PRACTICAL METHODS OF CONSTRUCTING BUILDING PERSPECTIVES

by

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Two methods will be presented for constructing building perspectives, motivated by two aspects:

— the architect constructing a perspective image has in general a definite idea of how the building is meant to appear, but in practice, he needs a perspective system granting the fancied building the desired appearance:

— generalization of minicomputers justifies to examine their possibilities in perspective construction.

In up-to-date architecture, masses of the majority of buildings are fairly approximated by an "enveloping prism". After having constructed the perspective image of the prism, the construction is easy to achieve even without rotated centre and vanishing points indispensable in usual constructions [3]. (In practice, perspective images without at least one accessible vanishing point are exceptional, lending fastness to the unconventional perspective construction.)

Now, two methods will be presented for constructing perspective images of prisms on a horizontal plane. In this paper perspective image will be meant as a perspective with two vanishing points and vertical image plane. To assume prisms to stand on a horizontal plane is no restriction, namely the image of the prism part above the horizontal plane is simple to complete to that of the whole prism.

The first construction method is based on the following theorem:

If the perspective image of one face of a prism having also edges parallel to the image plane is given, then the geometrical loci of images of missing vertices define a hyperbola.

Because of the constraint, the face image is a trapezium; for the sake of simplicity, the planes of the prism faces normal to the image plane will be considered as horizontal. Be the image of such a face the trapezium AHH_aA_a (Fig. 1).

Extending the trapezium edges and finding their intersection yields one vanishing point of the horizontal edges of the prism. Passing from this point a normal to the vertical edges yields the horizont line. According to the statement above, now simply the missing vertices of the perspective image of the prism part belonging to the orthogonal trapezium AHH_hA_h have to be found.

Let this image represent a prism with edge HH_h in the image plane, of a width $AH = a$ and length $BH = b$. Point A_0 is defined at a distance a from point H parallelly to the horizontal line; the intersection of line A_0-A and the horizontal line H is point O_b.

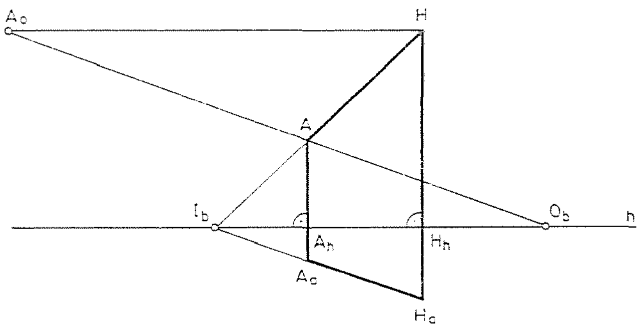

Fig. 1

Now, geometrical loci of centre C of the perspective system are on a circle K_1 in the horizontal plane, with centre I_b and radius I_bO_b, since the centre and the point O_b are at equal distance from the vanishing point.

Again, geometrical loci of point B are on a circle K_2 with centre H and radius b in the plane A passing through point H.

The prism edge AH being normal to the edge HB, also the radius I_bC is normal to HB; hence projection radii of point B have as geometrical loci the surface defined by connecting an arbitrary point of K_1 with the end point of a radius of K_2 normal to the radius belonging to that point.

Second-ordership of the defined surface will be understood by imagining the circle K_2 to be shifted in its plane to have point H on the normal of the horizontal plane at point I_b.

This transformation is a shear one, with the horizontal plane as basis. Now, the transformed surface defined as above is a hyperboloid of revolution if the two circles are not in the same plane. Since this transformation affects neither the surface order, nor the number of its points in infinity, also the original surface is a hyperbolic hyperboloid.

Geometrical locus of B^*, perspective image of point B, is on the intersection line of this surface with the image plane, a hyperbola as deduced by considerations similar to the former ones. The image plane turns out to be plane of symmetry for the hyperboloid of revolution, and the direction of shear transformation to be parallel to the image plane, consequently the image plane is a plane of symmetry for the original hyperbolic hyperboloid as well.

