THE EQUILIBRIUM EQUATIONS OF MEMBRANE SHELLS EXPRESSED IN GENERAL SURFACE COORDINATES

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Abstract

The aim of the paper is to derive the equations expressing the static equilibrium of membrane shells without introducing the bending theory. The first part gives a comprehensive introduction to the notions of tensor analysis which are needed in the forthcoming mechanical applications and contains a brief sketch of the background in classical differential geometry. The derived formulas are illustrated on some simple examples in the last chapter.

Introduction

The stresses in membrane shells are usually determined by applying Pucher's differential equation, which enables us to calculate the projections of the stress components onto an external coordinate system. In this paper a general equation is presented, by means of which the stress components can be expressed in an arbitrary surface coordinate system. This equilibrium equation is usually introduced as a special case of the bending theory. The aim of this paper is to derive the equation directly, applying as simple tools as possible. The geometry of the curved, two-dimensional surfaces, as the mathematical background of the equilibrium equation, is discussed. The paper is intended to be a comprehensive introduction for graduate students in civil engineering and architecture.

Pucher's differential equation excellently demonstrates the fact, that the application of coordinate systems is of advantage when describing physical phenomena mathematically. This fact is generally accepted, but we must not forget, that the physical phenomena are totally independent of our coordinate systems. The representation of physical phenomena in coordinate systems may be regarded therefore as a disturbing type of description.

It is self-evident, that if a natural law holds, its representation holds in an arbitrary coordinate system, as well. There are a couple of rules, which permit us to transform the representation of a natural law from a coordinate system into another one. If the coordinates of a phenomenon observed in several coordinate systems are transforming under these rules, the phenomenon is said to be coordinate-invariant or simply invariant.

To eliminate the mentioned disturbing effect of the coordinate systems, basically two ways are possible:

- The "direct" description, which doesn't use any coordinate systems. The technical application of this type of equations may be sometimes cumbersome.
- The formulation of general equations, where the form of the coordinate system itself is "blank". During technical applications any type of coordinate system may be "substituted" into the equation.

In the section 1.1 we will introduce the basic notions by the "direct" way for the sake of comparison. Further on the second way will be followed and the equivalence of the two different descriptions will be indicated at appropriate places.

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1. Mathematical preliminaries

1.1. Invariant quantities — tensors

1.1.1. Scalars

Physical phenomena are quantitatively described by numbers. In the simplest case a single numerical data identifies the observed variable. This type of physical variables are called scalars. For example: volume, mass, temperature.

If each point of the physical space is associated with a number, we arrive at a scalar field. For example if we measure the temperature at each point of a room, this data is represented by a scalar field. Two-dimensional scalar fields can be visualized as (generally curved) surfaces in the three-dimensional space.

Scalars can be interpreted as homogeneous, linear scalar-scalar functions. A function is called homogeneous and linear if the following equations hold:

$$f(a) + f(b) = f(a + b) \tag{1}$$

$$f(\lambda a) = \lambda f(a) \tag{2}$$

The scalar s defines for example the homogeneous linear scalar-scalar function $h(x) = sx$. For the sake of generality, scalars interpreted as homogeneous linear functions will be called 0th order tensors. The meaning of this will (hopefully) become evident in the following sections.

1.1.2. Vectors

Physical quantities identified by a number and a direction are called vectors. For example: velocity, acceleration, force. Vectors may be visualized as directed intervals. The n-dimensional vector space is the set of all n-dimensional vectors, where vectors can be added with each other in the usual way and can be multiplied by scalars. The three-dimensional Euclidean space is for example the vector space of the above-mentioned directed intervals. Two vectors are called equivalent in the vector space, if they have the same direction and magnitude.

If each point of the physical space is associated with a vector, we arrive at a vector field (not to be confused with the vector space). For example if we measure the magnitude and direction of velocity of the particles on the surface of a streaming liquid, this set of data is represented by a two-dimensional vector field. As an other example we can measure the principal stresses at each point of a three-dimensional elastic continuum to arrive at three different three-dimensional vector fields.

Vectors can be interpreted as homogeneous linear scalar-vector or vector-scalar functions. Vector $\mathbf{v}$ defines for example the scalar-vector function (= vector-valued function with independent scalar variable) $g(x) = \mathbf{v}x$ or the scalar-vector function (= scalar-valued function with independent vector variable) $k(\mathbf{x}) = \mathbf{v}\mathbf{x}$. Vectors interpreted as homogeneous linear functions will be called *1*st order tensors.

1.1.3. Second order tensors

Certain physical quantities can be described neither by scalars nor by vectors. This is the point, where second order tensors are introduced. Second order tensors are homogeneous linear vector-vector functions (= vector-valued functions with independent vector variable). If each point of the physical space is associated with a second order tensor, we arrive at a tensor field. A *2*nd order tensor field expresses the homogeneous linear connection between two vector fields. Second order tensors can hardly be directly visualised. We can form some image, however, by observing, that the application of a tensor to the unit sphere (formed by unit vectors of the three-dimensional Euclidean space) distortes the sphere into a general ellipsoid. During this transformation the unit vectors are rotated and their length changes, as well.

Two basic numbers are associated with a tensor: the order and the dimension. The order of the tensor fixes the number of vectors between which the tensor defines a functional relation. The dimension of the tensor fixes the dimension of the space where the above-mentioned vectors are interpreted.

To mention some examples for second order tensors:

The rotation tensor describes the rotation of a rigid body by defining a functional relation between the vectors associated with the points of the original and the rotated body.

The planar state of stress is described by the stress tensor defining a functional relation between a direction vector and the stress vector in that direction.

1.2. Representation in coordinate systems

The n-dimensional base is a system of n linearly independent vectors. We will use mainly 2-and 3-dimensional bases. If a tensor is given with respect to a base, we speak about representation in a coordinate system.

1.2.1. Orthogonal systems

In the simplest case the vectors of our base are mutually perpendicular unit vectors, this is called an orthogonal coordinate system. The base consisting of the vectors $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, and $\mathbf{e}_{(3)}$ will be denoted by K. The orthogonality of this base can be expressed by using the scalar (dot) product:

$$\mathbf{e}_{(1)}\mathbf{e}_{(1)} = \mathbf{e}_{(2)}\mathbf{e}_{(2)} = \mathbf{e}_{(3)}\mathbf{e}_{(3)} = 1 \tag{3}$$

and

$$\mathbf{e}_{(1)}\mathbf{e}_{(2)} = \mathbf{e}_{(2)}\mathbf{e}_{(3)} = \mathbf{e}_{(1)}\mathbf{e}_{(3)} = 0 \tag{4}$$

The above equations can be expressed more concisely by

$$\mathbf{e}_{(i)}\mathbf{e}_{(k)} = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases} \tag{5}$$

(In the forthcoming formulas the latin indices i, j, k, etc. are assumed to be equal to 1, 2 or 3.)

The symbol

$$\mathbf{e}_{(i)}\mathbf{e}_{(k)} = \delta_{ik} \tag{6}$$

is commonly used and called the Kronecker-delta. According to (5) and (6) the Kronecker-delta is defined by

$$\delta_{ik} = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases} \tag{7}$$

1.2.1.1. Vectors

Since the vectors of the base K are linearly independent, an arbitrary vector can be expressed with respect to this base as

$$\mathbf{v} = v_1\mathbf{e}_{(1)} + v_2\mathbf{e}_{(2)} + v_3\mathbf{e}_{(3)} \tag{8}$$

The set of numbers $v_i = (v_1, v_2, v_3)$ is called the coordinates of the vector $\mathbf{v}$ in the system K. We arrive at the geometrical interpretation of the coordinates if eq. (8) is multiplied by the vectors $\mathbf{e}_{(i)}$. ("Multiplication" will mean, that we form the dot product with each member in the equation)

$$\mathbf{v}\mathbf{e}_{(i)} = v_i \tag{9}$$

The above formula contains three equations, depending on, which base vector was eq. (8) multiplied with. Equation (9) demonstrates, that the coordinates v_i are the orthogonal projections of the vector onto the coordinate axes.

In calculations the vector $\mathbf{v}$ is often substituted by the coordinates v_i, which doesn't mean, that the two things are identical. If we change the coordinate system, the coordinates change, but the vector doesn't. We introduce the notation

$$v_i = K(\mathbf{v}) \tag{10}$$

expressing, that v_i is the image of $\mathbf{v}$ in the system K.

1.2.1.2. Second order tensors

A second order tensor is uniquely given if we know the transformed version of an arbitrary vector $\mathbf{v}$. It seems to be logical to deal with the transformations of the base vectors first, since if

$$\mathbf{T}\,\mathbf{e}_{(i)} = \mathbf{f}_{(i)} \tag{11}$$

and

$$\mathbf{u} = \sum_i u_i\,\mathbf{e}_{(i)} \tag{12}$$

holds, than obviously

$$\mathbf{T}\mathbf{u} = \mathbf{v} \tag{13}$$

where

$$\mathbf{v} = \sum_i u_i\mathbf{f}_{(i)} \tag{14}$$

Multiplying (14) by $\mathbf{e}_{(k)}$, we arrive at

$$\mathbf{v}\,\mathbf{e}_{(k)} = \sum_i u_i\mathbf{f}_{(i)}\mathbf{e}_{(k)} \tag{15}$$

The left hand side can be expressed by using (9):

$$\mathbf{v}\,\mathbf{e}_{(i)} = v_i \tag{16}$$

Introducing the notation

$$\mathbf{e}_{(i)}\mathbf{f}_{(k)} = t_{ik} \tag{17}$$

we arrive at

$$v_i = \sum_k t_{ik}\, u_k \tag{18}$$

Forming a table with the values t_{ik} in the following way is called the matrix of the tensor $\mathbb{T}$ in the system K, more concisely $K(\mathbb{T})$:

$$t_{ik} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \tag{19}$$

The determinant of the above matrix is called the determinant of the tensor $\mathbb{T}$ in the system K and is denoted by

$$\det t_{ik} = t \tag{20}$$

We will calculate as an illustrative example the elements of the matrix of the planar rotation tensor $\mathbb{F}$ in the K system. The vector transformation can be written as:

$$\mathbf{a}' = \mathbb{F}\,\mathbf{a} \tag{21}$$

With coordinates:

$$a'_\alpha = \sum_{\alpha=1}^{2} f_{\alpha\beta}\, a_\beta \tag{22}$$

(In the forthcoming formulas the greek indices will be equal to 1 or 2.) Similarly to (17):

$$f_{\alpha\beta} = \mathbf{e}_{(\alpha)}\,\mathbf{e}'_{(\beta)} \tag{23}$$

Figure 1 demonstrates, that by rotating the base vectors of the planar system K we arrive at the vectors

$$\begin{aligned} \mathbf{e}'_{(1)} &= \mathbf{e}_{(1)} \cos\varphi + \mathbf{e}_{(2)} \sin\varphi \\ \mathbf{e}'_{(2)} &= \mathbf{e}_{(1)} (-\sin\varphi) + \mathbf{e}_{(2)} \cos\varphi \end{aligned} \tag{24}$$

According to this the representation of the tensor $\mathbf{F}$ in the system K is given by

$$f_{\alpha\beta} = K(\mathbf{F}) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{25}$$

Using (25) an arbitrary vector can be transformed in K.

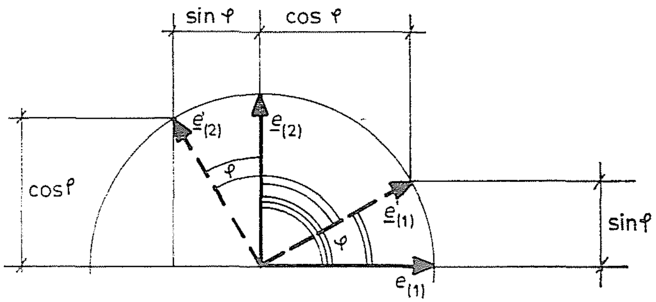


Fig. 1. The tensor of planar rotation

1.2.2. Skew systems

1.2.2.1. Vectors

The vector v can be expressed not only in the system K, but in an arbitrary system A determined by the base vectors $\mathbf{a}_{(i)}$ on the condition, that the base vectors are linearly independent:

$$\mathbf{a}_{(1)}(\mathbf{a}_{(2)} \times \mathbf{a}_{(3)}) = V \neq 0 \tag{26}$$

V denotes the volume of the parallelepyds spanned by the three vectors. The vector v can be expressed as

$$\mathbf{v} = \sum_i v_i \mathbf{a}_{(i)} \tag{27}$$

Similarly to (10) $v_i = A(\mathbf{v})$. In the skew system A the scalar product of the vector v and the base vectors isn't equal to the vector coordinates, since the scalar product of two different base vectors isn't zero. We will introduce therefore the reciprocal base $\bar{\mathbf{a}}_{(i)}$ by

$$\mathbf{a}_{(i)}\bar{\mathbf{a}}_{(k)} = \delta_{ik} \tag{28}$$

Multiplying eq. (27) by $\bar{\mathbf{a}}_{(k)}$ yields

$$\mathbf{v}\,\bar{\mathbf{a}}_{(k)} = v_k \tag{29}$$

which indicates, that the coordinates of v in the system A are equal to the orthogonal projections to the reciprocal base in the proper scale. This is illustrated in the plane by Fig. 2.

