MULTI-PARAMETER LOAD PROBLEMS WRITTEN IN TERMS OF CHARACTERISTIC LOAD AND ANGULAR ROTATION VECTORS

by

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Engineering structures are normally acted upon by loads needing several parameters to be described. There are dead loads and live loads, these latter include different kinds of applied working loads, those of meteorological origin etc. All these loads may assume different values in different parts of the structure, and the structure itself is exposed to a great variety of load redistributions during its service life. Specification of safety, simultaneity etc. factors for the basic design values of different loads in codes is intended essentially to outline the multiplicity, "spectrum" of load distributions to be considered.

Engineering practice is required to, and is attempting to, take the multiplicity of load distributions into consideration. Therefore design or checking of structures involves the analysis of several "critical load positions", in typical, mostly "corner" points of the domain of load distributions to be reckoned with. Often, however, critical load positions assumed numerically in design (e.g. of multistorey building frameworks) is often selected on the basis of practical experience, engineering sense, while exacter analyses fail to hint to a correct assumption. The actual knowledge is still less adequate to answer questions of what load types the structure can bear, how the change of the rigidity and load bearing conditions of the structure affects the load capacities for different types of loading, how to design rigidity and load bearing conditions of the structure from the aspect of probable load distributions, etc. In general, our knowledge of the relations between structures and multiparameter load systems can be stated to still need deepening. In what follows, a comprehensive approximation will be suggested, likely of help in understanding relations between structures and multiparameter load systems. It will be illustrated on a simple problem and a structural model rather clearly exhibiting its essential features.

1. Relation between stress and load capacity of a statically determinate structure

Assume first a statically determinate cantilever beam with arbitrarily varying cross-sectional dimensions subject to an arbitrary load system normal to the bar axis. This elementary problem will be further simplified by substituting a discrete model of elastic hinges connected by perfectly stiff bars. Loads on the discrete model act only at elastic hinges (Fig. 1). Assumption of discrete models is an extended analysis method for structures, acceptable also to the "engineering sense". In this paper, the problem of the relation between discrete models and continuous structures will not be dealt with any more, and exclusively the discrete model will be examined.

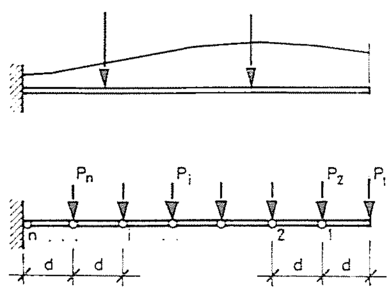

Fig. 1

All characteristics of the discrete model, intervening in the structural analysis, may be written by as many parameter values (say n) as the number of assumed elastic hinges, sets of n scalar values considered as vectors of n-dimensional *Euclidean* spaces for the sake of illustrativeness. Thus, description of a given load acting on the structure requires indication of loads $P_1, P_2, \ldots, P_n$ acting at each elastic hinge (or beam end), i.e. a vector $\mathbf{p}$ of the n-dimensional load space.

From among load-induced moments of the structure, only those in the hinges are of importance for the model, thus the moment diagram can be described by a vector $\mathbf{m}^* = [M_1, M_2, \ldots M_n]$. In case of small deformations, the vectors of the load and the moment spaces there is a mutually one to one linear correspondence, for all loads $P_1, P_2, \ldots P_n$, moments $M_1, M_2 \ldots, M_n$ in the elastic hinges can be determined by simple structural means, and vice versa:

$$\mathbf{A}\,\mathbf{p} = \mathbf{m}\,, \qquad \mathbf{A}^{-1}\,\mathbf{m} = \mathbf{p}\,.$$

In case of this model, load capacity of the beam is restricted by ultimate moments of the elastic hinges. For moment bearing capacities $M_{1H}, M_{2H}, \ldots, M_{nH}$ of the respective elastic hinges, effective moments have to meet conditions

$$|M_1| \leq M_{1H}\,, \quad |M_2| \leq M_{2H}\,, \ldots, \quad \text{and} \quad |M_n| \leq M_{nH}$$

determining a rectangular hyper-parallelepipedon in the n-dimensional space of moments with (one-dimensional) edges parallel to co-ordinate axes M_1, $M_2, \ldots, M_n$ and centered at the origin. By other words, the given structure can bear moment systems with a moment vector of the form:

$$\mathbf{m}^* = [c_1 M_{1H}\,, c_2 M_{2H}\,, \ldots, c_n M_{nH}] \tag{1}$$

where:

$$-1 \leq c_1 \leq 1, \quad -1 \leq c_2 \leq 1, \ldots, -1 \leq c_n \leq 1,$$

that is, parameters $c_1, c_2, \ldots, c_n$ are co-ordinates between -1 and $+1$.

