ON THE POSSIBILITY OF DESIGNING HYPERSTATIC STRUCTURES

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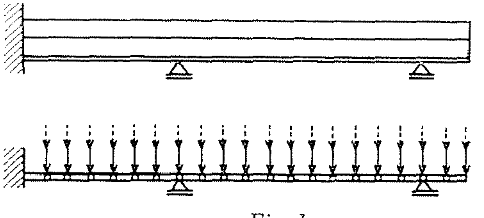
In engineering practice, there is hardly anything to go on for the expedient selection of cross-sectional dimensions of hyperstatic structures, a problem little concerned with in engineering education, either. In what follows, some simple principles and applications will be presented in the scope of selecting cross-sectional dimensions of hyperstatic beam structures, ideas intended by the Author to underlie preparation of introducing a more comprehensive study of this subject in graduate and post-graduate structural engineering courses.

1. Stating the problem

Let us consider the sketched continuous beam of ideal elastic material, clamped at the left end, subject to dead and live loads. Cross-sectional dimensions of this beam have to be selected under different stipulations.

To ease theoretical survey and computer treatment, the problem will be handled by the method of discretization. The original continuous structure (Fig. 1a) will be substituted by a discrete model of elastic hinges and perfectly rigid connecting bars (Fig. 1b). The discrete model is acted upon by loads in discretized form, i. e., by concentrated forces at bar ends (elastic hinges or cantilever ends). The supports may be placed similarly.



a)

b)

Fig. 1

The beam analysis will only concern flexural stresses and deformations. Perfectly rigid bars being assumed to support illimited stresses without deformation, only moments

$$M_2, M_3, \ldots, M_n$$

in the $(n-1)$ elastic hinges have to be considered.

Load capacity of the structure is considered to be adequate if absolute values of moments in each elastic hinge do not exceed the ultimate moment for that hinge:

$$-M_{2Ru} \leq M_2 \leq M_{2Ru}, \; \ldots, \; -M_{nRu} \leq M_n \leq M_{nRu}.$$

For this hyperstatic structure, moments arising in each hinge can only be determined by taking deformations into consideration. In the discrete model structure only elastic hinges are able to elastic deformation, to angular rotation proportional to developing bending moments. Deformabilities of elastic hinges will be described by their flexibilities:

$$\varphi_2 = h_2 M_2, \; \ldots, \; \varphi_n = h_n M_n.$$

After these preliminaries, the problem of determining cross-sectional dimensions of this hyperstatic structure can be closer formulated: beam behaviour being affected by bar cross sections through two parameters, the ultimate moment and the flexibility, the design problem consists in selecting ultimate moments and flexibilities

$$M_{2Ru}, \; \ldots, \; M_{nRu} \quad \text{and} \quad h_2, \; \ldots, \; h_n$$

so as to meet certain conditions still to be considered.

As concerns stating the problem, let us notice that deductions will not be restricted to the beam over three supports in Fig. 1 but affect a continuous beam clamped at the left end, discretized by inserting n elastic hinges, and supported at m elastic hinges (see Fig. 2). Since hinged support at the left end can be simulated by selecting h_n very high and M_{nRu} very low, while applying a support both at 1 and 2, right-end clamping is simulated, the instructions to be drawn can be stated to refer to continuous beams in general (or even, to elastic structures described by scalar quantities for each cross section from both stress and deformation aspects).

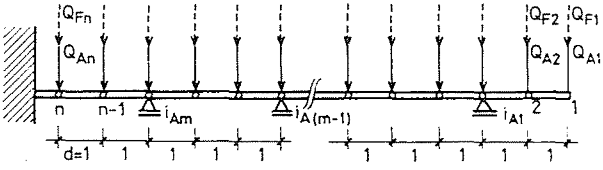


Fig. 2

2. Basic relationships

To write basic relationships, dividing points or elastic hinges of the discretized model structure will be numbered from right to left, beginning with 1. The right-side cantilever end will be numbered 1, here is no elastic hinge but a force may act (Fig. 2). To make formulae clearer, consecutive elastic hinges will be assumed to be equidistant, at unit spacings. (This is of course a stipulation on spacing proportions to be dissolved by multiplying the divisions.) The beam has m supports (beside clamping), their place is, however, fixed by numbering the elastic hinges supported by them against vertical displacement. Thus, the right-side end support is at elastic hinge i_{A1}, the next one at i_{A2}, at last that closest to clamping at i_{Am}.

This hyperstatic problem will be solved by the force method. Let cantilevered beam with left-end clamping be the primary beam. Now, according to the force method, there will be m unknowns, namely supporting forces $A_1, A_2, \ldots, A_m$, to be determined from m linear equations expressing zero vertical displacements of nodes $i_{A1}, i_{A2}, \ldots, i_{Am}$.