Since the base vectors of the reciprocal system are linearly independent, v can be expressed as

$$\mathbf{v} = \sum_i \bar{v}_i\, \bar{\mathbf{a}}_{(i)} \tag{30}$$

Multiplying (30) by $\mathbf{a}_{(i)}$ yields:

$$\bar{v}_i = \mathbf{v}\,\mathbf{a}_{(i)} \tag{31}$$

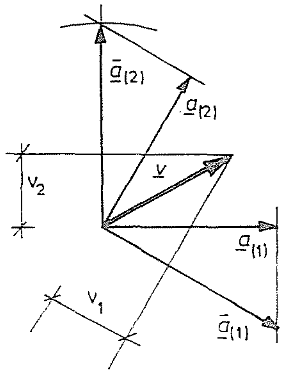


Fig. 2. The skew reciprocal base

That means, that in the skew system A the vector v can be equally given by the numbers v_i or $\bar{v}_i$. The numbers v_i will be called the contravariant coordinates, the numbers $\bar{v}_i$ the covariant coordinates of the vector v in the system A. In the orthogonal system K the contravariant and covariant coordinates naturally coincide. For further use we introduce the notation

$$\bar{v}_i = v_i = (v_1, v_2, v_3) \tag{32}$$

$$v_i = v^i = (v^1, v^2, v^3)$$

Let's now examine the geometrical interpretation of the symbols v_i and v^i. For the sake of simplicity we will work in the plane. Similarly to the notation introduced above, the reciprocal vectors $\bar{\mathbf{a}}_{(\alpha)}$ will be denoted by $\mathbf{a}^{(\alpha)}$. Rewriting now the two previous equations with the new notations yields

$$\mathbf{v} = \sum_{\alpha} v\, \mathbf{a}^{(\alpha)} \tag{33}$$

and

$$\mathbf{v}\, \mathbf{a}_{(\beta)} = v_\beta \tag{34}$$

Equation (34) contains the projections of v in the directions of the vectors $\mathbf{a}_{(\alpha)}$ expressed in proper units. Since the vectors $\mathbf{a}_{(\alpha)}$ aren't unit vectors, we have to choose the quantity $|\,\mathbf{a}_{(\alpha)}\,|$ as unit. If we don't, than the magnitude of the projection may be calculated by

$$v_{(\alpha)} = \frac{v_\alpha}{|\,\mathbf{a}_{(\alpha)}\,|} \tag{35}$$

Summarizing the above investigations: the covariant coordinates mean the orthogonal projections, the contravariant components the projections in the direction of the coordinate axes. We are now interested in the problem, how to calculate the covariant coordinates from the contravariant ones. Expressing v in both ways:

$$\mathbf{v} = \sum_i \mathbf{a}_{(i)} v^i = \sum_i \mathbf{a}^{(i)} v_i \tag{36}$$

Let's multiply eq. (36) by $\mathbf{a}_{(i)}$ yielding

$$v_k = \sum_i v^i \mathbf{a}_{(i)} \mathbf{a}_{(k)} \tag{37}$$

The above equation demonstrates, that the connection between the two representations is given by the scalar products of the base vectors. This products depend on two indices, we introduce the notation

$$\mathbf{a}_{(i)} \mathbf{a}_{(k)} = g_{ik} \tag{38}$$

The numbers g_{ik} are the elements of a matrix. Substituting (38) into (37) yields

$$v_k = \sum_i v^i g_{ik} \tag{39}$$

If the inverse of the matrix g_{ik} exists (let's denote it by g^{ik}), than it is easily derived, that

$$v^k = \sum_i v_i g^{ik} \tag{40}$$

There are some useful applications of the above derived results. Let's calculate the scalar product of two vectors in the skew coordinate system! We will treat the following two vectors:

$$\mathbf{u} = \sum_i \mathbf{a}_{(i)} u^i = \sum_i \mathbf{a}^{(i)} u_i \tag{41}$$

$$\mathbf{v} = \sum_i \mathbf{a}_i v^i = \sum_i \mathbf{a}^i v_i$$

The scalar product in contravariant representation:

$$\mathbf{u}\,\mathbf{v} = \sum_{ik} u^i v^k \mathbf{a}_{(i)} \mathbf{a}_{(k)} \tag{42}$$

By using the formerly introduced g_{ik} notation:

$$\mathbf{u}\,\mathbf{v} = \sum_{ik} u^i v^k g_{ik} \tag{43}$$

Deriving the same expression by using the co variant components: (44)

$$\mathbf{u}\,\mathbf{v} = \sum_{ik} u_i v_k g^{ik}$$

Now let's substitute eq. (39) into eq. (43):

$$\mathbf{u}\,\mathbf{v} = \sum_i u^i v_i \tag{45}$$

and similarly:

$$\mathbf{u}\,\mathbf{v} = \sum_i u_i v^i \tag{46}$$

Equations (45) and (46) closely resemble to their analogons in orthogonal coordinate systems.

1.2.2.2. Second order tensors

We will proceed as we did in orthogonal systems. A second order tensor is given in the most natural and simple way if we know the $\mathbf{b}_{(i)}$ transformed versions of the $\mathbf{a}_{(i)}$ base vectors. Knowing this vectors the transformation of an arbitrary vector may be executed, since if

$$\mathbf{T}\,\mathbf{a}_{(i)} = \mathbf{b}_{(i)} \tag{47}$$

and

$$\mathbf{u} = \sum_i u^i \mathbf{a}_{(i)} \tag{48}$$

then on the basis of (13) and (14) obviously

$$\mathbf{T}\,\mathbf{u} = \mathbf{v} \tag{49}$$

where

$$\mathbf{v} = \sum_i u^i \mathbf{b}_{(i)} \tag{50}$$

Let's multiply (50) with $\mathbf{a}_{(k)}$:

$$\mathbf{v}\,\mathbf{a}_{(k)} = \sum_i u^i \mathbf{b}_{(i)}\mathbf{a}_{(k)} \tag{51}$$

The left hand side can be written because of (31) and (32) as:

$$\mathbf{v}\,\mathbf{a}_{(k)} = v_k \tag{52}$$

Let

$$\mathbf{a}_{(i)}\mathbf{b}_{(k)} = t_{ik} \tag{53}$$

According to this:

$$v_i = \sum_k t_{ik} u^k \tag{54}$$

To calculate the quantities t_{ik} we used the base vectors $\mathbf{a}_{(i)}$, therefore the matrix t_{ik} will be called the covariant representation of the tensor $\mathbf{T}$ in the skew system A. The contravariant representation t^{ik} may be derived in a similar way. Remark, that transforming the components of the vector $\mathbf{u}$ by the matrices t_{ik} or t^{ik} we always arrive at $\mathbf{v}$ components of different representation. To avoid this inconvenience let's introduce the mixed representation of the tensor $\mathbf{T}$ by

$$t^{i}_{.k} = \mathbf{a}^{(i)}\mathbf{b}_{(k)} \tag{55}$$

and

$$t_{i}^{.k} = \mathbf{a}_{(i)}\mathbf{b}^{(k)} \tag{56}$$

where naturally

$$\mathbf{T}\,\mathbf{a}_{(i)} = \mathbf{b}_{(i)} \tag{57}$$

and

$$\mathbf{T}\,\mathbf{a}^{(i)} = b^{(i)} \tag{58}$$

$$v_i = \sum_k t_i^{.k} u_k \tag{59}$$

$$v^i = \sum_k t^{i}_{.k}\, u^k \tag{60}$$

The sequence order of the indices is not indifferent, since the matrices $t^{i}_{.k}$ and $t_{i}^{.k}$ are in general not identical. We will introduce now the so-called Einstein summation convention for the dummy indices:

$$a_i b^i = \sum_i a_i b^i \tag{61}$$

If the dummies are the indices of a mixed representation tensor, then this summation is called the contraction of the tensor resulting a tensor of order 0, that means, a scalar. This operation is equivalent to the summation of the components in the main diagonal.

In generality : by summing a tensor of order n to a single pair of dummy indices we arrive at a tensor of order $(n-2)$.

Let's examine now the meaning of the symbols g_{ik} and g^{ik} introduced in eq. (38). By comparing (53) with (38) we can observe that the symbol g_{ik} is the covariant representation of the $\mathbf{E}$ unit tensor, since on the basis of the equations (3) and (38) this tensor maps the base vectors onto themselves. Similarly the symbol g^{ik} is the contravariant representation of the unit tensor. In the orthogonal system we have naturally

$$g_{ik} = g^{ik} = \delta_{ik} \tag{62}$$

The symbols g^{ik} and g_{ik} are usually called the components of the metric tensor. This name will be explained later.

We are going to investigate the relationship between the four possible representations of a second order tensor $\mathbf{T}$ in the skew system A. Since the various representations can be computed by using the covariant and contravariant base vectors, the transformation rules between the representations can be derived from the relation between the base vectors. Let's express the vectors $\mathbf{a}^{(i)}$ as a linear combination of the vectors $\mathbf{a}_{(i)}$:

$$\mathbf{a}^{(i)} = n^{ik}\mathbf{a}_{(k)} \tag{63}$$

By multiplying eq. (63) with $\mathbf{a}^{(i)}$ we arrive at

$$n^{ik} = g^{ik} \tag{64}$$

According to this the correspondence between the two systems is given by

$$\text{(a)}\quad \mathbf{a}^{(i)} = g^{ik}\mathbf{a}_{(k)} \tag{65}$$

$$\text{(b)}\quad \mathbf{a}_{(i)} = g_{ik}\mathbf{a}^{(k)}$$

Equation (65) enables us to determine the relation between the different tensor representations. Let

$$t_{ik} = \mathbf{a}_{(i)}\mathbf{b}_{(k)} \tag{66}$$

If we substitute (65/b) into (66), we arrive at

$$t_{ik} = g_{ik}\mathbf{a}^{(i)}\mathbf{b}_{(k)} \tag{67}$$

On the basis of (55):

$$t_{ik} = g_{ik}t^{i}_{\cdot k} \tag{68}$$

The relation between two arbitrary representations of the tensor $\mathbb{T}$ may be derived in a similar way. Remark, that the multiplication with g^{ik} or g_{ik} results the "moving" of an index up or down, respectively. Applying this to the metric tensor $\mathbb{G}$:

$$g_{ik}g^{ik} = g^{i}_{\cdot k} = \delta^{i}_{k} \tag{69}$$

Developing this equation for two dimensions we arrive at the following formulas:

$$g^{11} = \frac{g_{22}}{g} \qquad g^{12} = g^{21} = -\frac{g_{12}}{g} \qquad g^{22} = \frac{g_{11}}{g} \tag{70}$$

where g denotes the determinant according to (20).

1.2.3. Curvilinear systems

The location of a point in space may be identified not only by the coordinates introduced before, but by the means of other parameters, as well. We will use the parameters Θ_i. Let $A(\mathbf{x}) = x_i$ the representation of the position vector $\mathbf{x}$ in the skew system A. The functional relation

$$x_i = f(\Theta_j) \tag{71}$$

has to exist. Let's consider the parameters Θ_i as the coordinates of the position vector x. If the function f is non-linear, then the quantities Θ_i are called the curvilinear coordinates of x. This fact will be denoted by

$$A^G(\mathbf{x}) = \Theta_i \tag{72}$$

The transformation of the coordinates to the system A^G can be carried out only in the case if the function f is invertible, i.e. the mapping is unique both ways. The curvilinear coordinate systems are often applied, for example the spherical coordinate system, called the spatial polar system, as well. For this special case eq. (71) may be written as

$$\begin{aligned} x_1 &= r\cos\varphi\ \sin\gamma \\ x_2 &= r\sin\varphi\ \sin\gamma \\ x_3 &= r\cos\gamma \end{aligned} \tag{73}$$

In the system A^G correspond to the constant value of any single parameter a curved surface in the three-dimensional space. If two parameters are simultaneously constant, then we arrive at space curves (lines) after which the A^G system was named. ("Curvilinear system") The system A^G can be treated locally as a skew system. In other words, the system A^G defines a skew system A at each point of the three-dimensional space. The base vectors of the local system A are given by the tangent vectors of the coordinate lines

$$\mathbf{a}_{(i)} = \frac{\partial x}{\partial \Theta_i} \tag{74}$$

(The derivation of vector fields will be discussed in section 4.2 in detail.) Up to now we were dealing with the curvilinear representation of the position vector x, but this doesn't answer the question about the curvilinear representation of an arbitrary vector v with origin differing from the origin of the coordinate lines. For convenience we define the curvilinear representation $A^G(\mathbf{v})$ as the representation of v in the skew system A determined by the system A^G at the origin of v by the equation (74). Remark, that the representation of the metric tensor depends on the coordinates, as well

$$g_{ik} = \frac{\partial \mathbf{x}}{\partial \Theta_i} \cdot \frac{\partial \mathbf{x}}{\partial \Theta_k} \tag{75}$$

1.3. Transformation of coordinates

Our aim is to determine the transformation rules for tensor coordinates if we switch from system A^G to $\bar{A}^G$. The base vectors are those defined by eq. (74):

$$\mathbf{a}_{(i)} = \frac{\partial \mathbf{x}}{\partial \Theta_i} \tag{76}$$

and the base vectors of the system $\bar{A}^G$

$$\bar{\mathbf{a}}_{(i)} = \frac{\partial \mathbf{x}}{\partial \bar{\Theta}_i} \tag{77}$$

where

$$\mathbf{x} = f(\Theta_i) \tag{78}$$

$$\Theta_i = h(\bar{\Theta}_i) \tag{79}$$

Let's express the vectors $\bar{\mathbf{a}}_{(i)}$ as the linear combinations of the vectors $\mathbf{a}_{(i)}$:

$$\bar{\mathbf{a}}_{(i)} = \beta_{\bar{i}}^{j} \mathbf{a}_{(j)} \tag{80}$$

According to this equation the matrix β (which is quadratic, of course) inherits the first index from the original system, the second one (with-) from the transformed system. Let

$$\mathbf{v} = v^i \mathbf{a}_{(i)} = \bar{v}^i \bar{\mathbf{a}}_{(i)} \tag{81}$$

On the basis of (80) and (81):

$$\text{(a)} \quad \bar{v}^i = v^k \beta_k^{\bar{i}} \tag{82}$$

$$\text{(b)} \quad \bar{v}_i = v_k \beta_{\bar{k}}^{i}$$

We can determine the relation between the representations g_{ik} and $\bar{g}_{ik}$ of the metric tensor:

$$g_{ik} = \mathbf{a}_{(i)} \mathbf{a}_{(k)} \tag{83}$$

$$\bar{g}_{ik} = \bar{\mathbf{a}}_{(i)} \bar{\mathbf{a}}_{(k)}$$

and on the basis of (80):

$$g_{ik} = \beta_{\bar{i}}^{j} \beta_{\bar{k}}^{l} g_{jl} \tag{84}$$

Based on the above formulas the general transformation rule for the coordinates of a second order tensor $\mathbf{t}$ is easily derived. Let

$$t_{ik} = \mathbf{a}_{(i)} \mathbf{b}_{(k)} \tag{85}$$

$$t_{ik} = \bar{\mathbf{a}}_{(i)} \mathbf{b}_{(k)}$$

and on the basis of (80) and (82):

$$\text{a)} \quad \bar{t}_{ik} = \beta_{\bar{i}}^{j} \beta_{\bar{k}}^{l} t_{jl} \tag{86}$$

$$\text{b)} \quad \bar{t}_{ik} = \beta_{\bar{i}}^{i} \beta_{\bar{k}}^{k} t_{ik}$$

Remark, that eq. (86) serves in many cases as an alternative definition for tensors. In practical applications we can decide often on the basis of this formula, whether the examined phenomenon can be described by a tensor or not. We have to measure in an experiment the coordinates in two different coordinate systems, and if the measured quantities transform under the rules prescribed by eq. (86), then they are the representations of a tensor. The connection between the two coordinate systems is given by the matrix β.

Based on the equations (76), (77), (78) and (79) we can define now the components of the matrix β by the equations

$$\beta_i^{\bar{k}} = \frac{\partial \Theta_k}{\partial \Theta_i} \tag{87}$$

and

$$\beta_{\bar{k}}^i = \frac{\partial \Theta_i}{\partial \bar{\Theta}_k} \tag{88}$$

Of course,

$$\beta_i^j \beta_j^{\bar{k}} = \delta_{ik} \tag{89}$$

holds, as well.

The above given definitions may be easily generalized for contravariant and mixed representations, the determinant of which is an invariant scalar. It is worth observing, that in the equations (80) and (82) the relation between the vector coordinates is the opposite of that between the vectors. This is the origin of the name "contravariant".

As an illustrative example for the transformation rules we will calculate the relation between the tensor coordinates given in an orthogonal system by eq. (17) and in a skew system by eq. (53) in two dimensions. Coordinates are illustrated in Fig. 3.