At the same time, the hyperbola section in the symmetry plane of the hyperbolic hyperboloid is the outline curve of the perpendicular projection of the surface on the symmetry plane; hence perpendicular projections of surface generators belonging to the plane section points are tangential to the plane section hyperbola. Thereby the geometric locus of point B is simple to indicate, namely points of circles K_1 and K_2 in the image plane fit the hyperbola, and the straight line connecting any of these points with the centre of the other circle is tangential to the hyperbola in question. (If any of the four points is inaccessible, then two tangents have to be drawn.)

Now, in possession of the geometrical locus of point B^*, another condition, e.g. the image width, or the direction of the third edge starting from H, permits to determine point B^*.

A possible way of determining the intersection between a conic section defined by four points and a tangent or by three points and two tangents consists in determining, first, five points of the conic section (in the first case, by the *Pascal* theorem, and in the second case, by the *Desargues* involution theorem), then determining the intersection point of the conic section given with five points, and the straight line by means of the so-called *Steiner* method of construction, double elements as conic section points are projected by sets of projection radii from two points of the conic section.

Since the method above would be somewhat tedious to follow graphically, let us try it in an analytic way: let the perspective image of point B be the intersection point B^* of the projection radius BC with the image plane; be $B'C'$ the normal projection of the projecting radius in the image plane, and the line $(C)(B_h)$ the rotated of the normal projection in the horizontal plane of the projection radius about the horizontal line into the image plane.

Let I_b be the origin of the co-ordinate system, and angle included between the line I_bC and the image plane the parameter α, R the radius of K_1, r that of K_2, and j the distance I_bH_h; now, the point co-ordinates are: (Fig. 2)

Due to the similarity of triangles $CC'B_h^*$ and $B_h\ B_h'\ B_h^*$:

$$\frac{x - R\cos\alpha}{-R\sin\alpha} = \frac{j - r\cdot\sin\alpha - x}{r\cos\alpha}\,.$$

Expressing x:

$$x = R\,\frac{r - j\sin\alpha}{r\cos\alpha - R\sin\alpha}\,.$$

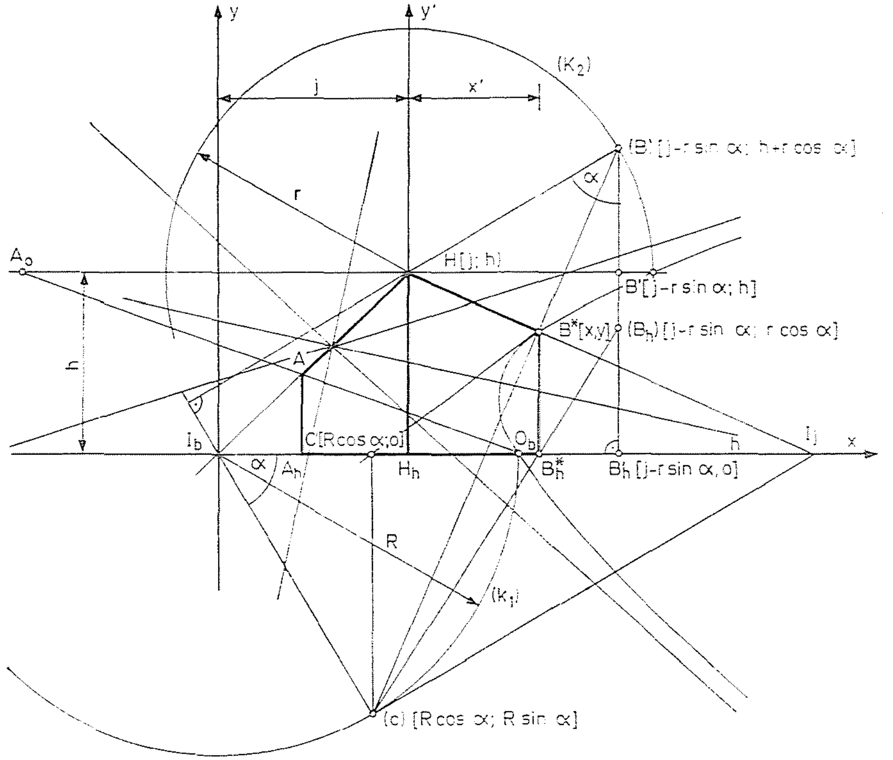

Fig. 2

Due to the similarity of triangles $C'B_h^*B^*$ and $C'B_h'B'$:

$$\frac{y}{x - R\cos\alpha} = \frac{h}{j - r\sin\alpha - R\cos\alpha}\,.$$

Hence:

$$y = h\,\frac{x - R\cos\alpha}{j - r\sin\alpha - R\cos\alpha}\,.$$

Substituting x from (1) and eliminating the compound fraction:

$$y = h\,\frac{R(r - j\sin\alpha) - R\cos\alpha(r\cos\alpha - R\sin\alpha)}{(j - r\sin\alpha - R\cos\alpha)\,(r\cos\alpha - R\sin\alpha)}\,.$$

Transforming the fraction numerator into a product:

$$R(r - j\sin\alpha) - R\cos\alpha(r\cos\alpha - R\sin\alpha) = - R\sin\alpha(j - r\sin\alpha - R\cos\alpha).$$

Resubstituting and simplifying:

$$y = -h \frac{R \sin \alpha}{r \cos \alpha - R \sin \alpha} . \tag{2}$$

Thus, the geometrical locus of point B^* is described by the set of Eqs (1) and (2) by means of parameter α. Eliminating α yields the equation of the geometrical locus. The quotient of (1) by (2):

$$\frac{x}{y} = \frac{r - j \sin \alpha}{-h \sin \alpha} .$$

Expressing $\sin \alpha$:

$$\sin \alpha = \frac{ry}{jy - hx} . \tag{3}$$

Expressing $\sin^2 \alpha$ from (2):

$$\sin^2 \alpha = \frac{r^2y^2}{R^2(y - h)^2 + r^2y^2} . \tag{4}$$

(3) and (4) yield the equation of the geometrical locus:

$$4^2x^2 - 2hjxy - (R^2 + r^2 - j^2)y^2 + 2R^2hy - R^2h^2 = 0. \tag{5}$$

Thus, the wanted geometrical locus is a second-order curve. Considering the subdeterminant of the quadratic coefficients,

$$A_{33} = -h^2(R^2 + r^2)$$

is seen to be negative if any of parameters h and R or r is non-zero, hence in fact, the wanted geometrical loci are on a hyperbola.

Therefrom, in possession of e.g. the image width, the missing co-ordinate y of the image of point B becomes:

$$y = h \frac{R^2 - jx \pm \sqrt{R^2(x - j)^2 + r^2(x^2 - R^2)}}{R^2 + r^2 - j^2} .$$

The co-ordinates are more convenient to handle by shifting the origin to point H_h. Now, after subtituting $x = x' + j$, and simplifying, the former co-ordinate y becomes:

$$y = h \frac{R^2 - j(x' + j) \pm \sqrt{R^2(x'^2 - r^2) + r^2(x' + j)^2}}{R^2 + r^2 - j^2} .$$

The presented method has the shortcoming that the image obtained by construction or calculation suits further construction only if the spatial position of the centre proves to be convenient.

Remark: In connection with the constructional method, it has been referred to that the generators of the hyperbolic hyperboloid projected normally onto the image plane are tangential to the hyperbola of geometrical loci. Now, adequate spatial location of the centre is provided by finding a hyperbola tangent intersecting the horizontal line at mid-spacing $A_h B_h$. Since the hyperbola has four such tangents, the problem is inaccessible to construction, the image width can only be assessed so that the wanted solution about meets the condition above.