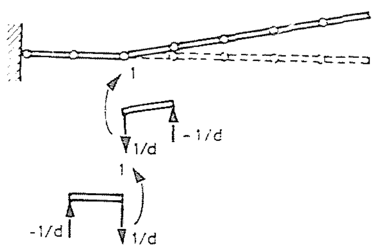

Fig. 2

Let us see now, what kind of loads $P_1, P_2, \ldots, P_n$ can be borne by our structure? Let us determine first the load vectors inducing unit moments in a single elastic hinge (in the first one, second one, n-th one, etc., in this order). These load vectors will be termed characteristic load vectors and denoted by $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, corresponding to the numeral of elastic hinge where they induce the unit moment. Let the unit moment act in the i-th hinge (Fig. 2). Now, two bar sections, the i-th and the $(i+1)$-th, are loaded. To maintain their equilibrium, the hinge moments have to be balanced by bar end forces. Now, to produce a unit moment in the i-th node, forces P_i, P_{i+1} and P_{i+2} have to act with values $-1/d$, $+2/d$ and $-1/d$, respectively, other nodal forces being zero. Accordingly (and taking d for 1) characteristic load vectors are:

$$\begin{aligned}
\mathbf{q}_1^* &= [-1, \quad 2, \; -1, \quad 0, \ldots, \quad 0, \quad 0, \quad 0]\,, \\
\mathbf{q}_2^* &= [\;\; 0, \; -1, \quad 2, \; -1, \ldots, \quad 0, \quad 0, \quad 0]\,, \\
&\;\vdots \\
\mathbf{q}_{n-2}^* &= [\;\; 0, \quad 0, \quad 0, \quad 0, \ldots, -1, \quad 2, \; -1]\,, \\
\mathbf{q}_{n-1}^* &= [\;\; 0, \quad 0, \quad 0, \quad 0, \ldots, \quad 0, \; -1, \quad 2]\,, \\
\mathbf{q}_n^* &= [\;\; 0, \quad 0, \quad 0, \quad 0, \ldots, \quad 0, \quad 0, \; -1]\,.
\end{aligned} \tag{2}$$

In the structure, moments may arise of max. absolute values of M_{1H}, $M_{2H}, \ldots, M_{nH}$ rather than unity. Hence, all load systems $\mathbf{p}$ which are admitted, can be written in the form:

$$\mathbf{p}^* = c_1 M_{1H}\mathbf{q}_1 + c_2 M_{2H}\mathbf{q}_2 + \ldots + c_n M_{nH}\mathbf{q}_n \ldots \tag{3}$$

where parameters $c_1, c_2, \ldots, c_n$ assume values between -1 and $+1$. [Eq. (3) results directly from combining (1) and (2).]

Let us have a closer look at the geometrical body, or better, at the n-dimensional "hyper-body" defined by Eq. (3) in the load space, taking the linear independence of vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ and limiting conditions of parameters $c_1, c_2, \ldots, c_n$ into consideration. This body will be confined by $(n-1)$-dimensional "hyper-planes". One "hyper-plane" is given by Eq. (3) by giving one of the parameter values one of the possible extreme values ($+1$ or -1), and letting the other $n-1$ parameters range throughout the entire value set. Thus, there are $2 \cdot n$ hyper-planes, pairs (belonging to $+1$ and -1 values of the same parameter) being parallel. One-dimensional edges of the permissible load body can be deduced from Eq. (3) by fixing all parameters but one as $+1$ or -1. Hence $2^{(n-1)}$ edges belong to each parameter, all being parallel to the characteristic load vector corresponding to the given parameter, and there are $n2^{(n-1)}$ edges in all, corresponding to the n parameters. The body of permissible loads is a "hyper-rhomboid" or "hyper-parallelepiped" centred at the origin, and in view of its edges being parallel to characteristic load vectors, this "hyper-parallelepiped" is in general a skew-angled one with respective edge lengths:

$$2M_{1H}\,,\; 2M_{2H}\,, \ldots, 2M_{nH}\,.$$

It is interesting to note the body representing the permissible loads to be inambiguously defined by n vectors $M_{1H}\mathbf{q}_1, M_{2H}\mathbf{q}_2, \ldots, M_{nH}\mathbf{q}_n$, products of characteristic load vectors by limit moments of elastic hinges. Direction of these vectors — hence of the hyper-parallelepiped edges — is independent of the load capacity characteristics of the beam (limit moments $M_{1H}, M_{2H}, \ldots, M_{nH}$) and it only depends on the overall geometry of the beam axis (rectilinearity, elastic hinge spacing). Limit moments $M_{1H}, M_{2H}, \ldots, M_{nH}$ affect the size, more exactly, the edge length of the load-capacity body.

The load-capacity parallelepiped includes all load types supported by the structure. It is, however, difficult to decide whether a given load $\mathbf{p}$ is inside or outside the load-capacity parallelepiped. To this aim, the given vector $\mathbf{p}$ is produced as a linear combination of characteristic load vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, hence its co-ordinates in the oblique co-ordinate system of a basis $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$:

$$\mathbf{p}^* = k_1\,\mathbf{q}_1 + k_2\,\mathbf{q}_2 + \ldots + k_n\,\mathbf{q}_n\,.$$

to see if all its parameters $c_1 = k_1/M_{1H}$, $c_2 = k_2/M_{2H}$, ... and $c_n = k_n/M_{nH}$ are between -1 and $+1$. This procedure differs only by its wording from that usual in engineering practice, consisting in the static determination of moments in each elastic hinge due to load $\mathbf{p}$, and consecutive confrontation with the corresponding limit moments.

Co-ordinates of the given load vector in the system of characteristic load vectors being exactly the moments in each elastic hinge,

$$k_1 = M_1, k_2 = M_2, \ldots, k_n = M_n$$

they will be computed by exactly the same steps.

Denoting by $\mathbf{A}^{-1}$ the matrix of characteristic load vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ as column vectors, its inverse $\mathbf{A}$ can be used to determine the co-ordinates of load vector $\mathbf{p}$ according to the bases $q_1, q_2, \ldots, q_n$:

$$\mathbf{A}\,\mathbf{p} = \mathbf{k}$$

(where $\mathbf{k}$ is the column vector of the $k_1, k_2, \ldots, k_n$ values). Since $\mathbf{k} = \mathbf{m}$, it is legitimate to denote the matrix of characteristic load vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ by $\mathbf{A}^{-1}$ and to consider it the inverse of the "equilibrium matrix" $\mathbf{A}$ determining the moment for given loads. (This follows also from the construction method of characteristic load vector $\mathbf{q}_1, \mathbf{q}, \ldots, \mathbf{q}_n$.)

The n row vectors of matrix $\mathbf{A}$ describing the moments for a given load are normal to the n hyper-planes of the load-capacity parallelepiped. Namely a hyper-plane is described by the $(n-1)$ column vectors of matrix $\mathbf{A}^{-1}$, and a corresponding row vector of matrix $\mathbf{A}$ is normal to all of them (the unit matrix being the product of inverted matrices $\mathbf{A}$ by $\mathbf{A}^{-1}$). In this meaning, row vectors of matrix $\mathbf{A}$ — with elements being arms involved in moment calculations in the original meaning — are load space vectors (hence to be designed $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$). Vector directions were previously seen to be strictly related to load-capacity hyper-parallelepipeds, being normal to their sides. Their size can be related to the load-capacity parallelepiped by successively determining intersection points between radii described by these vectors and the corresponding hyper-planes. Since at the side the corresponding, say i-th, elastic hinge will develop exactly the limit moment M_{iH}, the intersection point will be determined by considering $\mathbf{q}^i$ alternatively a row vector or a column vector and computing the scalar product as a matter of fact, the moment in the i-th hinge due to load $\mathbf{q}^i$, and reducing $\mathbf{q}^i$ in proportion of M_{iH}.