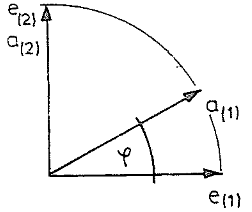


Fig. 3. Relative position of the planar K and A system

In the examined planar case

$$K(\mathbf{T}) = \mathbf{e}_{(\alpha)}\mathbf{f}_{(\beta)} \tag{90}$$

holds. Let

$$\mathbf{a}_{(1)} = \frac{\mathbf{e}_{(1)}}{\cos\varphi} \qquad \mathbf{a}_{(2)} = \mathbf{e}_{(2)} \tag{91}$$

The elements of the transformation matrix β are readily derived as:

$$\beta_{\bar{1}}^{1} = \frac{\mathbf{e}_{(1)}}{\mathbf{a}_{(1)}} = \cos\varphi \quad \beta_{\bar{1}}^{2} = \sin\varphi \tag{92}$$

$$\beta_{\bar{2}}^{1} = 0 \qquad\qquad \beta_{\bar{2}}^{2} = 1$$

Now we can calculate the components of the matrix $\bar{t}_{ik}$:

$$A(\mathbb{T})_{11} = \bar{t}_{11} = \beta_{\bar{1}}^{1}(\beta_{\bar{1}}^{1}t_{11} + \beta_{\bar{1}}^{2}t_{12}) + \beta_{\bar{1}}^{2}(\beta_{\bar{1}}^{1}t_{21} + \beta_{\bar{1}}^{2}t_{22}) = \tag{93}$$
$$= \cos^2\varphi\, t_{11} + \sin^2\varphi\, t_{22} + \sin\varphi\cos\varphi(t_{12} + t_{21})$$

$$A(\mathbb{T})_{12} = \bar{t}_{12} = \beta_{\bar{1}}^{1}(\beta_{\bar{2}}^{1}t_{11} + \beta_{\bar{2}}^{2}t_{12}) + \beta_{\bar{1}}^{2}(\beta_{\bar{2}}^{1}t_{21} + \beta_{\bar{2}}^{2}t_{22}) =$$
$$= \cos\varphi\, t_{12} + \sin\varphi\, t_{21}$$

$$A(\mathbb{T})_{21} = \bar{t}_{21} = \beta_{\bar{2}}^{1}(\beta_{\bar{1}}^{1}t_{11} + \beta_{\bar{1}}^{2}t_{12}) + \beta_{\bar{2}}^{2}(\beta_{\bar{1}}^{1}t_{21} + \beta_{\bar{1}}^{2}t_{22}) =$$
$$= \cos\varphi\, t_{21} + \sin\varphi\, t_{12}$$

$$A(\mathbb{T})_{22} = \bar{t}_{22} = \beta_{\bar{2}}^{1}(\beta_{\bar{2}}^{1}t_{11} + \beta_{\bar{2}}^{2}t_{12}) + \beta_{\bar{2}}^{2}(\beta_{\bar{2}}^{1}t_{21} + \beta_{\bar{2}}^{2}t_{22}) = t_{22}$$

By this example we wanted to underline, that the computation of the transformation has to do only with the different representations of the same physical quantity in different reference frames. If a tensor equation holds, than it holds in an arbitrary coordinate system, but in each system the form of the equation will be different, according to the rules of transformation derived in this section.

1.4. Differentiation of tensor fields

1.4.1. Scalar field

1.4.1.1. Directional derivative

For the sake of simplicity we will treat a two-dimensional scalar field, which can be visualized as a curved surface in the three-dimensional space. This surface will be denoted by Z.

We will investigate the surface at point $\bar{P}$, which corresponds to the point P of the scalar field. If we intersect the surface Z by a plane passing through $\bar{P}$ and orthogonal to S (the plane, on which we interprete the scalar field), then the result is a curve on the surface. The intersection of this orthogonal plane with S is a straight line, which will be denoted by $\mathbf{e}$. Let us proceed now on $\mathbf{e}$ by the distance ε. The value of the scalar field will be denoted by z', at the

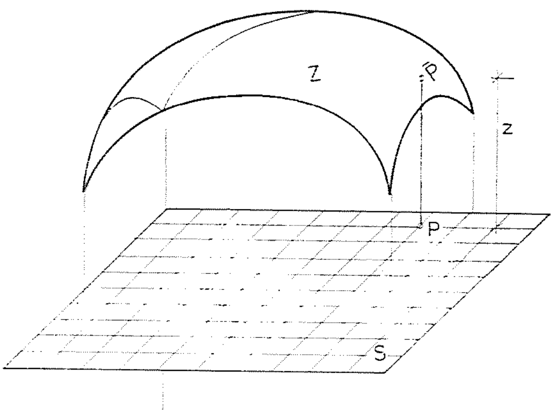


Fig. 4. Scalar field as curved surface

original point P with z. Now the directional derivative of the scalar field at point P in the direction e is defined by

$$\lim_{\varepsilon \to 0} \frac{(Z' - Z)}{\varepsilon} \tag{94}$$

This can be visualized as the directional tangent of the surface curve at point $\bar{P}$ in the plane of intersection. This is illustrated in Fig. 5:
Remark, that the directional derivative has been introduced without the use of any coordinate systems.

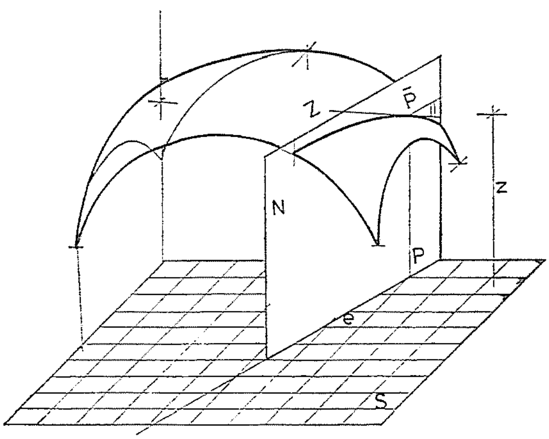


Fig. 5. Directional derivative of scalar field

1.4.1.2. *Partial derivative*

We will now use in the plane a coordinate system $x_{(\alpha)}$. In this system the scalar field can be interpreted as a scalar function in two variables in the form $z = f(x_{(\alpha)})$. If we calculate the directional derivatives in the directions of the coordinate lines, we arrive at the expressions

$$\frac{\partial z}{\partial x_{(\alpha)}} = z_{,\alpha} \tag{95}$$

which will be called the partial derivatives. The quantities $z_{,\alpha}$ can be represented by two scalar fields. (The partial derivatives of an n-dimensional scalar field are represented by n separate scalar fields in n dimensions.)

1.4.1.3. *Gradient field*

Despite the fact, that the partial derivatives of a scalar field depend on the coordinate system, we are able to define a coordinate-invariant quantity with the aid of them. Let's regard the partial derivatives as the components of a vector given in the same coordinate system as the original scalar field. We can decide, whether they are actually vector coordinates by the transformations rules derived in eq. (82/b). In the original system we have

$$z_{,\alpha} = \frac{\partial z}{\partial x_{(\alpha)}} \tag{96}$$

Transforming now to the new coordinate system $\bar{x}$ by the eq. (88) we arrive at

$$z_{,\bar{\alpha}} = \frac{\partial z}{\partial \bar{x}_{(\alpha)}} = \frac{\partial z}{\partial x_{(\beta)}} \frac{\partial x_{(\beta)}}{\partial \bar{x}_{(\alpha)}} = z_{,\beta}\beta^{\beta}_{\bar{\alpha}} \tag{97}$$

This illustrates, that the partial derivatives transform under the rule for vector coordinates. The physical invariant vector determined by the partial derivatives will be called the gradient of the scalar field. The gradient of an n-dimensional scalar field is an n-dimensional vector field. The gradient vector field will be denoted by **g**.

We will try to visualize the gradient field in two dimensions. Figure 6 demonstrates a two-dimensional scalar field as a curved surface $z = f(x, y)$. At point P of the surface the tangents parallel to the coordinate planes are indicated.

This tangents determine the tangent plane $P_1P_2P_3$. The partial derivatives are the directional tangents of the lines e_1 and e_2, therefore

$$z_{,x} = \frac{OP_3}{OP_1} \quad \text{and} \quad z_{,y} = \frac{OP_3}{OP_2} \tag{98}$$

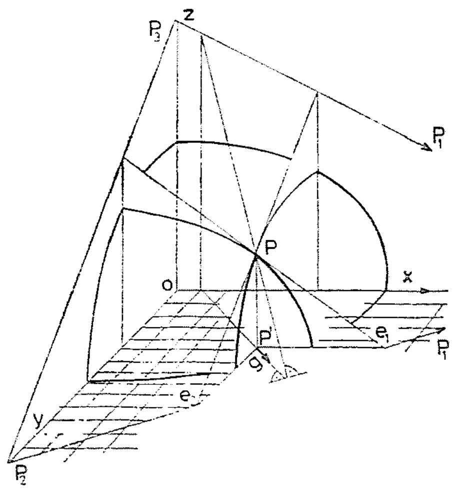


Fig. 6. The gradient

The tangent of the interval P_1P_2 on the plane xy is $\dfrac{OP_1}{OP_2}$. If we measure the vector g from the orthogonal P' projection of the point P, then we find, that g is orthogonal to P_1P_2, since the tangent of g can be expressed as

$$\frac{\dfrac{OP_3}{OP_2}}{\dfrac{OP_3}{OP_1}} = \frac{OP_1}{OP_2}$$

and is found to be the reciprocal value of the tangent of P_1P_2. The vector g indicates at each point the direction and magnitude of the fastest rate of change of the scalar field.

1.4.2. *Vector field*

1.4.2.1. *Directional derivative*

We will consider a three-dimensional vector field. This field defines the vector $\mathbf{v}$ at point P. Now we select an arbitrary straight line (direction) e passsing through P and proceed along this line by a distance ε to arrive at point P'. The vector defined by the vector field at this last mentioned point will be denoted $\mathbf{v}'$. The directional derivative of the vector field at point P in the direction e is then defined by

$$\lim_{\varepsilon \to 0} \frac{(\mathbf{v}' - \mathbf{v})}{\varepsilon} \tag{99}$$

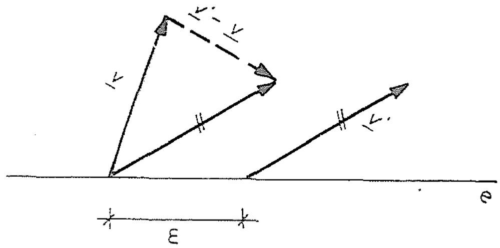


Fig. 7. Directional derivative of vector field

This is illustrated by Fig. 7.
Remark, that the directional derivative of the vector field has been introduced without the use of any coordinate systems, similarly to the directional derivative of the scalar field. The directional derivative of an n-dimensional vector field is a vector field of the same dimension.

1.4.2.2. Partial derivative

The vector field will be interpreted in the coordinate system x_i. In this system the vector field can be interpreted as a vector-vector function in one variable, since the vector $\mathbf{v}$ is the function of the position vector $\mathbf{x}$, both vectors given with their coordinates. We are going to determine the directional derivatives in the directions of the coordinate lines. In order to do this, we can write the vector field in the form

$$\mathbf{v} = v^i \mathbf{a}_{(i)} \tag{100}$$

on the basis of equations (27) and (32). Differentiating eq. (100) by the jth variable we arrive at

$$\mathbf{v}_{,j} = (v^i \mathbf{a}_{(i)})_{,j} \tag{101}$$

Applying the rule for product differentiation yields

$$\text{a)} \quad \mathbf{v}_{,j} = v^i{}_{,j}\mathbf{a}_{(i)} - v^i \mathbf{a}_{(i),j} \tag{102}$$

or resolved to covariant components

$$\text{b)} \quad \mathbf{v}_{,j} = v_{i,j}\mathbf{a}^{(i)} - v_i \mathbf{a}^{(i)}{}_{,j}$$

The first member contains the partial derivatives of the scalars v^i and v_i multiplied by the base vectors. This partial derivatives can be determined on the basis of section 1.4.1.

The second member contains the partial derivatives of the base vectors multiplied by the scalars v^i and v_i. In a straight (orthogonal or skew) coordinate system this derivatives disappear, of course, since the base vectors are of constant magnitude and direction. To visualize the meaning of the second member, let's regard Fig. 8:

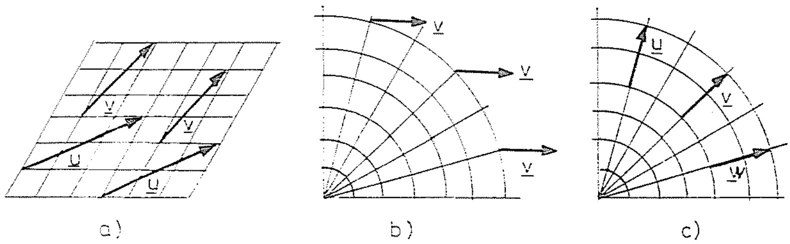


Fig. 8. Connection between vectors and vector coordinates

It can be observed, that in a straight coordinate system the change of the vector coordinates sufficiently describes the change of the vector, therefore the first member of the partial derivative contains enough information. In curvilinear systems this is not the case:

The vectors in Fig. 8/b are equal, but their components aren't. In Fig. 8/c the opposite happens, the vectors are not equal, but their components are. In section 1.4.1.2. we didn't meet this "second member", because the representation of scalar fields is independent of the base vectors.

1.4.2.3. Christoffel symbols

In section 1.4.2.1. we observed, that the directional derivative of a vector field is a vector field, as well. According to this the vector field $\mathbf{a}_{(i)},_j$ can be resolved to components in the bases $\mathbf{a}_{(i)}$ or $\mathbf{a}^{(i)}$:

$$\mathbf{a}_{(i)},_j = \Gamma_{ijk}\mathbf{a}^{(k)} = \Gamma^k_{ij}\mathbf{a}_{(k)} \tag{103}$$

Multiplying the above equation by $\mathbf{a}_{(k)}$ we arrive at

$$\mathbf{a}_{(i)},_j\mathbf{a}_{(k)} = \Gamma_{ijl}\mathbf{a}^{(l)}\mathbf{a}_{(k)} = \Gamma_{ijl}\delta^l_k = \Gamma_{ijk} \tag{104}$$

Equations (103) and (104) are equivalent definitions for the quantities Γ with three indices. The quantities Γ_{ijk} and Γ^k_{ij} are called the Christoffel symbols of the first and second kind, respectively. They were introduced by the mathematician Elvin Bruno Christoffel. The Γ symbols are cube-matrices, with 27 components in 3 dimensions. The first index of the Christoffel symbol refers to the variable (base vector field) to be differentiated, the second tells, in which direction it has to be differentiated and the third identifies the component of the directional derivative resolved already in the coordinate system. We are now going to derive some useful formulas in connection with Christoffel symbols.