Since the design problem involves a multiparameter load system, let us produce stresses for unit loads. Let us first consider the primary beam. Let hinge j be acted upon by unit downward force. Hence, moment at hinge i:

$$M_{0i} = -(i - j) \qquad \text{for } i > j,$$
$$= 0 \qquad \text{for } i \leq j.$$

Since similar functions will be frequent in these analyses, the moment above will be simply denoted as:

$$M_{0i} = \{i - j\}$$

remarking that term $\{l\}$ "zero parentheses l" has a value of l if l is positive, and zero if l is zero or negative.

Analysis by the force method needs primary beam deflections at nodes supported for the original beam, easy to write according to the notation above. Vertical displacement $f_{k,j}$ of the primary beam due to unit force at j, at support k sited at i_{Ak} is given by:

$$f_{k,j} = \sum_{i=1}^{n} (\{i - j\}\{i - i_{Ak}\} h_i).$$

These $f_{k,j}$ values include displacements at supports due to unit forces acting at the supported nodes, hence to supporting forces of unit value. These being preferential in analyses by the force method, displacement at support

i_{Ak} due to unit value of upward supporting force at i_{A1} will be differentiated by denoting $e_{1,k}$:

$$e_{1,k} = -\sum_{i=1}^{n} \left(\{i - i_{A1}\}\{i - i_{Ak}\} h_i\right) = f_{k, i_{A1}}.$$

To ease survey of formulation, let us introduce matrices

$$\mathbf{A} = [e_{1,k}] \quad (l = 1, 2, \ldots, m; k = 1, 2, \ldots, \mathrm{m})$$

and

$$\mathbf{F} = [f_{k,j}] \quad (k = 1, 2, \ldots, m; j = 1, 2, \ldots, n).$$

A being a quadratic matrix (non-singular in compliance with its physical meaning in practically important problems), its inverted $\mathbf{A}^{-1}$ may be formed. Elements of this inverted matrix are supporting forces belonging to unit vertical displacements of each supporting point as kinematic loads. Multiplying this inverted by matrix $\mathbf{F}$ of support displacements in the primary beam from the right, i.e., forming matrix product

$$\mathbf{A}^{-1} \cdot \mathbf{F}$$

yields a rectangular matrix of m rows corresponding to the number of supports, and of n columns corresponding to the number of divisions. Every column contains supporting force values arising at m different supports due to unit force acting at a given dividing point. This is, in fact, solution of the hyperstatic structure problem according to the force method, delivering, for unit value of each dividing point force, the corresponding redundant quantities, i. e., supporting forces.

In knowledge of the former, let us determine final moments in the elastic hinges due to unit forces acting at each dividing point, simply by multiplying the unit force and the pertaining, already available supporting forces by the corresponding lever arms, and summing the products. For a moment arising in elastic hinge i due to unit force acting at the jth dividing point, the arm will be $\{i - j\}$, and the arm of the kth supporting force $\{i - i_{Ak}\}$. Let us introduce for the overall notation of these arms the quadratic matrix $\mathbf{R} = [-\{i - j\}]$ size $n \times n$, and the rectangular matrix $\mathbf{R}_A = [-\{i - i_{Ak}\}]$ size $n \times m$ (this latter is obtained from the former by omitting columns relating to unsupported nodes). Making use of them, and reminding that the needed supporting forces have been produced in form of matrix product $\mathbf{A}^{-1}\mathbf{F}$, the moment block

$$\mathbf{N} = \mathbf{R}_A \mathbf{A}^{-1} \mathbf{F} + \mathbf{R}$$

may be written, a quadratic matrix size $n \times n$, where an element in the ith row and the jth column indicates moment in elastic hinge i due to unit force acting at dividing point j (obviously, in the case of the examined hyperstatic structure, rather than of the primary beam).

Matrix N lends itself to determine moments for any real load. Denoting the entity of loads $Q_1, Q_2, \ldots, Q_n$ at dividing points $1, 2, \ldots, n$ by load vector $\mathbf{q} = [Q_1, Q_2, \ldots, Q_n]$, and the thereby produced moments $M_1, M_2, \ldots, M_n$ in elastic hinges $1, 2, \ldots, n$ by moment vector $\mathbf{m}^*$, they are related as:

$$\mathbf{m} = \mathbf{N}\mathbf{q}.$$

3. Load capacity of the structure

Analysis of the load capacity of the structure may rely on row vectors of matrix N:

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1^* \\ \mathbf{N}_2^* \\ \vdots \\ \mathbf{N}_n^* \end{bmatrix}.$$