The above can be recapitulated by stating that the load-capacity hyper-parallelepiped can be described by either of two different vector sets. Either the modified characteristic load vectors

$$\mathbf{k}_i = M_{iH}\,\mathbf{q}_i \ (i = 1, 2, \ldots, n)$$

computed from column vectors $\mathbf{q}_i$ of matrix $\mathbf{A}^{-1}$ can be used or the modified normal vectors of the hyper-planes

$$\mathbf{k}^i = \frac{1}{M_{iH}} \mathbf{q}^i \quad (i=1, 2, \ldots, n)$$

produced from row vectors of matrix $\mathbf{A}$. System $\mathbf{k}_i$ may be used as a basis (oblique co-ordinate system); the structure is able to support a load defined by vector $\mathbf{p}$ if none of the co-ordinates of $\mathbf{p}$ in the system $\mathbf{k}_i$ exceeds 1 in absolute value. The system $\mathbf{k}^i$ defines the characteristic width values of the load-capacity parallelepiped, the system lends itself for bearing a load defined by vector $\mathbf{p}$ if none of its scalar products with vectors of the system $\mathbf{k}^i$ exceeds 1 in absolute value.

2. Restrictions of deformations

Now, let us consider the elastic deformations of the structure. Rotations and moments of the elastic hinges are proportional to each other, thus, $M_{1H}, M_{2H}, \ldots, M_{nH}$ values above may limit either moments or angular rotations. Anyhow, proportionality factor between angular rotations and moments may differ between elastic hinges. Hence effective deformations are advisably discussed by indicating, in addition to moment limit values $M_{1H}, M_{2H}, \ldots, M_{nH}$, also the respective angular rotation limits $\varphi_{1H}, \varphi_{2H}, \ldots, \varphi_{nH}$ for each elastic hinge.

Let us now consider elastic hinge displacements normal to the beam axis. Denoting by $f_1, f_2, \ldots, f_n$ the displacements of free beam edges i.e. of elastic nodes, all these displacements may be represented by a vector $\mathbf{f}^* = [f_1, f_2, \ldots, f_n]$ in the n-dimensional space of displacements. In case of mutually independent limits for elastic hinge displacements normal to the beam axis, that is:

$$f_{1aH} \leq f_1 \leq f_{1fH}, \; f_{2aH} \leq f_2 \leq f_{2fH}, \ldots, f_{naH} \leq f_n \leq f_{nfH},$$

their entity defines a rectangular hyper-solid or box in the space of displacements. To transform this solid into the space of rotations $\varphi_1, \varphi_2, \ldots, \varphi_n$ of elastic hinges it is convenient to construct the rotations belonging to the unit displacement of the i-th elastic hinge, i.e. to the beam shape

$$f_k = 1 \text{ for } k = i$$

$$f_k = 0 \text{ for } k \neq i \;\; (k = 1, 2, \ldots, n)$$

as characteristic rotation vectors, as follows from Fig. 3:

$$
\begin{aligned}
\mathbf{s}_1^* &= [\ \ 1, \quad 0, \quad 0,\ 0,\ \ldots, 0, \quad 0,\ 0]\,, \\
\mathbf{s}_2^* &= [-2, \quad 1, \quad 0,\ 0,\ \ldots, 0, \quad 0,\ 0]\,, \\
\mathbf{s}_3^* &= [\ \ 1, \ -2, \quad 1,\ 0,\ \ldots, 0, \quad 0,\ 0]\,, \\
\mathbf{s}_4^* &= [\ \ 0, \quad 1, \ -2,\ 1,\ \ldots, 0, \quad 0,\ 0]\,, \\
&\ \ \vdots \\
\mathbf{s}_n^* &= [\ \ 0, \quad 0, \quad 0,\ 0,\ \ldots, 1, \ -2,\ 1]\,,
\end{aligned}
$$

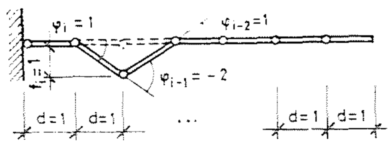

Fig. 3

Characteristic rotation vectors $\mathbf{s}_i$ and characteristic load vectors $\mathbf{q}_i$ discussed earlier are of rather similar structure. It is easy to understand that displacement limitations represented by a rectangular hyper-solid in the space of displacements have — as counterpart in the space of rotations — a hyper-parallelepiped with edges parallel to characteristic rotation vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ of lengths $f_{1aH}+f_{1fH}, f_{2aH}+f_{2fH}, \ldots, f_{naH}+f_{nfH}$. It is rather simple to pass from the space of rotations to that of moments by changing the scale (diminishing or increasing) corresponding to proportions $r_1 = M_{1H}/\varphi_{1H}$, $r_2 = M_{2H}/\varphi_{2H}, \ldots, r_n = M_{nH}/\varphi_{nH}$ in each co-ordinate direction. Thus, if limitations both for moments and rotations, and for displacements of a structure are indicated, then the former can be indicated by a rectangular hyper-solid and the latter by a skew angled hyper-parallelepiped in the space of moments, and moment vectors meeting both condition systems will be in the part common to both.