Multiplying (103) with the base vectors we arrive at

$$\Gamma_{ijl} = \Gamma^k_{ij}g_{kl} \text{ and } \Gamma_{ijk}g^{kl} = \Gamma^l_{ij} \tag{105}$$

This illustrates, that the third index of the Christoffel symbols can be "moved" up or down by the method first described in eq. (68). This does not hold for the first two indices, since the Christoffel symbols are not third order tensors.
Differentiating eq. (74) by the jth variable yields

$$\mathbf{a}_{(i)},_j = \mathbf{x},_{ij} = \mathbf{x},_{ji} = \mathbf{a}_{(j)},_i \tag{106}$$

which displays the symmetry of the Christoffel symbols with respect to the first two indices. Differentiating eq. (38) by the kth variable yields

$$g_{ij},_k = \mathbf{a}_{(i)},_j \mathbf{a}_{(k)} + \mathbf{a}_{(i)} \mathbf{a}_{(j)},_k \tag{107}$$

Comparing this with (103) we find, that

$$g_{ij},_k = \Gamma_{ikj} + \Gamma_{jki} \tag{108}$$

Writing the above result cyclically thrice, we arrive at the equation system

$$\begin{aligned} &\text{a)} \quad \Gamma_{kij} + \Gamma_{jki} = g_{ij},_k \\ &\text{b)} \quad \Gamma_{kij} + \Gamma_{ijk} = g_{jk},_i \\ &\text{c)} \quad \Gamma_{jki} + \Gamma_{ijk} = g_{ki},_j \end{aligned} \tag{109}$$

Composing now the equation (b/) + (c/) — (a/) yields

$$2\Gamma_{ijk} = g_{jk},_i + g_{ki},_j - g_{ij},_k \tag{110}$$

This formula is convenient when calculating the Christoffel symbols in a coordinate system where the coordinates of the metric tensor are already known. By using eq. (70) we arrive at

$$\text{a)} \quad \Gamma^i_{ir} = \frac{1}{2g} \frac{\partial g}{\partial g_{is}} \frac{\partial g}{\partial \Theta} r = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \Theta} r \tag{111}$$

by using the result

$$\text{b)} \quad \frac{\partial g}{\partial g_{is}} = g g^{is}$$

1.4.2.4. Tensor derivative

Applying the symbols introduced in (104) to (102) yields

$$\mathbf{v},_j = v^i,_j \mathbf{a}_{(i)} + v^i \Gamma^k_{ij} \mathbf{a}_{(k)} \tag{112}$$

By changing the indices of the last two members we arrive at

$$\mathbf{v},_j = (v^i,_j + v^k \Gamma^i_{jk}) \mathbf{a}_{(i)} = v^i |_j \mathbf{a}_{(i)} \tag{113}$$

Writing the equation for the components only (by multiplying with $\mathbf{a}^{(k)}$)

$$v^i|_j = v^i{}_{,j} + v^k \Gamma^i_{jk} \tag{114}$$

$$v_i|_j = v_{i,j} - v_k \Gamma^k_{ij}$$

The above expression will be called the covariant derivative of the vector field v. The covariant derivatives $v^i|_j$ are analogons to the partial derivatives of the scalar field. They depend on the choice of coordinate system, but, as we did with the partial derivatives, it is possible to define a coordinate-invariant quantity by using them as coordinates. This invariant will be called the second order tensor derivative of the vector field. The second index of the tensor derivative is always covariant, this is the reason to call the quantities $v^i|_j$ the covariant derivatives. This characteristic was inherited from the fact, that we interpreted the position vector x in a contravariant (traditional) way. Resolving the position vector into covariant coordinates we could derive the contravariant derivative, but this has no practical reason.

In the case of the gradient vector we proved, that the partial derivatives of the scalar field transform under the rule prescribed for vector coordinates. Now we are going to do the same for the coordinates of the second order tensor derivative. In order to do this we calculate the covariant derivative of the vector v in the system $\bar{\mathbf{a}}_{(i)}$:

$$\mathbf{v}_{,\bar{j}} = v_{\bar{i}}|_{\bar{j}} \bar{\mathbf{a}}^{(i)} \tag{115}$$

Applying the "chain rule" yields

$$\mathbf{v}_{,\bar{j}} = \mathbf{v}_{,j} \frac{\partial \mathbf{a}^{(j)}}{\partial \bar{\mathbf{a}}^{(i)}} = v_i|_j \mathbf{a}^{(i)} \beta^{i}_{\bar{j}} \tag{116}$$

Comparing the right hand side of the above two equation yields

$$v_{\bar{i}}|_{\bar{j}} \bar{\mathbf{a}}^{(i)} = v_i|_j \mathbf{a}^{(i)} \beta^{i}_{\bar{j}} \tag{117}$$

Multiplying this with $\bar{\mathbf{a}}_{(k)} = \beta^k_{\bar{k}} \mathbf{a}_{(k)}$ we arrive at

$$v_{\bar{k}}|_{\bar{j}} = v_k|_j \beta^k_{\bar{k}} \beta^j_{\bar{j}} \tag{118}$$

Comparing the above equation with (86/b) we see, that our statement is proven, the covariant derivatives are actually the coordinates of a tensor.

Summarizing our results we can say, that the covariant differentation is an analogon of the partial differentiation and differs from the latter only in the curvilinear representation of tensor fields of order higher than zero.

1.4.3. *Second order tensor field*

We are going to introduce the derivative of the second order tensors. On the basis of the previously derived equations it will not be quite surprising, that the derivative of a second order tensor field is a third order tensor field.

We are going to deliver a rather formal description, but later we will examine the derivatives of specific second order tensor fields.
Multiplying the covariant t_{ij} representation of the tensor $\mathbb{T}$ with the contravariant vector components u^i and v^i we arrive at the scalar s:

$$s = t_{ij}u^i v^j \tag{119}$$

Differentiating the above expression by the kth variable and considering eq. (114) yields

$$\begin{aligned} s_{,k} &= t_{ij,k}u^i v^j + t_{ij}u^i{}_{,k}v^j + t_{ij}u^i v^j{}_{,k} = \\ &= t_{ij,k}u^i v^j + t_{ij}u^i\big|_k v^j + t_{ij}u^i v^j\big|_k - t_{ij}u^l \Gamma^i_{kl} v^j - t_{ij}u^i v^l \Gamma^j_{kl} \end{aligned} \tag{120}$$

This can be written in the following form, as well:

$$s_{,k} = (t_{ij}u^i v^j)_{,k} = t_{ij}\big|_k u^i v^j + t_{ij}u^i v^j\big|_k + t_{ij}u^i\big|_k v^j \tag{121}$$

by accepting the following definition

$$t_{ij}\big|_k u^i v^j = t_{ij,k}u^i v^j - t_{ij}u^l v^j \Gamma^i_{kl} - t_{ij}u^i v^l \Gamma^j_{kl} \tag{122}$$

Changing the dummy indices we arrive at the form

$$t_{ij}\big|_k u^i v^j = (t_{ij,k} - t_{lj}\Gamma^l_{ik} - t_{il}\Gamma^l_{kj})u^i v^j \tag{123}$$

This equation holds for an arbitrary matrix t_{ij} and given components u^i and v^j if and only if

$$t_{ij}\big|_k = t_{ij,k} - t_{lj}\Gamma^l_{ik} - t_{il}\Gamma^l_{kj} \tag{124}$$

The above expression is the definition of the covariant derivative of the tensor $\mathbb{T}$ in the representation t_{ij}. This can be expressed by

$$\mathbb{T}_{,k} = t_{ij}\big|_k \tag{125}$$

As mentioned before, we assume, that the quantities $t_{ij}\big|_k$ represent a third order tensor. To prove this, we use the same procedure as we did in the equations (115)—(118):

$$\mathbb{T}_{,\bar{k}} = t_{\bar{i}\bar{j}}\big|_{\bar{k}}\bar{\mathbf{a}}^{(i)}\bar{\mathbf{a}}^{(j)} \tag{126}$$

$$\mathbb{T}_{,\bar{k}} = \mathbb{T}_{,k}\frac{\mathbf{a}^{(k)}}{\bar{\mathbf{a}}^{(k)}} = t_{ij}\big|_k \mathbf{a}^{(i)}\mathbf{a}^{(j)}\beta^k_{\bar{k}} \tag{127}$$

$$t_{\bar{i}\bar{j}}\big|_{\bar{k}}\,\bar{\mathbf{a}}^{(i)}\bar{\mathbf{a}}^{(j)} = t_{ij}\big|_k \mathbf{a}^{(i)}\mathbf{a}^{(j)}\beta^k_{\bar{k}} \tag{128}$$

$$t_{\bar{i}\bar{j}}\big|_{\bar{k}} = t_{ij}\big|_k \beta^i_{\bar{i}}\beta^j_{\bar{j}}\beta^k_{\bar{k}} \tag{129}$$

Similarly to eq. (124) we arrive at the following expressions:

$$t^{i}_{\cdot j}\big|_{k} = t^{i}_{\cdot j},_{k} + t^{l}_{\cdot j}\Gamma^{i}_{kl} - t^{i}_{\cdot l}\Gamma^{l}_{jk} \tag{130}$$

$$t_{i}^{\cdot j}\big|_{k} = t_{i}^{\cdot j},_{k} - t_{l}^{\cdot j}\Gamma^{l}_{ik} + t_{i}^{\cdot l}\Gamma^{j}_{kl} \tag{131}$$

$$t^{ij}\big|_{k} = t^{ij},_{k} + t^{lj}\Gamma^{i}_{kl} + t^{il}\Gamma^{j}_{kl} \tag{132}$$

Based on our experiences with second order tensors we can generalize for higher order ones: The covariant derivative of an nth order tensor in a given representation can be computed by calculating the partial derivatives of the scalar tensor components and adding n members with Christoffel symbols. The result is an $(n + 1)$ th order tensor.

1.4.4. The Riemann—Christoffel tensor

Equation (118) demonstrates, that the components of the covariant derivative of the vector v are the representation of a second order tensor. In eq. (124) the covariant derivative of second order tensors is introduced. Based on this, we are going to execute covariant differentiation on the second order tensor derivative of v. Let

$$v_{i}\big|_{j}\big|_{k} = v_{i}\big|_{jk} \tag{133}$$

Applying eq. (124) to the above expression yields

$$v_{i}\big|_{jk} = (v_{i},_{j} - v_{m}\Gamma^{m}_{ij}),_{k} - (v_{l},_{j} - v_{m}\Gamma^{m}_{lj})\Gamma^{l}_{ik} - (v_{i},_{l} - v_{m}\Gamma^{m}_{il})\Gamma^{l}_{kj} \tag{134}$$

We want to investigate, whether the indices in the covariant derivative can be changed or not, in other words, whether $v_{i}\big|_{jk} = v_{i}\big|_{kj}$ or not. In the case simple partial differentiation this can be done. The quantities $v_{i}\big|_{kj}$ can be expressed by the simple change of indices:

$$v_{i}\big|_{kj} = (v_{i},_{k} - v_{m}\Gamma^{m}_{ik}),_{j} - (v_{l},_{k} - v_{m}\Gamma^{m}_{lk})\Gamma^{l}_{ij} - (v_{i},_{l} - v_{m}\Gamma^{m}_{il})\Gamma^{l}_{jk} \tag{135}$$

Expressing now the difference of the investigated quantities we arrive at

$$v_{i}\big|_{jk} - v_{i}\big|_{kj} = v_{i},_{jk} - v_{i},_{kj} - v_{m},_{k}\Gamma^{m}_{ij} + \tag{136}$$

$$+ v_{m},_{j}\Gamma^{m}_{ik} - v_{m}\Gamma^{m}_{ij},_{k} + v_{m}\Gamma^{m}_{ik},_{j} - v_{l},_{j}\Gamma^{l}_{ik} + v_{l},_{k}\Gamma^{l}_{ij} + v_{m}\Gamma^{m}_{lj}\Gamma^{l}_{ik} - v_{m}\Gamma^{m}_{lk}\Gamma^{l}_{ij}$$

Since the indices in the simple partial derivatives can be changed, the first two terms cancel each other and finally we have

$$v_{i}\big|_{jk} - v_{i}\big|_{kj} = v_{m}(\Gamma^{m}_{ik},_{j} - \Gamma^{m}_{ij},_{k} - \Gamma^{m}_{lj}\Gamma^{l}_{ik} - \Gamma^{m}_{lk}\Gamma^{l}_{ij}) \tag{137}$$

The left hand side of the above equation is obviously the representation of a third order tensor. This fact implies, that the bracet on the right hand side has to be the representation of a fourth order tensor, the first index of which is contravariant and the following three covariant.

Up to now we had to do only with tensors of order equal or lower than three. The appearance of a fourth order tensor doesn't imply difficulties, because all our former definitions for tensors are easily generalized. Returning now to eq. (137), let's denote the fourth order tensor by

$$v_i\big|_{jk} - v_i\big|_{kj} = v_m r^{m}_{\cdot ijk} \tag{138}$$

The quantities $r^{m}_{.ijk}$ will be called the representation of the fourth order Riemann—Christoffel tensor. Now it is easy to answer our previous question: the indices of the covariant derivative can be changed if and only if

$$r^{m}_{\cdot ijk} = 0 \tag{139}$$

The Riemann— Christoffel tensor is of course invariant under the transformation of coordinates, therefore if an equation holds in an arbitrary coordinate system, then it holds in each one. If we choose the orthogonal coordinate system, then eq. (139) is trivial, therefore it holds always. Now we have to ask, whether the orthogonal coordinate system exists in the examined space or not. This is not a trivial question, since in an equivalent way we may ask, whether the structure of the examined space satisfies the euclidean axioms, or not. The two dimensional case is discussed in the following section. The three dimensional case goes beyond the range of this paper, but remark, that the first man to establish a non-euclidean geometry without contradictions was János Bólyai. His geometry is the so-called hyperbolic geometry. Later the elliptic geometry was elaborated. In the hyperbolic geometry the curvature of space is a negative constant, in the elliptic geometry a positive constant. The most general geometry is due to Bernhard Riemann. In the Riemann geometry the curvature of space is non-constant. The general relativity theory of Albert Einstein was based on the Riemann geometry.

This illustrates, that the Riemann-Christoffel tensor is closely related to the curvature of space, therefore it is called the Riemann—Christoffel curvature tensor.

1.5. Geometry of curved surfaces

When Carl Friedrich Gauss was asked to participate in the geodesic surveying of the county Hannover, the great german mathematician meditated for a long time over the sufficient and necessary condition of the existence of a measure-preserving planar map of a hilly landscape. His investigations resulted in one of the most outstanding theorem of his career, he himself called it "Theorema egregium". He proved, that at each point of the surface a scalar quantity can be calculated which is invariant under the transformation of coordinates. This scalar is now called the Gauss-curvature of the surface. The necessary and sufficient condition for the surface to have a measure-preserving map in the euclidean plane is the disappearance of the Gauss curvature. It is surprising, that despite the fact, that the surface is enbedded in the three-dimensional euclidean space, it is theoretically possible, to measure the Gauss curvature "in the surface" for example flat, two-dimensional creatures moving exclusively in the surface could do that.

If the Gauss curvature doesn't disappear, then the surface can't be mapped in a measure-preserving way onto the euclidean plane, that means, that the euclidean geometry doesn't hold on the surface. In this section we will try to get acquainted with the intristic geometry of this non-euclidean surfaces.

1.5.1. *Interpretation of the metric tensor*

We will investigate the geometrical meaning of the metric tensor introduced in eq. (38). We consider a plane with coordinates $x_{(\alpha)}$ and the infinitesimal line element $d\mathbf{s}$ will be resolved to components in this coordinate system.

$$d\mathbf{s} = dx^{\alpha}\mathbf{a}_{(\alpha)} \tag{140}$$

We will now multiply $d\mathbf{s}$ with itself arriving at

$$d\mathbf{s}\; d\mathbf{s} = dx^{\alpha} dx^{\beta}\mathbf{a}_{(\alpha)}\mathbf{a}_{(\beta)} = dx^{\alpha} dx^{\beta} g_{\alpha\beta} \tag{141}$$

which is the square of the length of the line element. Equation (141) is a straightforward generalization of the Pythagoras formula, in differential geometry it is called the first fundamental form of the surface. In the usual orthogonal coordinate system the coordinates of the metric tensor are represented by the unit matrix, and the well-known form of the Pythagoras theorem holds. If we introduce an other coordinate system in the plane and transform the components of the metric tensor under the rule given in eq. (86) then the validity of the euclidean geometry will not be disturbed.