Namely scalar product $\mathbf{N}_i^*\mathbf{q}$ yields the moment for any load system $\mathbf{q}$ in the elastic hinge i. The load capacity of the structure is sufficient if for all i (i.e., in every elastic hinge) and for every load system $\mathbf{q}$ in the structure:

$$M_{iRu} \geq \left| N_i^* \mathbf{q} \right| .$$

In the actual case, load Q_j at point j may assume any value between lower and upper limits Q_{Aj} and Q_{Fj}, and loads acting at different points may be considered as independent (i.e., also partial load in the fields is allowed). Under these conditions, it is sufficient to examine two load vectors, $\mathbf{q}_{Ai}$ and $\mathbf{q}_{Fi}$, for any elastic hinge i. These are obtained as:

jth element of $\mathbf{q}_{Ai}$ is Q_{Aj}, and jth element of $\mathbf{q}_{Fi}$ is Q_{Fj} for $N_{i,j} \geq 0$;

while

jth element of $\mathbf{q}_{Ai}$ is Q_{Fi}, and jth element of $\mathbf{q}_{Fi}$ is Q_{Aj} for $N_{i,j} < 0$.

Utilizing the obtained vectors $\mathbf{q}_{Ai}$ and $\mathbf{q}_{Fi}$, load capacity of the structure may be stated to be adequate if inequalities

$$-M_{iRu} \leq \mathbf{N}_i^* \mathbf{q}_{Ai} \quad \text{and} \quad M_{iRu} \geq \mathbf{N}_i^* \mathbf{q}_{Fi}$$

are met for every $i = 1, 2, \ldots, n$.

4. A design possibility

The considered problem is how to assume ultimate moments $M_{2Ru}, \ldots, M_{nRu}$ and flexibility values $h_2, \ldots, h_n$. Of course, a fundamental requirement is the sufficient load capacity of the structure.

It will be seen how easy it is — at least theoretically — to design structures of adequate load capacity. Assume arbitrary $h_1, h_2, \ldots, h_n$ values (e.g. be flexibilities of every elastic hinge of given, equal values). Flexibility values unambiguously yield vectors $\mathbf{N}_i^*$ as described above. They will be used to form maximum and minimum moments $\mathbf{N}_i^* \mathbf{q}_{Fi}$ and $\mathbf{N}_i^* \mathbf{q}_{Ai}$ in every elastic hinge. Finally, assume M_{iRu} to equal the absolute value of either the maximum or the minimum moment depending on which of them has the higher one.

This design method is rather similar to that applied for statically determined structures: extreme stresses are determined and cross sections of the needed load capacity are selected accordingly. Another similarity is that in both cases a fully stressed design is achieved in the sense that every element of the structure is stressed to its ultimate strength, for at least one possible load combination. There is, however, a decisive difference: while for statically determined structures, extreme stresses depend only on the structural arrangement (e.g. hinge locations), rather than on cross-sectional dimensions, extreme stresses in hyperstatic structures can only be calculated in knowledge of the flexibility values, and flexibility, just as ultimate moment, is function of cross-sectional dimensions.

Let us examine when to apply the design method relying on the anticipation of flexibilities, if it has any significance at all. Formulae of flexibility and of ultimate moment are, respectively:

$$h = \frac{d}{EJ} \quad \text{and} \quad M_{Ru} = R_u W$$

where d is an interval assumed in establishing the discrete model (1 in the actual example), by no means a design parameter. E and R_u are values dependent on the building material. This latter could theoretically be subject to design but practically there is no point about varying ultimate strength and/or modulus of elasticity within the same beam, so in analysing the distribution of flexibility and of ultimate moment along the beam axis, E and R_u have to be considered as constant, leaving I and W alone to be considered as design variables.

Both I and W are magnitudes assigned to the plane configuration describing the beam cross section, strictly interrelated as:

$$W = \frac{I}{y_{\max}}.$$

The feasibility of the design method based on the anticipation of flexibilities depends on what is the set of cross sections to be selected from in design. For cross section families where members can be described by one parameter (e.g. square section by edge size, circular section by radius, rectangular section with invariable width by depth), the assumption of flexibilities unambiguously

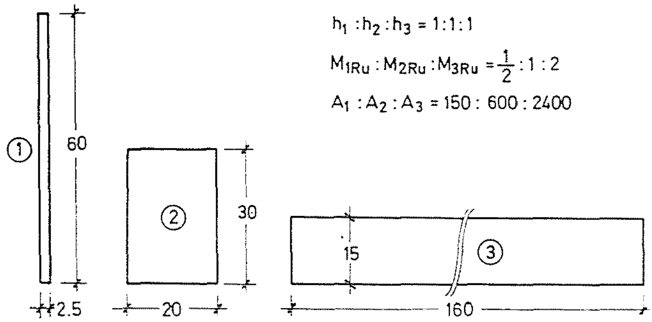


Fig. 3

defines the value of this singular free cross-sectional parameter, counteracting adequate assumption of M_{Ru}. Thus, the outlined design method is out of question for the case of single-parameter cross section families. The same holds for constant y_{max} (with any number of parameters of the cross section family), namely then assumption of flexibility I unambiguously determines ultimate moment W.