The second procedure starts from the requirement to have the normal projection of the centre at the mid-point of the perspective image. The angle included between one prism face and the image plane, the bisector of the visual cone and the image width are considered to be given (Fig. 3).

Let $AHBB_hH_hA_h$ be the perspective image of a prism on a horizontal plane with edge HH_h in the image plane. Since this prism is derivable by mirroring the original prism to scale about centre C, its corresponding edges are $AH = \lambda a$ and $BH = \lambda b$ and its height $HH_h = \lambda h$.

Passing through points A and B perspective lines parallel to the bisector of angle AHB: A_0 and B_0 will be the intersections of these parallel lines with the image plane, now, obviously:

$$\frac{A_0H}{\lambda a} = \frac{B_0H}{\lambda b} = \mu .$$

Let us take a horizontal plane passing through point H (A and B being spatial points).
Let o_A, o_H and o_B be lines parallel to the angle bisector fitting points A, H and B, resp. Lines BH and o_A intersect at O. Since $AHo_H \sphericalangle = o_HHB \sphericalangle = o_HHO \sphericalangle$, AHO is an equilateral triangle, hence $AH = OH$. Our statement follows from the theorem of parallel secants.

The μ value will be calculated from the triangle HB_0B. By convention, α being the angle included between the direction BH and the normal to the image plane $BHB_0 \sphericalangle = 90° - \alpha$ hence:

$$\mu = \frac{B_0H}{\lambda b} = \frac{\sin 45°}{\sin (45° + \alpha)} = \frac{1}{\sin \alpha + \cos \alpha} .$$

Parameters needed for further calculations are, according to Fig. 5:

$$Q_0 = -AH = \frac{\lambda a}{\sin \alpha + \cos \alpha}$$

$$b_0 = HB = \frac{\lambda b}{\sin \alpha + \cos \alpha}$$

$$d = (C)F = \frac{s}{2 \operatorname{tg} \beta}$$

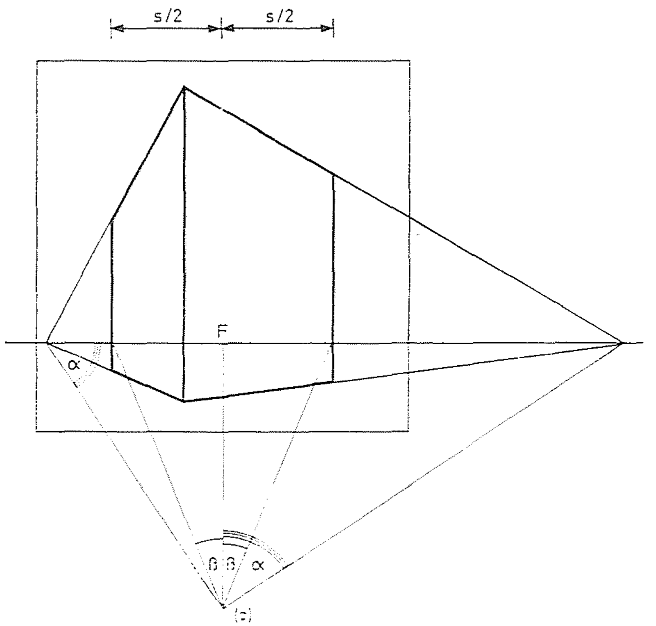

Fig. 3

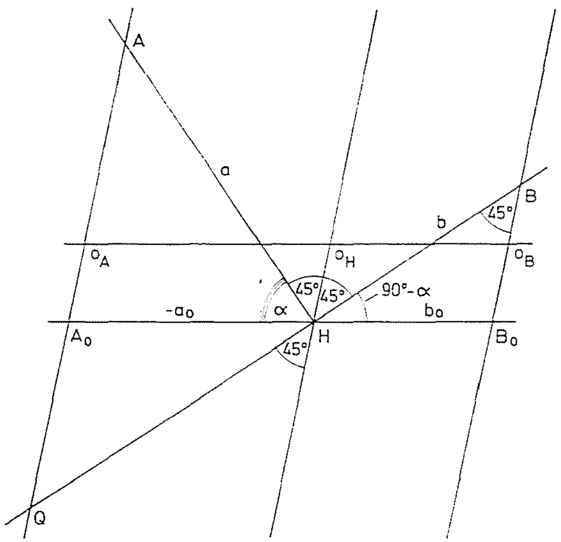

Fig. 4

$$f = d\,\mathrm{tg}(\alpha - 45°) = \frac{s}{2\,\mathrm{tg}\,\beta} \cdot \frac{\mathrm{tg}\,\alpha - 1}{\mathrm{tg}\,\alpha + 1}$$

$$i = d \cdot \mathrm{tg}\,\alpha = \frac{s}{2\,\mathrm{tg}\,\beta} \cdot \mathrm{tg}\,\alpha$$

$$j = -d\frac{1}{\mathrm{tg}\,\alpha} = -\frac{s}{2\,\mathrm{tg}\,\beta} \cdot \frac{1}{\mathrm{tg}\,\alpha}\,.$$

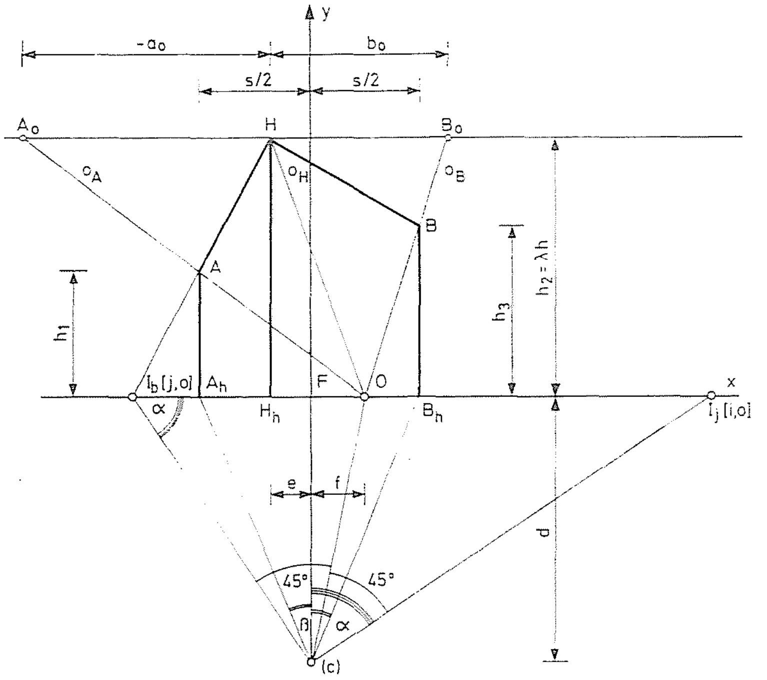

Fig. 5

Let us fit the perspective image to an orthogonal co-ordinate system with the horizontal line as x-axis, and the principal point as origin (Fig. 5). Let us calculate abscissae of A and B. Lines HI_j and B_0O intersect at B. Their equations are:

$$\lambda hx - (b_0 + e - f)y - \lambda hf = 0$$

$$\lambda hx - (e - i)y - \lambda hi = 0\,.$$

Solving the set of equations yields the co-ordinate x of B:

$$x_B = \frac{e(f-i) - b_0 i}{f - i - b_0}\,.$$

Similarly:

$$x_A = \frac{e(f-j) - a_0 j}{f - j - a_0}.$$

Conditions:
— point F in the centre of the perspective image,
— image width s
are met by the equation system:

$$x_A = \frac{s}{2}$$

$$x_B = \frac{s}{2}$$

where variables e and λ are unknown. Substituting into the second equation and expressing $\lambda (b_0 = \mu\lambda b$ and $a_0 = -\mu\lambda a)$:

$$\lambda = \frac{(f-i)\left(e - \frac{s}{2}\right)}{\mu b\left(i - \frac{s}{2}\right)}.$$