Orthogonal coordinate systems are, all the same, equivalent to any other coordinate system, therefore we can prescribe, in which arbitrary coordinate system we wish the metric tensor to be represented by the unit matrix.

1.5.2. *Classification of two-dimensional surfaces*

Now we ask the inverse question as before: what happenes, if we define in a region of the plane the components of the metric tensor arbitrarily in a given coordinate system, and there is no coordinate transformation, under which the representation becomes identic with the unit matrix? In this case the given g metric tensor field defines a non-euclidean geometry in the plane. This can be visually realized by bending the plane into the three-dimensional plane. This bending must include stretching, as well. This is the reason, why non-euclidean surfaces are often called curved surfaces. This name refers to the enbedding of a surface with non-euclidean metric into a higher dimensional euclidean space. If the mentioned bending doesn't include stretching, then we arrive at the well-known developable surfaces with euclidean metric.

We conclude from this, that the metric uniquely determines the geometry of the surface, but it doesn't uniquely determine the form of the surface in the embedding euclidean space.

It is hard to visualize curved spaces if their dimension is higher than two, because for the visualization we need the embedding euclidean space, the dimension of which is always higher than the dimension of the curved space. In the case described just before the embedding space had one dimension more than the curved surface. This is not bound to be so, since a one-dimensional wire can be bent in a way, that it can't be embedded in a two-dimensional surface. (Remark, that the intristic geometry of a wire doesn't change by bending.)

The two-dimensional surfaces will be classified on the bases of the minimally necessary dimension of the embedding euclidean space. If this dimension is two, then the surface is called a plane, if it is three, then the surface is called a hypersurface, if it is larger than three, then it is called a general two-dimensional surface. In this general case the intristic geometry of the surface is described by the Riemann-Christoffel tensor. To calculate the components of this tensor we need the coordinates of the metric tensor and their derivatives only, therefore we can say, that the intristic geometry of the surface is completely described by the metric tensor. However, this calculations are rather cumbersome, so it is difficult to see the connection between the given metric tensor field and the intristic geometry.

We are now especially interested in the description of hypersurfaces, which is a special case. The intristic geometry of a hypersurface can be described by a tensor field, which is much simpler than the Riemann—Christoffel tensor, but can't be applied to general two-dimensional surfaces. We are going to get acquainted with this simpler tensor field.

1.5.3. The second order curvature tensor

We are going to investigate a region of a two-dimensional hypersurface embedded in the three-dimensional euclidean space, with coordinate system $x_{(\alpha)}$ in the surface. At point P we can regard the tangent plane of the surface and the normal vector of the tangent plane. This normal vector is called the normal vector of the surface at point P. With the aid of this we are able to define a normal vector field $\mathbf{a}_{(3)}$ with

$$\mathbf{a}_{(3)}\mathbf{a}_{(3)} = 1 \tag{142}$$

The orthogonality condition with the base vectors yields

$$\mathbf{a}_{(\alpha)}\mathbf{a}_{(3)} = 0 \tag{143}$$

The direction of the $\mathbf{a}_{(3)}$ vector field depends on the surface coordinates, but not the magnitude. Therefore the partial derivatives of the unit normal vector field are surface vector fields, that can be resolved in the surface coordinate system:

$$\mathbf{a}_{(3)},_{\alpha} = - b_{\alpha\beta}\mathbf{a}^{(\beta)} \tag{144}$$

Multiplying this with $\mathbf{a}_{(\gamma)}$ we arrive at

$$\mathbf{a}_{(3)},_{\alpha}\mathbf{a}_{(\gamma)} = - b_{\alpha\gamma}\mathbf{a}^{(\gamma)}\mathbf{a}_{(\beta)} = - b_{\alpha\gamma}\delta^{\gamma}_{\beta} = - b_{\alpha\beta} \tag{145}$$

The quantities $b_{\alpha\beta}$ are the representation of a second order tensor, this can be demonstrated by the transformation equations. Now we will derive some useful formulas in connection with this tensor. Differentiating (143) and by using (145) we arrive at

$$\mathbf{a}_{(\alpha)},_{\beta}\mathbf{a}_{(3)} = - \mathbf{a}_{(\alpha)}\mathbf{a}_{(3)},_{\beta} = b_{\beta\alpha} \tag{146}$$

Writing eq. (104) in the above introduced coordinate system yields

$$\mathbf{a}_{(\alpha)},_{\beta} = \Gamma_{\alpha\beta\gamma}\mathbf{a}^{(\gamma)} + \Gamma_{\alpha\beta 3}\mathbf{a}^{(3)} \tag{147}$$

On the basis of the two previous equations we have

$$b_{\alpha\beta} = \mathbf{a}_{(\alpha)},_{\beta}\mathbf{a}_{(\beta)} = \Gamma_{\alpha\beta 3} \tag{148}$$

Using the derived formulas for the Christoffel symbols the last equation can be re-formulated as

$$b_{\alpha\beta} = \Gamma_{\alpha\beta 3} = \Gamma^{3}_{\alpha\beta} = - \Gamma_{3\alpha\beta} = - \Gamma_{3\beta\alpha} = - \Gamma_{\alpha 3\beta} \tag{149}$$

The tensor **B** represented by the matrix $b_{\alpha\beta}$ will be called the second order curvature tensor. Other representations of the tensor are found by multiplication with the metric tensor:

$$b^{\alpha}_{\beta} = b_{\gamma\beta}g^{\gamma\alpha} \qquad\qquad b^{\alpha\beta} = b^{\alpha}_{\gamma}g^{\gamma\beta} \tag{150}$$

The mixed representation can be derived by interpreting eq. (144) in an other representation:

$$\mathbf{a}_{(3)},_{\alpha} = - b^{\beta}_{\alpha}\mathbf{a}_{(\beta)} \tag{151}$$

From the above equation follows, that

$$d\mathbf{a}_{(3)} = \mathbf{a}_{(3)},_{\alpha}dx^{\alpha} = - b_{\alpha\beta}\mathbf{a}^{(\beta)}dx^{\alpha} \tag{152}$$

Let's multiply the infinitesimal vector with the line element ds by using equations (140) and (152):

$$d\mathbf{a}_{(3)}ds = -b_{\alpha\beta}\mathbf{a}^{(\beta)}dx^{\alpha}\mathbf{a}_{(\gamma)}dx^{\gamma} = -b_{\alpha\beta}\delta^{\beta}_{\gamma}dx^{\alpha}dx^{\gamma} \tag{153}$$

$$d\mathbf{a}_{(3)}ds = -b_{\alpha\beta}dx^{\alpha}dx^{\gamma} \tag{154}$$

In classical differential geometry the right hand side of eq. (154) is called the second fundamental form of the surface. The coefficients b_{11}, $b_{12} = b_{21}$, b_{22} were denoted by E, F, G by Carl Friedrich Gauss.

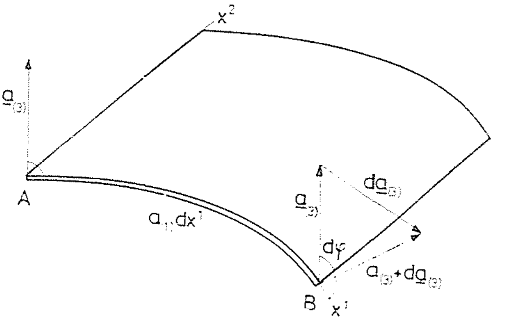


Fig. 9. Connection between the curvature and the unit normal vector

We will now try to visualize the components of the second order curvature tensor in mixed representation. We assume, that all components but b_1^1 disappear, and we intersect the surface with a normal plane along a $x^{(1)}$ line. This is demonstrated in Fig. 9.

Since $\mathbf{a}_{(3)} = 1$, the length of the infinitesimal vector

$$d\mathbf{a}_{(3)} = \mathbf{a}_{(3),1}dx^1 = -b_1^1\mathbf{a}_{(1)}dx^1 \tag{155}$$

is equal to the angle $d\varphi$ between the surface normals at point A and B. If this angle is divided by the length of the vector $\mathbf{a}_{(1)}dx^1$, then we arrive at the curvature of the surface line $x^{(1)}$, which is equal to the curvature of the surface in the direction $x^{(1)}$.

$$\frac{d\varphi}{ds} = \frac{|d\mathbf{a}_{(3)}|}{|\mathbf{a}_{(1)}|^{dx^1}} = b_1^1 \tag{156}$$

Now let's assume, that $b_1^1 = 0$ but $b_1^2 \neq 0$

Figure 10 illustrates, that

$$d\mathbf{a}_{(3)} = \mathbf{a}_{(3),1}dx^1 = -b_1^2\mathbf{a}_{(2)}dx^1 \tag{157}$$

so the twist of the surface is

$$\frac{d\vartheta}{ds} = \frac{|d\mathbf{a}_{(3)}|}{|\mathbf{a}_{(1)}{}^{dx^1}|} = |b_1^2|\frac{|\mathbf{a}_{(2)}|}{|\mathbf{a}_{(1)}|} \tag{158}$$

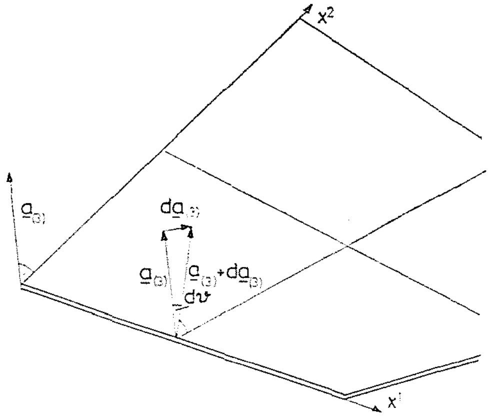


Fig. 10. Connection between the twist and the unit normal vector

If $\mathbf{a}_{(1)} = \mathbf{a}_{(2)}$, then $b_1^2 = b_2^1$ and both components are equal to the twist of the surface along the coordinate lines. If the opposite holds, that the two components aren't equal, but they are closely related by eq. (158) to the twist.

1.5.4. Covariant derivative of surface tensor fields

A general vector field with origin on the surface can be resolved into in-surface and normal components:

$$\mathbf{v} = v_i \mathbf{a}^{(i)} = v_\alpha \mathbf{a}^{(\alpha)} + v_3 \mathbf{a}^{(3)} \tag{159}$$

This equation is differentiated according to eq. (113), yielding

$$\mathbf{v}_{,\beta} = v_i \big|_\beta \mathbf{a}^{(i)} = v_\alpha \big|_\beta \mathbf{a}^{(\alpha)} + v_3 \big|_\beta \mathbf{a}^{(3)} \tag{160}$$

We conclude from equations (143) and (144), that

$$\mathbf{a}_{(3),\alpha} \mathbf{a}_{(3)} = 0 \tag{161}$$

therefore on the basis of (105)

$$\Gamma_{3\alpha 3} = \Gamma_{\alpha 33} = 0 \tag{162}$$

Similarly

$$\Gamma_{33\alpha} = \Gamma_{333} = 0 \tag{163}$$

The previous equations demonstrate, that the Christoffel symbols disappear on the surface, if they have more than one index equal to three. Using this fact and equations (148) and (160) we arrive at

$$v_\alpha \big|_\beta = v_{\alpha,\beta} - v_\gamma \Gamma^\gamma_{\alpha\beta} - v_3 b_{\alpha\beta} \tag{164}$$

and

$$v_3 \big|_\beta = v_{3,\beta} + v_\gamma b^\gamma_\beta \tag{165}$$

The first two terms in eq. (164) are the plane analogons of the covariant derivative defined in eq. (114). We introduce the following notation for them:

$$v_\alpha||_\beta = v_{\alpha\cdot\beta} - v_\gamma \Gamma^\gamma_{\alpha\beta} \tag{166}$$

According to this

$$v_\alpha|_\beta = v_\alpha||_3 - v_3 b_{\alpha\beta} \tag{167}$$

and similarly

$$v^\alpha||_\beta = v^\alpha{}_{\cdot\beta} + v^\gamma \Gamma^\alpha_{\beta\gamma} \tag{168}$$

In the case, when v is an in-surface vector field *($v_3 = 0$)* there is no difference between the two covariant derivatives.

$$\mathbf{v}_{\cdot\beta} = v^\alpha|_\beta \mathbf{a}_{(\alpha)} + v^\gamma b_{\gamma\beta} \mathbf{a}_{(3)} \tag{169}$$

Remark, that despite the fact, that v is an in-surface (or tangent) vector field, the partial derivatives have a normal component, as well. We are going to deal with second order surface tensor fields only, which transform tangent vectors into tangent vectors. For this special type of tensors the results for vector fields are easily generalized, and by using equations (132) and (168) we arrive at

$$t^{\alpha\beta}|_\gamma = t^{\alpha\beta}||_\gamma = t^{\alpha\beta}{}_{\cdot\gamma} + t^{\alpha\beta}\Gamma^\alpha_{\gamma\delta} + t^{\alpha\beta}\Gamma^\beta_{\gamma\delta} \tag{170}$$

1.5.5. Connection between the quantities related to the curvature

Up to now we mentioned three quantities, which are related to the curvature of the surface: the Gauss curvature, the Riemann—Christoffel curvature tensor and the second order curvature tensor. The first and the third one can be applied in the analysis of hypersurfaces, more general surfaces can be investigated by the Riemann—Christoffel tensor. The Riemann-Christoffel tensor has in two dimensions only one independent component, the $r^{12}_{\cdot\cdot 12}$. This component is equal to the determinant of the mixed representation second order curvature tensor and to the Gauss curvature of the surface! We are going now to prove the above statement formally, as well.

