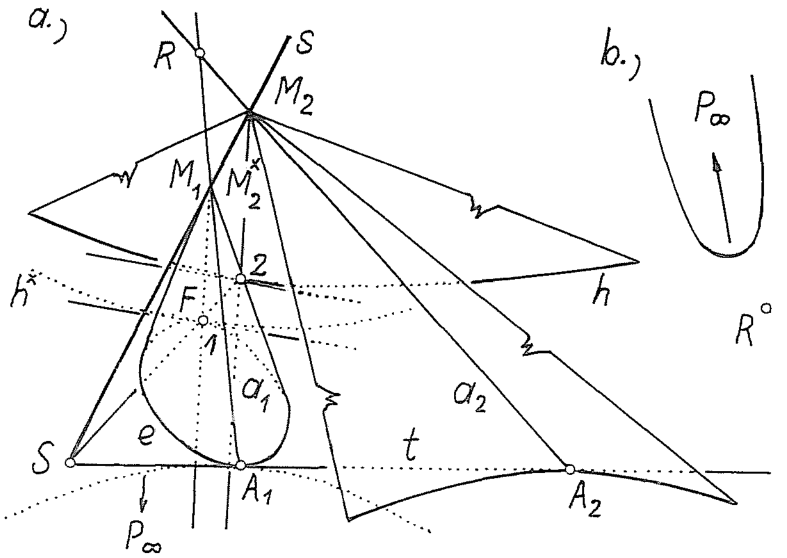

Fig. 28.

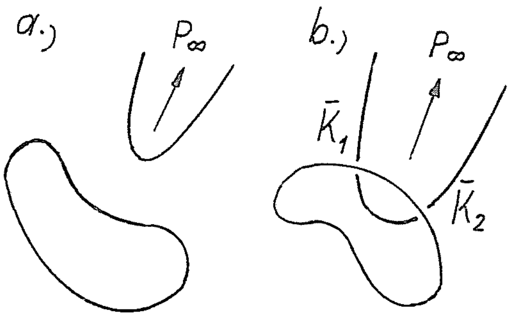

Fig. 29.

ered as branch of bifurcated pair-wise g^4 shrunk to a point, while P_∞ lies on the other branch, as seen from the projection under *b*). A bifurcated pair-wise projection of g^4, with parabolic point P_∞ is seen in *Fig. 29,* in case

a) without singular point,

b) with apparent double points K_1 and K_2.

5.3 g^4 with Four Elliptic Points

In this case, the pair of directional cones has two common conjugated complex pairs of generatrices. Depending on the definition, single or double