This common part is again a "hyper-solid", i.e., a "hyper-polyhedron" of max. $4n$ "sides". Provided in the moment space the "hyper-solid" of vectors meeting the system of requirements is available, the load space can be passed to, to construct the load-capacity hyper-solid including load vectors permissible for the requirement system. Moment space co-ordinate unit vectors have characteristic load vectors as counterparts in the load space, thereby the passage (linear mapping) is inambiguously settled.

3. Hyperstatic structures

Among displacement limitations, the case where certain displacements are specified as zero merits attention. Namely thereby the constraint by supports may be expressed, and the hyperstatic structures can be described by the special system of deformation limitations for statically determinate structures. In conformity with this train of thought, hyperstatic structures are involved in statements of the previous item as special cases of deformation limitations. Because of their practical importance and peculiarities, they merit to be looked at closer.

Assume a structure with the former moment (angular rotation) conditions

$$-M_{iH} \leq M_i \leq M_{iH}, \quad \text{or} \quad -\varphi_{iH} \leq \varphi_i \leq \varphi_{iH} \quad (i = 1, 2, \ldots, n)$$

in n elastic hinges, while m hinges (with subscripts $j1, j2, \ldots, jm$) are supported, zeroing displacements normal to the beam axis:

$$f_{j1} = 0, \; f_{j2} = 0, \ldots, f_{jm} = 0.$$

leading to a structure with m redundancies.

These deformation limitations are described by an $(n-m)$-dimensional subspace in the space of angular deformations. This linear subspace is obtained by omitting angular rotation vectors with subscripts $j1, j2, \ldots, jm$ corresponding to the specified supports from among characteristic rotation vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ and letting the remaining $(n-m)$ characteristic rotation vectors span the given linear subspace, termed subspace of permissible deformations.

The structure is simultaneously subject to deformation and moment (rotation) limitations. In the space of rotations, the limitations of moments (rotations) were seen to be described by a rectangular solid. Our hyperstatic structure admits rotation systems with vectors inside (or marginal to) the $(n-m)$-dimensional solid cut out by the rectangular hyper-solid from the linear subspace of permissible deformations.

Let us have a clooser look at the form of the $(n-m)$-dimensional solid representing the permissible angular deformation systems. To this aim, to simplify writing, characteristic rotation vectors will be rearranged with changed subscripts of an order that the first $(n-m)$ vectors span the linear subspace of permissible deformations so that an arbitrary vector of this subspace can be written as:

$$\mathbf{t} = f_1\mathbf{s}_1 + f_2\mathbf{s}_2 + \ldots + f_{(n-m)}\mathbf{s}_{(n-m)}$$

where $f_1, f_2, \ldots, f_{(n-m)}$ are arbitrary parameters. Position of the subspace of permissible deformations in the n-dimensional space of rotations is seen to be perfectly independent of the strength characteristics of the structure (i.e. of limit angular rotations), of any but the most general geometry and support conditions of the structure, expressed by vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots \mathbf{s}_{n-m}$. Enforcing angular rotation limit conditions:

$$-\varphi_{1H} \le f_1 S_{1,1} + f_2 S_{2,1} + \ldots + f_{(n-m)} S_{(n-m),1} \le \varphi_{1H}$$
$$-\varphi_{2H} \le f_1 S_{1,2} + f_2 S_{2,2} + \ldots + f_{(n-m)} S_{(n-m),2} \le \varphi_{2H}$$
$$\vdots$$
$$-\varphi_{nH} \le f_1 S_{1,n} + f_2 S_{2,n} + \ldots + f_{(n-m)} S_{(n-m),n} \le \varphi_{nH}.$$

These inequalities cut out of the $(n-m)$-dimensional subspace of permissible deformations a solid confined by $(n-m-1)$-dimensional hyperplane side pairs corresponding to each pair of inequalities.