By differentiating the base vector field twice we arrive at

$$\mathbf{a}_{(\alpha)\cdot\beta\gamma} = \Gamma^\delta_{\alpha\beta\cdot\gamma}\mathbf{a}_{(\delta)} + \Gamma^\xi_{\alpha\beta}(\Gamma^\delta_{\xi\gamma}\mathbf{a}_{(\delta)} + b_{\xi\gamma}\mathbf{a}_{(3)}) + b_{\alpha\beta\cdot\gamma}\mathbf{a}_{(3)} - b_{\alpha\beta}b^\delta_\gamma \mathbf{a}_{(\delta)} \tag{171}$$

which is equivalent to

$$\mathbf{a}_{(\alpha)\cdot\beta\gamma} = (\Gamma^\delta_{\alpha\beta\cdot\gamma} + \Gamma^\xi_{\alpha\beta}\Gamma^\delta_{\xi\gamma} - b_{\alpha\beta}b^\delta_\gamma)\mathbf{a}_{(\delta)} + (\Gamma^\xi_{\alpha\beta}b_{\xi\gamma} + b_{\alpha\beta\cdot\gamma})\mathbf{a}_{(3)} \tag{172}$$

The expression for $\mathbf{a}_{(\alpha)\cdot\beta\gamma}$ can be derived by changing the indices in the expression. Equating the coefficients of $\mathbf{a}_{(\delta)}$ in the two expressions yields

$$\Gamma^\delta_{\alpha\gamma\cdot\beta} - \Gamma^\delta_{\alpha\beta\cdot\gamma} + \Gamma^\xi_{\alpha\gamma}\Gamma^\delta_{\xi\beta} - \Gamma^\xi_{\alpha\beta}\Gamma^\delta_{\xi\gamma} = b_{\alpha\gamma}b^\delta_\beta - b_{\alpha\beta}b^\delta_\gamma \tag{173}$$

Comparing the left hand side of the above equation with the coefficient of v_m in eq. (137) yields for the two-dimensional representation of the Riemann—Christoffel tensor

$$r^{\delta}_{\cdot\alpha\beta\gamma} = \Gamma^{\delta}_{\alpha\gamma\cdot\beta} - \Gamma^{\delta}_{\alpha\beta\cdot\gamma} + \Gamma^{\xi}_{\alpha\gamma}\Gamma^{\delta}_{\xi\beta} - \Gamma^{\xi}_{\alpha\beta}\Gamma^{\delta}_{\xi\gamma} \tag{174}$$

Let us "lift" the index α and substitute into eq. (173)

$$r^{\delta\alpha}_{\cdot\cdot\beta\gamma} = b^{\alpha}_{\gamma}b^{\delta}_{\beta} - b^{\alpha}_{\beta}b^{\delta}_{\gamma} \tag{175}$$

The symmetry properties of the tensor $\mathbf{B}$ imply, that only the component $r^{12}_{\cdot\cdot12}$ can be independent. By substituting

$$\alpha = 2 \quad \beta = 1 \quad \gamma = 2 \quad \delta = 1 \tag{176}$$

we arrive at

$$r^{12}_{\cdot\cdot12} = b^{2}_{2}b^{1}_{1} - b^{2}_{1}b^{1}_{2} \tag{177}$$

The right-hand side is determinant of the mixed representation of the second order curvature tensor, which is equal to the Gauss curvature. Thus we demonstrated the connection between the curvature quantities.

2. Theory of membrane shells

Shells, in which bending moments can be neglected are called membrane shells. Membrane theory assumes the following things:

- The material of the shell is isotropic and obeyes Hooke's law.
- The thickness of the shell (and the bending moments) can be neglected.
- The state of stress of the shell is fully described by the membrane forces acting in the mean surface.
- Supports along the boundary are tangent to the mean surface.
- Deformations due to membrane forces are not hindered by boundary conditions.

The above conditions hold, of course, only approximately in the reality, therefore membrane theory can provide only information with restricted accuracy about a real structure. According to the above assumptions the shell is in plane state of stress, membrane forces having normal (N_x, N_y) and shear $(N_{xy} = N_{yx})$ components only in the mean surface. The state of stress is completely described by the three mentioned membrane force fields. To determine them we need the equilibrium equations and the boundary conditions. The latter ones go beyond the range of this paper.

Results based on membrane theory may serve as a particular solution for the bending theory.

2.1. *The stress tensor*

We will investigate surface on which external forces are acting. By intersecting the surface with a line Δs we need a force $\Delta \mathbf{r}$ to retain the surface in the original position. Point P of the surface is contained in line Δs, and the vector

$$t = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{r}}{\Delta s} = \frac{d\mathbf{r}}{ds} \tag{178}$$

defines the stress vector in the given direction at point P. By changing the direction of Δs, the direction and magnitude of $\mathfrak{t}$ changes, as well. The state of stress in the surface is described by the mathematical relation between the direction and the stress vector. We will try to derive this relation in a closed formula.

2.1.1. *Introduction of the surface stress tensor in orthogonal coordinate systems*

We assume, that the intristic geometry of the investigated hypersurface is euclidean, therefore we can estabilish an orthogonal coordinate system. We investigate the equilibrium of the infinitesimal orthogonal triangle illustrated in Fig. 11.
The sides of the triangle will be represented by their outer normals:

$$d\mathbf{s}_{(\alpha)} = \mathbf{e}_{(\alpha)} ds_{(\alpha)} \tag{179}$$

The outer unit normal to the side P_1P_2 is $\mathbf{n}$, therefore this side is represented by the vector

$$d\mathbf{s} = \mathbf{n}\, ds \tag{180}$$

Of course, it would be possible to represent the triangle sides vectorially by themselves. Our previous approach is easy to generalize to curvilinear coor-

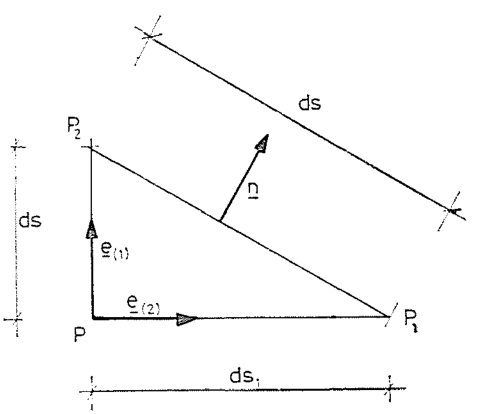


Fig. 11. Shell element in orthogonal coordinates

dinates, but at the end of the next section we will demonstrate the last mentioned method, as well.

The sum of the vectors defined in equations (179) and (180) is zero, because the vector triangle is similar to the original one.

$$\mathbf{n}\, ds = \mathbf{e}_{(1)} ds_1 + \mathbf{e}_{(2)} ds_2 \tag{181}$$

The stress vector $\mathbf{t}$ will act upon the side P_1P_2, the vectors $\mathbf{t}_{(\alpha)}$ on the sides PP_α. The equilibrium condition for the triangle is given by

$$\mathbf{t}\ ds = \mathbf{t}_{(1)}\, ds_1 + \mathbf{t}_{(2)}\, ds_2 \tag{182}$$

Dividing equations (180) and (181) by ds and comparing them yields

$$\mathbf{n} = \frac{ds_1}{ds}\, \mathbf{e}_{(1)} + \frac{ds_2}{ds}\, \mathbf{e}_{(2)} \tag{183}$$

$$\mathbf{t} = \frac{ds_1}{ds}\, \mathbf{t}_{(1)} + \frac{ds_2}{ds}\, \mathbf{t}_{(2)}$$

The coordinates of the vectors $\mathbf{n}$ and $\mathbf{t}$ are identic in the two bases ($\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$ and $\mathbf{t}_{(1)}$, $\mathbf{t}_{(2)}$). This means, that the matrix transforming the vectors into each other is identic with the matrix of the base transformation, which is a representation of a tensor. This tensor will be denoted by N and called the surface stress tensor. The coordinates of the surface stress tensor are in the orthogonal representation identic with the stress vector components in the coordinate directions:

$$K(\mathrm{N}) = n_{\alpha\beta} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} \tag{184}$$

This relation holds only in orthogonal systems. We are now going to investigate the more general case.

2.1.2. *Introduction of the surface stress tensor in general coordinate systems*

We will proceed as we did in the previous section, but without making any restrictions to the intristic geometry of the hypersurface, therefore the investigated triangle in Fig. 12 isn't orthogonal any more

The contravariant base vectors coincide with the sides of the triangle. The sides will be represented by their outer normals, as before. On the basis of (38) we know, that

$$|\mathbf{a}_{(\alpha)}| = \sqrt{g_{\alpha\alpha}} \tag{185}$$

According to this

$$d\mathbf{s}_2 = \frac{\mathbf{a}^{(2)}}{\sqrt{g^{22}}}\, ds_2 \qquad d\mathbf{s}_1 = \frac{\mathbf{a}^{(1)}}{\sqrt{g^{11}}}\, ds_1 \tag{186}$$

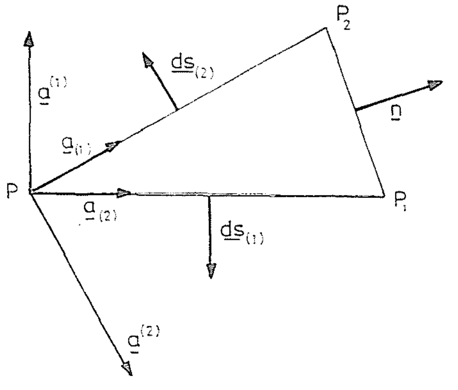


Fig. 12. Shell element in general coordinates

The outer unit normal to the side P_1P_2 will be denoted by $\mathbf{n}$ as before, this side will be represented (similarly to (180)) by

$$ds = \mathbf{n} ds \tag{187}$$

The geometry of the triangle is expressed by

$$\mathbf{n}\, ds = \frac{\mathbf{a}^{(1)}}{\sqrt{g^{11}}} \quad ds_1 + \frac{\mathbf{a}^{(2)}}{\sqrt{g^{22}}} ds_2 \tag{188}$$

Resolving $\mathbf{n}$ into covariant components yields

$$\mathbf{n} = n_\alpha \mathbf{a}^{(\alpha)} \tag{189}$$

Substituting into (188) yields (190)

$$n_\alpha \sqrt{g^{\alpha\alpha}}\, ds = ds_\alpha$$

The force equilibrium is expressed similarly to eq. (182) by

$$\mathbf{t}\, ds = \mathbf{t}_{(1)}\, ds_1 + \mathbf{t}_{(2)}\, ds_2 \tag{191}$$

By comparing (190) with (191) we arrive at

$$\mathbf{t} = n_\alpha \mathbf{t}_{(\alpha)} \sqrt{g^{\alpha\alpha}} \tag{192}$$

The left hand side of the above equation is coordinate-invariant, therefore the right hand side has to be invariant, as well. This is only possible, if the expression $\mathbf{t}_{(\alpha)}\sqrt{g^{\alpha\alpha}}$ transforms under the same laws as a tensor in contravariant representation. Resolving into contravariant components yields

$$t_{(\beta)} = t^{\alpha}_{(\beta)} \mathbf{a}_{(\alpha)} \tag{193}$$

Calculating the right hand side of (192):

$$t_{(\beta)} \sqrt{g^{\beta\beta}} = t^{\alpha}_{(\beta)}\, \mathbf{a}_{(\alpha)} \sqrt{g^{\beta\beta}} \tag{194}$$

By introducing the notation

$$t^{\alpha}_{(\beta)}\sqrt{g^{\beta\beta}} = n^{\beta\alpha} \tag{195}$$

we arrive at

$$\mathbf{t}_{(\alpha)}\sqrt{g^{\alpha\alpha}} = n^{\alpha\beta}\,\mathbf{a}_{(\beta)} \tag{196}$$

Summarizing our results: The contravariant components of the surface stress tensor express the stress components in the direction of the coordinate axes multiplied by the magnitude of the covariant base vectors. Comparison between (192) and (196) yields

$$\mathbf{t} = n^{\alpha\beta} n_{\alpha}\mathbf{a}_{(\beta)} \tag{197}$$

which is equivalent to the scalar equation

$$t^{\alpha} = n^{\alpha\beta} n_{\alpha} \tag{198}$$

We will now investigate the case, when the triangle sides represent themselves vectorially illustrated in Fig. 13.

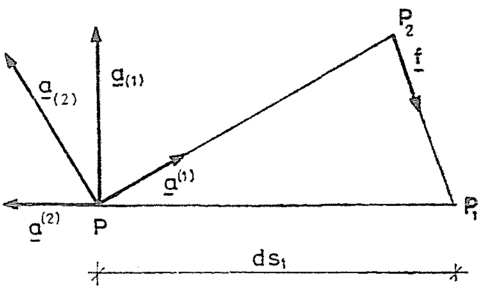


Fig. 13. Shell element in transformed general coordinates

We will denote the covariant representation of the unit vector in the direction of the side P_1P_2 by f_{α}. Equation (190) can be directly expressed by this components. Since **f** is orthogonal to **n**, the covariant coordinates of the two vectors will be identic, if we change the covariant and contravariant base, and the sign of one of them. In this new system the coordinates of the two vectors are equivalent and can be changed in the formulas. We are now going to give a formal description of the above described conclusion. The analogous reference for the equations is given.

$$d\mathbf{s}_1 = \frac{\mathbf{a}^{(1)}}{\sqrt{g^{11}}}\,ds_1 \quad d\mathbf{s}_2 = \frac{\mathbf{a}^{(2)}}{\sqrt{g^{22}}}\,ds_2 \qquad \text{see (186)} \tag{199}$$

$$d\mathbf{s} = \mathbf{f}\,ds \qquad \text{see (187)} \tag{200}$$

$$\mathfrak{f}\,ds = \frac{\mathbf{a}^{(1)}}{\sqrt{g^{11}}}\ ds_1 + \frac{\mathbf{a}^{(2)}}{\sqrt{g^{22}}} ds^2 \qquad \text{see (188)} \tag{201}$$

$$f_\alpha \sqrt{g^{\alpha\alpha}}\, ds = ds_\alpha \tag{202}$$

By comparing (202) with (190) we see, that our previous conclusion was correct.

2.1.3. Connection between the physical components and the tensor components

In connection with eq. (195) we already investigated the physical interpretation of the covariant components of the stress tensor. This investigation is necessary, because the final aim of our calculations is the determination of the physical components. Equation (196) can be transformed to

$$\mathfrak{t}_{(\alpha)} = \frac{1}{\sqrt{g^{\alpha\alpha}}} \cdot \frac{\sqrt{g_{\beta\beta}}}{\sqrt{g_{\beta\beta}}} \cdot n^{\alpha\beta}\,\mathbf{a}_{(\beta)} \tag{203}$$

Introducing a new notation this can be written as

$$\mathfrak{t}_{(\alpha)} = \frac{N_{\alpha\beta}\,\mathbf{a}_{(\beta)}}{\sqrt{g_{\beta\beta}}} \tag{204}$$

Comparing the above formula with eq. (185) we see, that the quantities $N_{\alpha\beta}$ are the physical components of the vector $\mathbf{t}$ along the skew axes. On the basis of (203) and (204) we arrive at

$$N_{\alpha\beta} = \sqrt{\frac{g_{\beta\beta}}{g^{\alpha\alpha}}}\, n^{\alpha\beta} \tag{205}$$

We will now investigate by using the transformation rule (86), whether the quantities $N_{\alpha\beta}$ are the representation of a tensor, or not. The transformation equation will be written for the component N_{11}:

$$\sqrt{\frac{\beta_{\bar 1}^{1}\beta_{\bar 1}^{1}g_{11} + \beta_{\bar 1}^{1}\beta_{\bar 1}^{2}(g_{12}+g_{21}) + \beta_{\bar 1}^{2}\beta_{\bar 1}^{2}g_{22}}{\beta_{1}^{\bar 1}\beta_{1}^{\bar 1}g^{11} + \beta_{1}^{\bar 1}\beta_{1}^{\bar 2}(g^{12}+g^{21}) + \beta_{1}^{\bar 2}\beta_{1}^{\bar 2}g^{22}}} \cdot \beta_{\bar 1}^{1}\beta_{\bar 1}^{1}n_{11} + \beta_{\bar 1}^{1}\beta_{\bar 1}^{2}(n_{12}+n_{21}) +$$
$$+ \beta_{\bar 1}^{2}\beta_{\bar 1}^{2}n_{22} = \beta_{\bar 1}^{1}\beta_{\bar 1}^{1}\sqrt{\frac{g_{11}}{g^{11}}}\,n_{11} + \beta_{\bar 1}^{1}\beta_{\bar 1}^{2}\left(\sqrt{\frac{g_{11}}{g^{22}}}\,n_{12} + \sqrt{\frac{g_{22}}{g^{11}}}\,n_{21}\right) + \beta_{\bar 1}^{2}\beta_{\bar 1}^{2}\sqrt{\frac{g_{22}}{g^{22}}}\,n_{22} \tag{206}$$

By applying the $x^{1'} = x^2$, $x^{2'} = x^1$ inverse transformation, where

$$\beta_{\bar 1}^{1} = \beta_{1}^{\bar 1} = 0 \quad \beta_{\bar 1}^{2} = \beta_{1}^{\bar 2} = 1 \tag{207}$$

holds and by substituting (207) into (206) yields the condition

$$\sqrt{\frac{g_{22}}{g^{22}}} = 1 \tag{208}$$

which holds exclusively in orthogonal systems. This indicates, that the physical components don't represent a tensor, but in a special case they coincide with the tensor components. In the general case the physical components may be computed on the basis of eq. (205) from the tensor components.