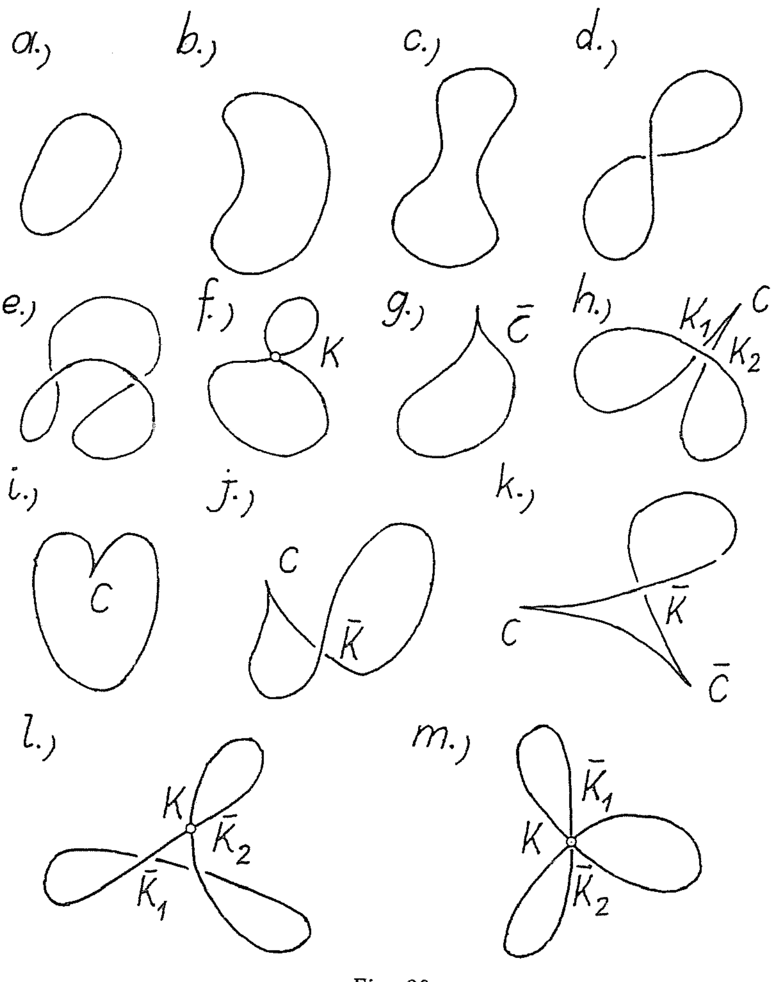

Fig. 30.

branched g^4 may result, maybe presenting singular points both real and apparent in the finite.

Figs 30 and *31* show various projections of single-branched and bifurcated g^4, respectively, showing also their singular points. In case j) of *Fig. 31,* bifurcated intersection is reduced to hermit points R_1 and R_2, namely the two intersecting cone shells contact only at these two points. For the sake of completeness, case k) is an example for g^4 decomposing to two cone sections, where the number of possible real or apparent double points cannot exceed two.

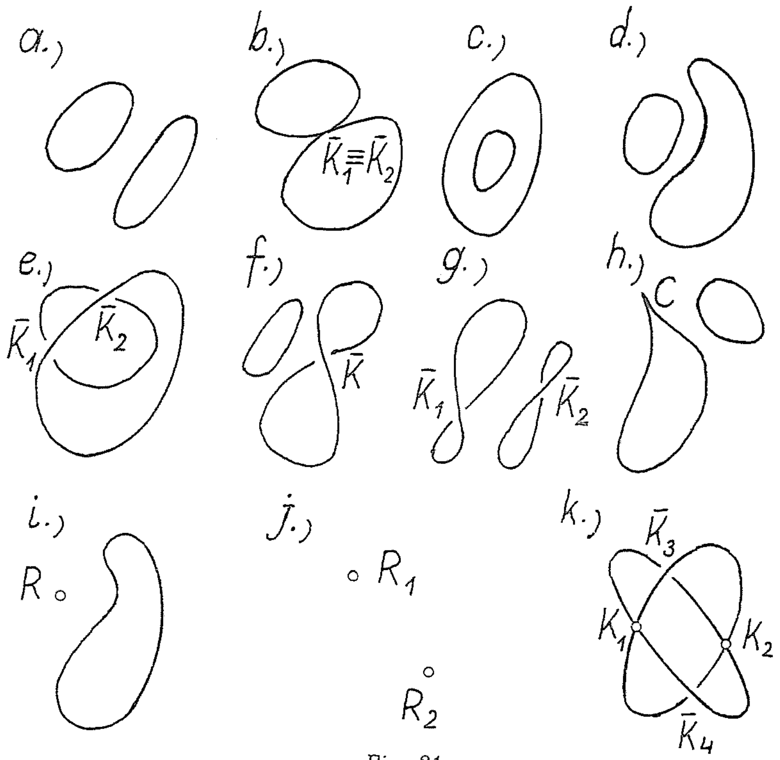

Fig. 31.

Conclusive Remarks

There are too many of various space curves g^4 possible according to the classification to be presented all. Spatial conditions were seen to be restricted to two-dimensional problems, where directrices of the pair of directional cones are assumed properly, true to specifications. By the way, the so-called degenerating intersections were only referred to, rather than to be explained. In the formal analysis of g^4, also symmetry conditions might be disclosed. For instance, in *Fig. 23,* g^4 has a single plane of symmetry parallel to the elevation plane, where its projection is a double projection. g^4 with two, or even three planes of symmetry may be constructed, in the latter case there is a central symmetry.

Practical applications of first-kind, fourth-order space curves are possible primarily in cases of decomposition of g^4, for boundary and shell surfaces. Application possibilities of g^4 in metric geometry may be also of interest. There is a rich literature on first and second-kind fourth-order space curves in chapters on projective and constructive geometry.