Substituted into the first equation:

$$\frac{e(f-j) + \frac{(f-i)(e-s/2)}{\mu b(i-s/2)}\mu a j}{f - j + \frac{(f-i)(e-s/2)}{\mu b(i-s/2)}\mu a} = -\frac{s}{2}.$$

Expressing e:

$$e = \frac{s}{2}\,\frac{-\frac{a}{b}\,\frac{s/2+j}{f-j}\,\frac{f-i}{s/2-i} - 1}{-\frac{a}{b}\,\frac{s/2+j}{f-j}\cdot\frac{f-i}{s/2-i} + 1} = \frac{s}{2}\,\frac{Q-1}{Q+1}. \tag{8}$$

Let us possibly simplify the compound fractional term encountered in both the numerator and the denominator.
In the orthogonal triangle $I_b C I_j$, $f-i$ and $f-j$ are sections of the hypotenuse:

$$f - i = -I_j O \text{ and } f - j = O I_b.$$

Point O being the intersection of the angle bisector starting from corner (C) with the opposite side:

$$\frac{f-i}{f-j} = -\frac{I_j O}{O I_b} = -\frac{I_j(C)}{I_b(C)} = -\operatorname{tg}\alpha$$

namely $I_j(C)F \sphericalangle = (C)I_bF' \sphericalangle$. Substituting:

$$Q = \frac{a}{b} \cdot \frac{\dfrac{s}{2} - \dfrac{s}{2\,\mathrm{tg}\,\alpha} \cdot \dfrac{1}{\mathrm{tg}\,\alpha}}{\dfrac{s}{2} - \dfrac{s}{2\,\mathrm{tg}\,\beta} \cdot \mathrm{tg}\,\alpha}\,\mathrm{tg}\,\alpha .$$

Simplifying:

$$Q = \frac{a}{b}\,\frac{1 - \mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta}{\mathrm{tg}\,\alpha - \mathrm{tg}\,\beta} .$$

Enlarging the fraction by $\cos\alpha\cos\beta$:

$$Q = \frac{a}{b}\,\frac{\cos(\alpha + \beta)}{\sin(\alpha - \beta)} . \tag{9}$$

Substituting into (8) and multiplying by denominators of the compound fractions:

$$e = \frac{s}{2}\,\frac{a\cos(\alpha + \beta) - b\sin(\alpha - \beta)}{a\cos(\alpha + \beta) - b\sin(\alpha - \beta)} . \tag{10}$$

Substituting into (7):

$$\lambda = \frac{\left(\dfrac{s}{2\,\mathrm{tg}\,\beta}\,\dfrac{\mathrm{tg}\,\alpha - 1}{\mathrm{tg}\,\alpha + 1} - \dfrac{s}{2\,\mathrm{tg}\,\beta} \cdot \mathrm{tg}\,\alpha\right)\left(\dfrac{s}{2} \cdot \dfrac{Q-1}{Q+1} - \dfrac{s}{2}\right)}{\dfrac{1}{\sin\alpha + \cos\alpha}\,b\left(\dfrac{s}{2\,\mathrm{tg}\,\beta} - \dfrac{s}{2}\right)} .$$

Expressing and simplifying:

$$\lambda = \frac{s(\sin\alpha + \cos\alpha)(\mathrm{tg}^2\,\alpha + 1)}{b(Q+1)(\mathrm{tg}\,\alpha + 1)(\mathrm{tg}\,\alpha - \mathrm{tg}\,\beta)} .$$