2.2. *Equilibrium equations*

The equilibrium of the structure is expressed by the equilibrium of the parts. In our case it is sufficient to investigate the force equilibrium of the infinitesimal shell element illustrated in Fig. 14. because in membranes there are no bending moments.

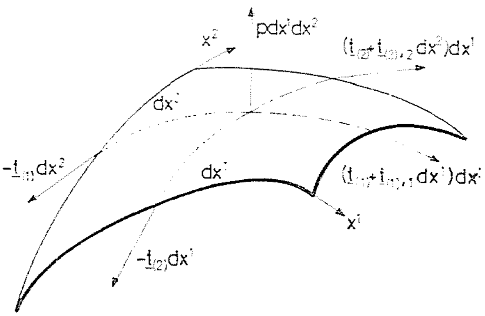


Fig. 14. The equilibrium of the shell element

The area dS of the shell element is given by

$$dS = \sqrt{g_{11} g_{22}} \cos \varphi = \sqrt{g_{11} g_{22}} \sqrt{1 - \frac{(g_{12})^2}{g_{11} g_{22}}} = \sqrt{g} \tag{209}$$

on the basis of (141). The distributed external load will be denoted by p, the resultant of the load acting upon dS is on the basis of (209)

$$P = p\sqrt{g} \tag{210}$$

The equilibrium is expressed by the formula

$$T_{(\alpha),\alpha} + P = 0 \tag{211}$$

where

$$T_{(\alpha)} = t_{(\alpha)} ds_{(\alpha)} \tag{212}$$

On the basis of Fig. 12 and eq. (141) we have

$$ds_{(\alpha)} = \sqrt{g_{(3-\alpha)(3-\alpha)}} \tag{213}$$

which can be written according to (70) as

$$ds_{(\alpha)} = \sqrt{g g^{\alpha\alpha}} \tag{214}$$

Substituting now into (212) and by using (196) we arrive at

$$T_{(\alpha)} = \sqrt{g g^{\alpha\alpha}}\, \mathbf{t}_{(\alpha)} = \sqrt{g n^{\alpha\beta}}\, \mathbf{a}_{(\beta)} \tag{215}$$

To differentiate the surface vector field $T_{(\alpha)}$ we use the following form of eq. (169):

$$\mathbf{v}_{,\beta} = \mathbf{a}_{(\gamma)}(v^{\alpha}{}_{,\beta} + v^{\alpha}\Gamma^{\gamma}_{\alpha\beta}) + v^{\alpha} b_{\alpha\beta}\mathbf{a}_{(3)} \tag{216}$$

By using this equation we can write (211) for the in-surface equilibrium and the normal equilibrium separately:

$$[(\sqrt{g}n^{\alpha\beta})_{,\beta} + \sqrt{g}n^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta}]\mathbf{a}_{(\gamma)} - P^{\alpha}\mathbf{a}_{(\alpha)}\sqrt{g} = 0 \tag{217}$$

$$\sqrt{g}n^{\alpha\beta} b_{\alpha\beta}\mathbf{a}_{(3)} + \sqrt{g}P^{3}\mathbf{a}_{(3)} = 0 \tag{218}$$

Writing now scalar equations yields

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^{(\alpha)}} n^{\alpha\beta} + n^{\alpha\beta}{}_{,\beta} + n^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta} - P^{\alpha} = 0 \tag{219}$$

$$n^{\alpha\beta} b_{\alpha\beta} - P^{3} = 0 \tag{220}$$

The above equations can be written on the basis of (11) and (117) in the following concise form:

$$n^{\alpha\beta}|_{\beta} - P^{\alpha} = 0 \tag{221}$$

$$n^{\alpha\beta} b_{\alpha\beta} - P^{3} = 0 \tag{222}$$

By substituting the expressions for the metric tensor components and the Christoffel symbols into the above formulas we arrive at a differential equation system with three unknowns for the three independent components of the stress tensor. Since (221) contains a free greek index it contains two independent equations, which means, that we have three equations for three unknowns.

3. Applications

3.1. Computation of the metric and curvature quantities of a surface described by a scalar-scalar function in two independent variables

In the praxis the shape of the shell is usually given in the form $z = f(x,y)$, which is a scalar-scalar function in two independent variables. Since the shape of the shell will not be restricted in this section we will stick to a special type of coordinate system: the orthogonal projection of the xy lines onto the surface. This system is illustrated in Fig. 15.

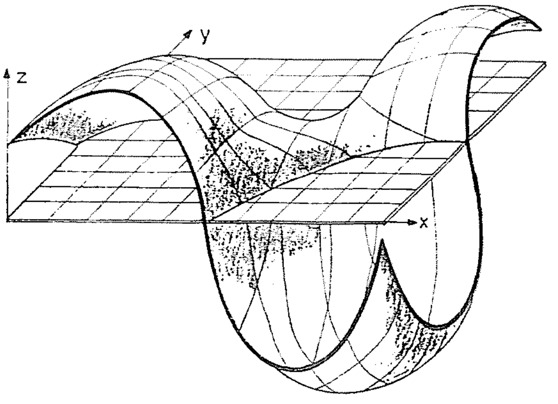


Fig. 15. Orthogonally projected coordinates

Let's determine the coordinates of the base vector fields $\mathbf{a}_{(\alpha)}$ as two-dimensional scalar fields: on the basis of eq. (74):

$$\mathbf{a}_{(1)} = 1 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + f_x \cdot \mathbf{k} \tag{223}$$

$$\mathbf{a}_{(2)} = 0 \cdot \mathbf{i} + 1 \cdot \mathbf{j} + f_y \cdot \mathbf{k} \tag{224}$$

The symbols $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ describe the unit vectors at the coordinate axes, the symbols f_x, f_y the partial derivatives of the function f. By using (38) and the above equations the components of the metric tensor are expressed as

$$g_{11} = \mathbf{a}_{(1)} \cdot \mathbf{a}_{(1)} = 1 + f_x^2 \tag{225}$$

$$g_{12} = \mathbf{a}_{(1)} \cdot \mathbf{a}_{(2)} = f_x f_y = g_{21}$$

$$g_{22} = \mathbf{a}_{(2)} \cdot \mathbf{a}_{(2)} = 1 + f_y^2$$

The determinant of the metric tensor is

$$g = f_x^2 + f_y^2 + 1 \tag{226}$$

The investigated surface inherits the metric of the embedding euclidean space, this was the condition we used to determine the coordinates of the metric tensor. This condition means, that the scalar (dot-) product of two vectors is identic in the surface and the embedding space, or (equivalently), the surface can be locally substituted by the tangent plane. The coordinates of the curvature tensor will be determined by (143). First we need the coordinates of the $\mathbf{a}_{(3)}$ vector field, which is orthogonal to both base vectors. By using (143), (223) and (224) we arrive at

$$1 \cdot x + 0 \cdot y + f_x \cdot z = 0 \tag{227}$$

$$0 \cdot x + 1 \cdot y + f_y \cdot z = 0 \tag{228}$$

The coordinates of $\mathbf{a}_{(3)}$ were denoted by x, y and z. On the basis of (142)

$$x^2 + y^2 + z^2 = 1 \tag{229}$$

By solving the above equations to the three unknowns x, y and z we arrive at

$$\mathbf{a}_{(3)} = \frac{-f_x}{\sqrt{f_x^2 + f_y^2 + 1}} \cdot \mathbf{i} - \frac{f_y}{\sqrt{f_x^2 + f_y^2 + 1}} \cdot \mathbf{j} + \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \cdot \mathbf{k} \tag{230}$$

The partial differentiation of the base vectors yields

$$\begin{aligned} \mathbf{a}_{(1)'x} &= 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + f_{xx}\, \mathbf{k} \\ \mathbf{a}_{(1)'y} &= 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + f_{xy}\, \mathbf{k} = \mathbf{a}_{(2)'x} \\ \mathbf{a}_{(2)'y} &= 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + f_{yy}\, \mathbf{k} \end{aligned} \tag{231}$$

After substituting into (148) we arrive at

$$b_{\alpha\beta} = \frac{f_{\alpha\beta}}{\sqrt{g}} \tag{232}$$

(in this case the greek indices can be equal to x or y). Now we are able to express the Gauss curvature of the surface as

$$k = \frac{f_{xx} f_{yy} - f_{xy^2}}{g^2} \tag{233}$$

The mixed representation components of the curvature tensor can be visualized only under the restrictions we used in equations (156) and (158). In general they are complicated, for example

$$b_1^1 = \frac{f_y^2 f_{xx} + f_{xx} - f_x f_y f_{xy}}{(f_x^2 + f_y^2 + 1)^{\frac{3}{2}}} \tag{234}$$

With the help of eq. (110) and eq. (225) the Christoffel symbols can be determined as

$$\Gamma_{\alpha\beta}^{\gamma} = f_{\alpha\beta} f_{\gamma} \tag{235}$$

The Christoffel symbols in mixed representation can be found by the contravariant representation of the metric tensor. This is expressed on the basis of (70), (225) and (226) as

$$\begin{aligned} g^{11} &= \frac{1 + f_y^2}{f_x^2 + f_y^2 + 1} \\ g^{12} &= \frac{-f_x f_y}{f_x^2 + f_y^2 + 1} = g^{21} \\ g^{22} &= \frac{1 + f_x^2}{f_x^2 + f_y^2 + 1} \end{aligned} \tag{236}$$

By using (106) we arrive at

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{f_{\alpha\beta} f_{\gamma}}{g} \tag{237}$$

Now we have the metric and curvature quantities necessary to construct the differential equations.

3.2. *Equilibrium equations of the hyperbolic paraboloid*

We will now show a possible application of the derived equations. The surface investigated is the hyperbolic paraboloid. At each point of this surface two straight lines can be drawn which are contained in the surface. We choose at first this straight lines as coordinates. We will now have to place the surface into a three-dimensional euclidean system, where the coordinate lines are exactly the projections of the mentioned straight lines, because the formulas in the previous section were derived for this type of coordinate systems. This condition will be fulfilled if the surface is given by the equation

$$z_1 = \frac{x_1 y_1}{c} \tag{238}$$

(The lower indices are there to distinguish this system from the following ones.) The relative position of surface and coordinate system is illustrated in Fig. 16. Developing equations (221) and (222) yields

$$n^{11}{}_{,1} + n^{11}(2\Gamma^1_{11} + \Gamma^2_{21}) + n^{12}{}_{,2} + n^{12}(3\Gamma^1_{12} + \Gamma^2_{22}) + n^{22}\Gamma^1_{22} = P^1 \tag{239}$$

$$n^{22}{}_{,2} + n^{22}(2\Gamma^2_{22} + \Gamma^1_{12}) + n^{21}{}_{,1} + n^{21}(3\Gamma^2_{21} + \Gamma^1_{11}) + n^{11}\Gamma^2_{11} = P^2 \tag{240}$$

$$n^{11} b_{11} + 2n^{12} b_{12} + n^{22} b_{22} = P^3 \tag{241}$$

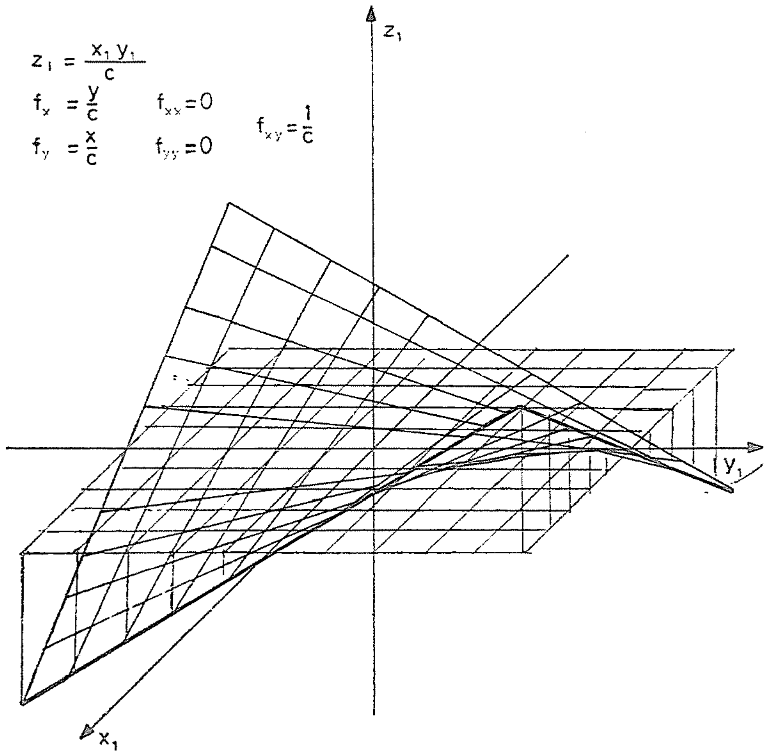
If we substitute the quantities computed on the basis of the last section for the equation (238) and demonstrated in Fig. 16, then we arrive at the following system of differential equations:

$$\frac{\partial n^{11}}{\partial x} + \frac{x}{x^2 + y^2 + c^2} + \frac{\partial n^{12}}{\partial y} + n^{12} \frac{3y}{x^2 + y^2 + c^2} = P^1 \tag{242}$$

$$\frac{\partial n^{22}}{\partial y} + n^{22} \frac{y}{x^2 + y^2 + c^2} + \frac{\partial n^{21}}{\partial x} + n^{21} \frac{3x}{x^2 + y^2 + c^2} = P^2 \tag{243}$$

$$n^{12} \cdot \frac{2}{(x^2 + y^2 + c^2)^{\frac{1}{2}}} = P^3 \tag{244}$$

All variables have the index 1, therefore this isn't indicated. Figure 16 and eq. (244) illustrates, that in this coordinate system $b_{11} = b_{22} = 0$. The reason for this is, that the curvature of the surface along the coordinate line disappears,



$$g_{11} = 1 + \frac{y^2}{c^2} \qquad g_{12} = g_{21} = \frac{xy}{c^2} \qquad g_{22} = 1 + \frac{x^2}{c^2}$$

$$b_{11} = b_{22} = 0 \qquad b_{12} = b_{21} = \frac{1}{(x^2 + y^2 + c^2)^{1/2}}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{y}{x^2+y^2+c^2} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{x}{x^2+y^2+c^2}$$

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = 0$$

Fig. 16. Hyperbolical paraboloid and related quantities in the $x_1y_1z_1$ system

because this lines are straight. In formal computation this results in the disappearence of the second partial derivatives of the same variable. The mixed second partial derivative refers to the twist. The Gauss curvature is computed by using (233):

$$k = \frac{-c^2}{[(x_1)^2 + (y_1)^2 + c^2]^2} \tag{245}$$

The Gauss curvature proved to be negative, which refers to the hyperbolic character of the surface. Formula (245) indicates, that the orthogonal projection of the points with constant Gauss curvature is a circle on the x_1y_1 plane.