This solid is seen to have at most n such side pairs, i.e. $2n$ sides, corresponding to the n pairs of inequalities. Not all n side pairs are, however, absolutely existing. Let us consider, e.g. the first $n-m$ pairs of inequalities, defining — because of the linear independence of characteristic rotation vectors — a range of parameters f:

$$-f_{1H} \le f_1 \le f_{1H},$$
$$-f_{2H} \le f_2 \le f_{2H},$$
$$\vdots$$
$$-f_{(n-m)H} \le f_{(n-m)} \le f_{(n-m)H}\,.$$

For all the m remaining angular rotations, the greatest and the least angular rotations φ_j' and $-\varphi_j'$, respectively, producible with parameter values within the given range, can be calculated. For $\varphi_j' \le \varphi_{jH}$ the specified limit of the rotation of the j-th elastic hinge — on the hyperstatic structure of limited deflections — is no real restriction any more (a condition always met because of the redundancy of the structure and of the limited rotations of other hinges). Sides corresponding to these hinges are missing. Thus, the solid cut out of the subspace of permissible deformations by the angular rotation limits has at most n side pairs, but not all of them will take part in the real confinement of the solid of permissible rotations, some of them may be missing. On the other hand, existence of $(n-m)$ side pairs is certain, it being the number of inequalities absolutely needed to define the range of parameters $t_1, t_2, \ldots, t_{n-m}$.

Let us see now how to describe one side pair of the body of permissible angular rotations — say that for the first pair of limiting inequalities — in terms of characteristic angular rotation vectors. Begin with vectors

$$\mathbf{d}_k = \mathbf{s}_k - S_{k,1}/S_{i,1}\mathbf{s}_i$$

where $k = 1, 2, \ldots (i-1), (i+1), \ldots (n-m)$, and s_i an is arbitrary vector with $S_{i,1} \neq 0$. For all the resulting vectors $\mathbf{d}_k$, $D_{k,1} = 0$.

Be $\mathbf{e}_1$ the co-ordinate unit vector corresponding to the angular rotations of the examined elastic hinges. The vector written as

$$\mathbf{t} = u_0\mathbf{e}_1 + (u_1\mathbf{d}_1 + u_2\mathbf{d}_2 + \ldots + u_{i-1}\mathbf{d}_{i-1} + u_{i+1}\mathbf{d}_{i+1} + \ldots + u_{n-m}\mathbf{d}_{n-m})$$

will obviously meet the conditions limiting the rotations of the tested elastic hinge until

$$-\varphi_{1H} \leq u_0 \leq \varphi_{1H} \,.$$

Thus. a side confining the body of permissible angular rotations will be described by the end points of vectors in the form

$$\mathbf{t} = \varphi_{1H}\, \mathbf{e}_1 + (u_1\mathbf{d}_1 + \ldots + u_{i-1}\mathbf{d}_{i-1} + u_{i+1}\mathbf{d}_{i+1} + \ldots + u_{n-m}\mathbf{d}_{n-m})$$

meanwhile parameters $u_1, \ldots, u_{i-1}, \ldots u_{n-m}$ ($n-m-1$ in number) pass through their entire set of values. In the formula of vector t the generalized "direction" of the $(n-m-1)$-dimensional side is defined by terms in brackets or the vectors $\mathbf{d}_k$ ($k = 1, \ldots, i-1, i+1, \ldots, n-m$) therein. The side is always parallel to the $(n-m-1)$-dimensional subspace spanned by vectors $\mathbf{d}_k$, irrespective of the rotation limits φ_{1H}. Considering that vectors $\mathbf{d}_k$ have been constructed from characteristic rotation vectors by using purely the properties of these vectors, and that the set of characteristic rotation vectors depends purely on the general geometry and support conditions of the beam, it can be stated that "directions" of permissible rotation solid sides are independent of any structural data (φ_H or M_H). Rotation limits φ_{iH} of elastic hinges define the size of the permissible rotation solid, the distance of its sides from the origin, and the number of sides effectively confining the solid (and designate the real side among the possible ones if not all sides of any possible direction take part in the confinement).

In knowledge of the permissible rotation solid, definition of the load-capacity polyhedron in the presented manner may follow. Let us first pass from the space of rotations to that of moments by co-ordinate scale changes corresponding to factors $r_1 = M_{1M}/\varphi_{1H}$, $r_2 = M_{2H}/\varphi_{2H}, \ldots, r_n = M_{nH}/\varphi_{nH}$, then assigning characteristic load vectors to moment co-ordinate unit vectors, the body of permissible moments is transformed to the load space to produce the load-capacity polyhedron.