Enlarging by $\cos^2\alpha \cdot \cos\beta$:

$$\lambda = \frac{s\cos\alpha}{b\sin(\alpha - \beta)(Q+1)} .$$

Substituting (9) and multiplying:

$$\lambda = \frac{s\cos\alpha}{a\cos(\alpha + \beta) + b\sin(\alpha - \beta)} . \tag{11}$$

Determination of the perspective image of the prism still needs ordinates of points A, H and B (height above the horizontal line). Among these vertical edges, that containing H is in the image plane, hence:

$$h_2 = \lambda h = h \frac{s \cos \alpha}{a \cos(\alpha + \beta) + b \sin(\alpha - \beta)} . \tag{12}$$

h_1 will be determined from similar triangles $I_b A A_h$ and $I_b H H_h$:

$$h_1 : h_2 = \left| -\frac{s}{2} - j \right| : (e - j)$$

hence:

$$h_1 = \lambda h \cdot \frac{-\frac{s}{2} - j}{e - j} .$$

Substituting (8) and arranging:

$$h_1 = \lambda h \cdot \frac{Q + 1}{Q \dfrac{\frac{2j}{s} - 1}{\frac{2j}{s} + 1} + 1} .$$

Coefficient of Q in the denominator:

$$\frac{\frac{2j}{s} - 1}{\frac{2j}{s} + 1} = \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} .$$

Substituting it with (9) and (11) into (13):

$$h_1 = \frac{h s \cos \alpha}{a \cos(\alpha + \beta) + b \sin(\alpha - \beta)} \cdot \frac{\dfrac{a}{b} \dfrac{\cos(\alpha + \beta)}{\sin(\alpha - \beta)} + 1}{\dfrac{a}{b} \dfrac{\cos(\alpha + \beta) \cos(\alpha - \beta)}{\sin(a - \beta) \cos(\alpha + \beta)} + 1} .$$

Hence:

$$h_1 = \frac{h s \cos \alpha}{a \cos(\alpha - \beta) + b \sin(\alpha - \beta)} . \tag{14}$$

Again, from triangles $I_j B B_h$ and $I_b H H_h$:

$$h_3 = \frac{h s \cos \alpha}{a \cos(\alpha + \beta) + b \sin(\alpha + \beta)} . \tag{15}$$

For the sake of clearness, let us introduce notations:

$$\begin{aligned} \gamma &= \alpha + \beta \\ \delta &= \alpha - \beta \\ A &= a \cos \gamma \\ B &= b \sin \delta \\ H &= hs \cos \beta \end{aligned}$$

Now, according to (10), (14), (12) and (15), the wanted co-ordinates are:

$$e = \frac{s}{2} \cdot \frac{A - B}{A + B}$$

$$h_1 = \frac{H}{a \cdot \cos \delta + B}$$

$$h_2 = \frac{H}{A + B}$$

$$h_3 = \frac{H}{A + b \cdot \sin \gamma} .$$

Remark: It is easy to show that the geometric loci of point H are on a straight line with the equation:

$$y = x \frac{h(a \sin \alpha + b \cos \alpha)}{ab \cos 2\alpha} - \frac{h s(a \sin \alpha - b \cos \alpha)}{2ab \cos 2\alpha} .$$

Modifying the β value in the range 20° to 35° generally does not affect the h_1 and h_3 values by more than 3 to 5%, so that after having calculated two images, the new point H can be assumed anywhere along the line connecting the two former points H, and the h_1 and h_3 values can be assessed at an accuracy meeting graphic requirements.

Last but not least, application of a minicomputer much accelerates calculations. Another bonus is to eliminate rotation of the centre, often limiting the size of the image in traditional construction.

Summary

Motivations to find new methods for perspective construction include the attempts to invert the mental process of the architect trying to find a perspective system helping him realizing in a perspective image the building he definitely fancied as well as to have rather quick methods for practical perspective construction by applying popular minicomputers.

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