The hyperbolic paraboloid is a translational surface, as well. The translation of a parabola along another parabola in an orthogonal plane and inverse direction describes the same surface we were investigating until now. We are looking for an external $x_2y_2z_2$ system, where the orthogonal projection of the two sets of parabolas coincides with the coordinate lines in the x_2y_2 plane. By rotating our previous system around the $z_1 = z_2$ axis we arrive at the wanted equation:

$$z_2 = \frac{(x_2)^2 - (y_2)^2}{2c} \tag{246}$$

The relative position of the surface and the coordinate system is illustrated in Fig. 17.

The equilibrium equations are the following:

$$\frac{\partial n^{11}}{\partial x} + n^{11}\frac{2x}{x^2+y^2+c^2} + \frac{\partial n^{12}}{\partial y} - n^{12}\frac{y}{x^2+y^2+c^2} + n^{22}\frac{x}{x^2+y^2+c^2} = P^1 \tag{247}$$

$$\frac{\partial n^{22}}{\partial y} - n^{22}\frac{2y}{x^2+y^2+c^2} + \frac{\partial n^{21}}{\partial x} + n^{12}\frac{x}{x^2+y^2+c^2} - n^{11}\frac{y}{x^2+y^2+c^2} = P^2 \tag{248}$$

$$(n^{11} + n^{22})\frac{1}{(x^2+y^2+c^2)^{\frac{1}{2}}} = P^3 \tag{249}$$

(The indices of the variables are always 2.)

The above formulas demonstrate, that the coordinate directions are now identical with the directions of the main curvatures, because the mixed partial derivative disappears. Our computations can be checked by calculating the coordinate-invariant scalar Gauss curvature in the new system:

$$k = \frac{-c^2}{[(x_2)^2 + (y_2)^2 + c^2]^2} \tag{250}$$

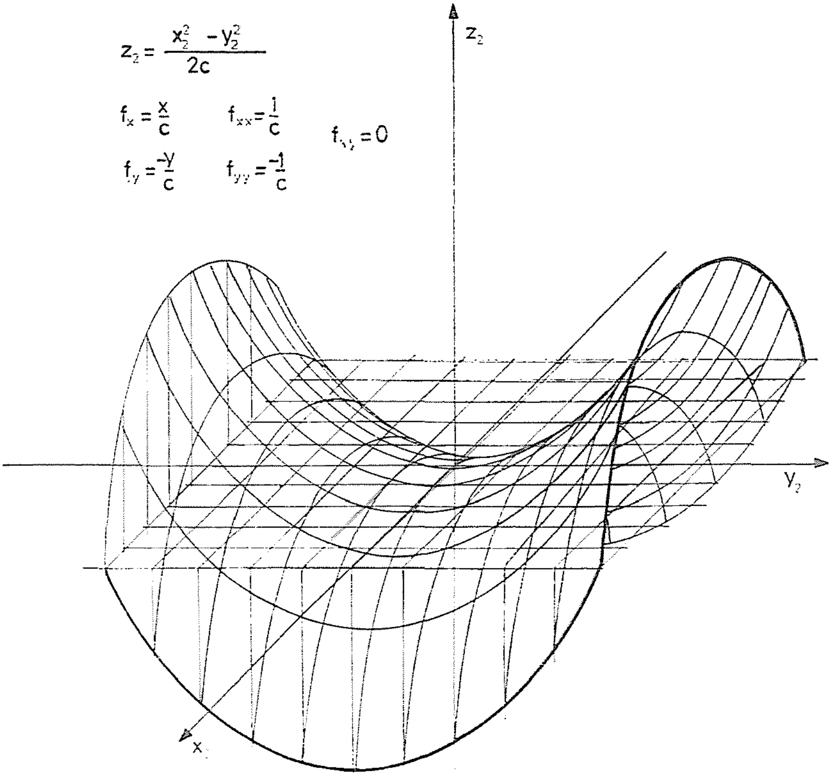
The above formula is identical with (245) if, and only if

$$(x_1)^2 + (y_1)^2 = (x_2)^2 + (y_2)^2 \tag{251}$$

holds. This relation is always true, because

$$(x_1)^2 + (y_1)^2 = (x_2)^2 + (y_2)^2 = r^2 \tag{252}$$

where "r" means the horizontal distance from the z axis, which is invariant under any rotation around this axis.



$$g_{11} = 1 + \frac{x^2}{c^2} \qquad g_{12} = g_{21} = -\frac{xy}{c^2} \qquad g_{22} = 1 + \frac{y^2}{c^2}$$

$$b_{11} = b_{22} \quad \frac{1}{(x^2+y^2+c^2)^{1/2}} \qquad b_{12} = b_{21} = 0$$

$$\Gamma^1_{11} = \Gamma^1_{22} = \frac{x}{x^2+y^2+c^2} \qquad \Gamma^2_{11} = \Gamma^2_{22} = \frac{-y}{x^2+y^2+c^2}$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \Gamma^2_{21} = \Gamma^2_{12} = 0$$

Fig. 17. Hyperbolical paraboloid and related quantities in the $x_2y_2z_2$ system

We already referred to the hyperbolic character of the surface. The origin of this name is the fact, that the plane sections orthogonal to the axis are hyperbolas. We want this hyperbolas to be coordinate lines. In order to do this we simply have to change the vertical axis with any horizontal one in the $x_1y_1z_1$ system arriving at

$$z_3 = \frac{cy_3}{x_3} \tag{253}$$

The relative position of surface and coordinate system is illustrated in Fig. 18. We have now the following equilibrium equations:

$$\frac{\partial n^{11}}{\partial x} - n^{11}\frac{c^2(4y + x^2)}{x(c^2x^2 + c^2y^2 + x^4)} + \frac{\partial n^{12}}{\partial y} + n^{12}\frac{3c^2y}{c^2x^2 + c^2y^2 + x^4} = P^1 \tag{254}$$

$$\frac{\partial n^{22}}{\partial y} + n^{22}\frac{c^2y}{c^2x^2 + c^2y^2 + x^4} + \frac{\partial n^{21}}{\partial x} - n^{21}\frac{c^2(2y + 3x^2)}{x(c^2x^2 + c^2y^2 + x^4)} + n^{11}\frac{2c^2y}{c^2x^2 + c^2y^2 + x^4} = P^2 \tag{255}$$

$$(n^{11}y - n^{12}x)\frac{2c}{x(c^2x^2 + c^2y^2 + x^4)^{\frac{1}{2}}} = P^3 \tag{256}$$

(The constant index 3 of the variables was neglected here again for the sake of simplicity.)

The Gauss curvature is calculated as

$$k = \frac{-c^2(x_3)^4}{[c^2(y_3)^2 + c^2(x_3)^2 + (x_3)^4]^2} \tag{257}$$

By transforming the above expression in the original system with

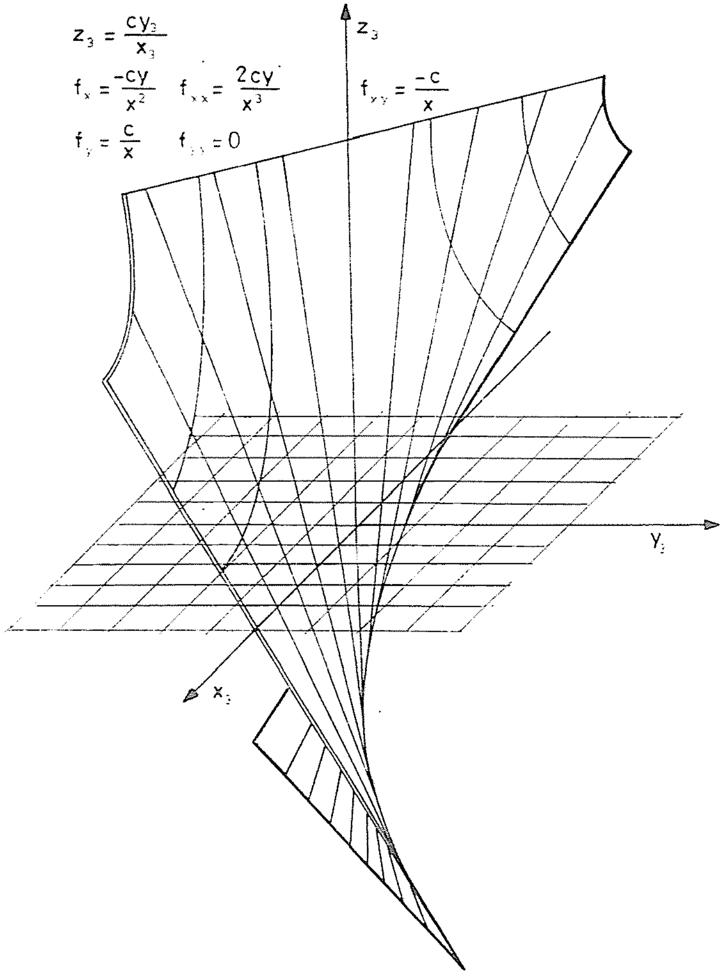
$$y_3 = z_1 = \frac{x_1y_1}{c} \quad x_3 = x_1 \tag{258}$$

we arrive the eq. (250).

The equivalence of the equilibrium equations in the different coordinate systems can be investigated only after having solved the equations. After solution we either transform the tensor components under the rule (86) or we compute the eigenvectors, but this goes beyond the range of this paper. As a closing word we'd like to refer to more general coordinate systems. As a first possibility we mention the case, when the orthogonal projections of the coordinate lines are straight lines, but not orthogonal any more. In this case the coordinates of the base vector fields are either calculated by directional derivatives, or the equation of the surface has to be expressed in the skew system. The most general coordinate lines are space curves, which cannot be transformed into a position, where the orthogonal projections are straight lines. In this case the tangent of the projected curve describes the direction, along which the surface has to be differentiated.

All surface coordinate systems, which are the projections of the coordinate lines of the embedding space don't have the "natural" arch length as parameter along their coordinate lines. This is a disadvantage from the physical point of view, but the equations are much simpler that way, which is the engineer's point of view.

We finished our calculations, and we'd like to investigate, how do our results correspond to the promises we made at the beginning of this paper.



$$g_{11} = 1 + \frac{c^2 y^2}{x^4} \qquad g_{12} = g_{21} = -\frac{c^2 y}{x^3} \qquad g_{22} = \frac{c^2}{x^2}$$

$$b_{11} = \frac{2cy}{x(c^2x^2 + c^2y^2 + x^4)^{1/2}} \qquad b_{12} = b_{21} = \frac{-c}{(c^2x^2 + c^2y^2 + x^4)^{1/2}} \qquad b_{22} = 0$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{c^2}{c^2x^2 + c^2y^2 + x^4} \qquad \Gamma^1_{11} = \frac{-2c^2 y}{x(c^2x^2 + c^2y^2 + x^4)} \qquad \Gamma^1_{22} = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{-c^2 x}{c^2x^2 + c^2y^2 + x^4} \qquad \Gamma^2_{11} = \frac{2c^2 y}{c^2x^2 + c^2y^2 + x^4} \qquad \Gamma^2_{22} = 0$$

Fig. 18. Hyperbolical paraboloid and related quantities in the $x_3y_3z_3$ system

The main aim of our investigations was to eliminate the coordinate systems from the physical description of membranes as far as possible. In order to do this, we had to develope a mathematical tool, which enables us to make direct computations with curved surfaces. Despite the fact, that in technical applications one usually prefers a formal description which is easy to handle, this doesn't guarantee, that the same formal description is easy to understand, as well. The type of description we tried to demonstrate in this paper can hardly be handled easily by the practical engineer, but we hope, that we could provide a deep insight for the interested reader in the physical and mathematical background of membrane theory, which is often camouflaged by the usual methods in computational mechanics. To illustrate, that the derived results are not "pure theory" we illustrated them on the hyperbolical paraboloid — and they seem to work.

List of symbols

1. Symbols without indices

$\mathbf{e}$	unit vector
$\mathbf{t}$	stress vector
p	physical load vector
$\mathbf{E} = \mathbf{Q}$	unit tensor = metric tensor
$\mathbf{F}$	rotation tensor
$\mathbf{B}$	second order curvature tensor
$\mathbf{N}$	stress tensor
K	orthogonal coordinate system
A	skew coordinate system
A^G	curvilinear coordinate system
g	determinant of the metric tensor
k	Gauss curvature

2. Indices

i, j, k, l, m	can be equal to 1, 2 or 3
α, β	can be equal to 1 or 2

(By quantities, where both latin and greek indices may occur, we will explain only the version with latin indices.)

3. Symbols with a single index

$\mathbf{e}_{(i)}$	vectors of the orthogonal unit base
$\mathbf{f}_{(i)}$	transformed versions of the above vectors
$\mathbf{a}_{(i)}, \mathbf{a}^{(i)}$	contra- and covariant base vectors

$\mathbf{t}_{(\alpha)}$	stress vectors
$T_{(\alpha)}$	physical resultant vectors
$z_{,i}$	partial derivatives of a scalar field
$\mathbf{v}_{,i}$	partial derivatives of a vector field

4. Symbols with two indices

g_{ij}, g^{ij}	covariant and contravariant representation of the metric tensor
$g^i_j = \delta^i_j$	mixed representation of the metric tensor = Kronecker delta
$f_{\alpha\beta}$	representation of the rotation tensor
$n_{\alpha\beta}, n^{\alpha\beta}$	representation of the stress tensor
$\beta^i_{\bar{k}}, \beta^{\bar{i}}_k$	matrices of coordinate-transformations
$b_{\alpha\beta}, b^{\alpha\beta}, b^\alpha_\beta$	representations of the second order curvature tensor
$v_{i,j}$	partial derivatives of vector field
$v_i\|_j, v^i\|_j$	representations of second order tensor derivative
$\mathbf{a}_{(i),j}$	partial derivatives of base vector fields
$t^\beta_{(\alpha)}$	contravariant coordinates of the stress vectors

5. symbols with three indices

$\Gamma_{ijk}, \Gamma^k_{ij}$	Christoffel symbols
$t_{ij,k}$	partial derivatives of a second order tensor
$t_{ij}\|_k$	representation of the third order tensor derivative
$\mathbf{a}_{(i),jk}$	partial derivatives of the base vector fields

6. Symbols with four indices

r_{ijkm}	representation of the Riemann—Christoffel tensor
$\Gamma^i_{jk,m}$	partial derivatives of Christoffel symbols

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