The design method based on anticipating the flexibilities emerges only for cross section families of at least two parameters, where I and W can be independently assumed. This is the case of e.g. rectangular sections where both b_s and h_s may be freely selected in design. Assuming flexibility I and ultimate moment W, formula

$$W = \frac{I}{h_s/2}$$

yields depth h_s, in its knowledge known formulae of either I or W yield section widths b_s, hence to each flexibility and ultimate moment value a rectangular section can be assigned. Practical value of the obtained rectangular cross sections is a different problem. Sections of identical flexibilities and of load capacities interrelated as 1/2:1:2 are seen in Fig. 3. Rather curious, practically almost irrealizable configurations are seen to result, beams with sections so much varying by depth and width dimensions along the axis are unlike to be practical. Thus, the design method anticipating flexibility values would be theoretically feasible for an adequate choice of sections, in general (that is, assuming flexibilities without preliminary considerations) it cannot lead to direct practical results.

5. Theoretical conclusions

Although the design method of anticipating flexibilities does not lead to direct practical conclusions, it may be of help in the theoretical consideration of problems concerning hyperstatic structural design.

To elucidate theoretical conclusions, let us first survey and refine the involved fundamental concepts.

The entity of cross sections available for design are called a family of cross sections, said to be complete if assuming h and M_{Ru} with any value not opposite to their physical meaning, there is at least one member of the family of cross sections with the assumed flexibility and ultimate moment values. A family of cross sections may have one or several parameters, depending on whether a given cross section can be selected from the family by indicating one, two or more scalars. A complete family of cross sections has two or more parameters.

Designing the structure means to assign a cross section of the family to each examined element of the structure (in the discretized problem to each hinge or section). Stresses in a designed structure may be calculated for all possible load combinations. Selecting among them that of the highest absolute value at a beam section under investigation yields the extreme stress at the given point. The designed structure is adequate if the absolute value of extreme stresses nowhere exceeds the ultimate strength. The structure is fully stressed if the extreme stress equals the ultimate strength at any point tested. In other words, for any section of a fully stressed structure there is a stress equal in absolute value to the ultimate strength. Obviously, a fully stressed structure is always adequate.

In designing a structure, there are always design conditions to be considered. Maybe, simply an adequate structure has to be designed, a design problem with several solutions. Practical design conditions are, however, more restricted than that. The structure may be required to be fully stressed, a design condition subject to the following statements:

— If cross sections can be selected out of a complete family of cross sections, a fully stressed design can always be achieved.
— For a complete family of cross sections, a fully stressed structure may be designed for any set of arbitrarily assumed flexibility values, that is, the design problem of a fully stressed structure has several solutions.

The condition of design may be to design an optimum structure, in the sense as follows. A characteristic value c is assigned to every cross section in the given family of cross sections. If value c of the cross section with the highest load capacity from among those of identical flexibility but different load capacities is always the greatest, characteristic values c are said to be well arranged. Characteristic values c express some important aspect of structural design, e.g. specific material consumption, costs, or their combinations. Obviously, the concept of well arranged characteristic values c involves that from among sections of identical flexibilities, those of higher load capacity cost more, and consume more of material as a rule. In designing a structure, a section is chosen for all nodes examined. Summing up values c of cross sections chosen for nodes

$i = 1, 2, \ldots, n$ yields the so-called objective function of the structure:

$$C = \sum_{i=1}^{n} c_i .$$

(It may be called a function because its value depends on the cross sections chosen in design.) The adequate structure having the least value of the objective function C is the optimum one.

Concepts of optimum and of fully stressed structures are somehow related, namely:

— In the case of a complete family of cross sections with well arranged characteristic values c, from among structures with a given set of specified flexibilities, the fully stressed is the optimum one.

— For a complete family of cross sections and well arranged characteristic values c, the optimum structure is a fully stressed one.

These statements directly follow from concept definitions and from observations made with the design method relying on the anticipation of flexibilities, making special justifications superfluous. The last statement, however, may be completed by remarking that the optimum structure may be found by taking all possible different sets of flexibility values, determining the optimum structure in case of each given set of flexibilities and searching out the structure with the least objective function value among these sub-optima. Since under the conditions considered, the optimum design in case of a given set of flexibility values is always a fully stressed design (last but one statement), the overall optimum is a fully stressed design, too (in accordance with the last statement).

Summary

Some aspects of structural design of continuous beams and similar hyperstatic structures are dealt with. The influence of moment bearing capacity and flexibility of the individual members (cross sections) on the overall load capacity of the whole structure is investigated. Criteria are established for the feasibility of fully stressed design and some relationships are presented between fully stressed and optimum designs.

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