The load capacity polyhedron of the structure under investigation of n elastic hinges with m redundancies is situated in an $(n-m)$-dimensional linear subspace of the n-dimensional load space. (This linear subspace is the transformed of the subspace of permissible deformations.) As a matter of fact, it is no "real" body in the n-dimensional load space. Omitting from the load space all co-ordinate directions corresponding to restrained, i.e. supported beam axis points is equivalent to omit supporting forces and to consider only the subspace of active loads, a $(n-m)$-dimensional linear co-ordinate subspace. Projecting the load capacity polyhedron on this subspace (zeroing supporting force co-ordinates), the projected load-capacity polyhedron in the $(n-m)$-dimensional subspace of active loads confined by $(n-m-1)$-dimensional sides is already a real body. As a matter of fact, this projected load-capacity polyhedron has to be considered the effective load capacity range defining the active load combinations supported by our structure. The original load-capacity polyhedron differs only by assigning the corresponding supporting force values to the permissible combinations of active loads.

The outlined procedure of constructing the active load capacity polyhedron points out how structural characteristics affect the load-capacity. The characteristic rotation vectors and the characteristic load vectors are the most general and most invariant structural characteristics, depending exclusively on the rectilinearity of the structure axis and the uniform spacing of elastic hinges. Supports define what are the characteristic rotation vectors spanning the subspace of permissible deformations. Limit rotations φ_{iH} define the number and the relative position of side pairs confining the load capacity range. Finally, limiting moments M_{iH} define the scale of each co-ordinate axis of the oblique co-ordinate system determined on the basis of characteristic load vectors, setting the scale of the effective dimensions of the load capacity range. All these are felt to offer a means of a visualized survey of the correlation between the structure, the strength characteristics, the load and the load capacity, to assist both analysis and construction. This is proposed to be expounded in a subsequent paper, while now, an illustrative example will help recapitulating the statements above, and understanding application possibilities, as a conclusion.

4. Illustrative example

Let us consider the model structure with three elastic hinges in Fig. 4. Characteristic load vectors are:

$$\mathbf{q}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{q}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

column vectors of the inverse of equilibrium matrix A:

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}.$$

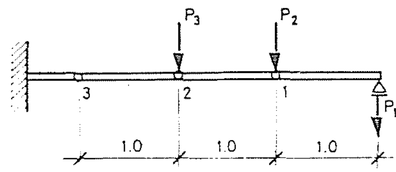

Fig. 4

This matrix defines the load system **p** belonging to the given moment system $\mathbf{m}^* = [M_1, M_2, M_3]$ of the elastic hinges:

$$\mathbf{p} = \mathbf{A}^{-1}\ \mathbf{m}\ ,$$

that is,

$$\mathbf{p} = M_1\, \mathbf{q}_1 + M_2\, \mathbf{q}_2 + M_3\, \mathbf{q}_3\ .$$

Moment bearing limits

$$|M_1| \leq M_{1H}\ ,\ |M_2| \leq M_{2H}\ \text{ and }\ |M_3| \leq M_{3H}$$

of elastic hinges define a rectangular solid in the three-dimensional moment space, transferred by transformation $\mathbf{p} = \mathbf{A}^{-1}\mathbf{m}$ to the — equally three-dimensional — load space to become there an oblique parallelepiped. Characteristic load vectors $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ are presented in the load space by Fig. 5, while Fig. 6 shows the load-capacity parallelepiped corresponding to the case of $M_{1H} = M_{2H} = M_{3H} = 1$ in continuous line. The load-capacity parallelepiped in dashed line results by increasing M_{3H} from 1 to 2.

Deformation analysis of our structure starts from characteristic displacement vectors

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{s}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\ \mathbf{s}_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

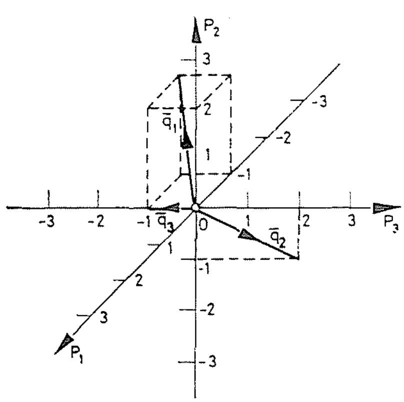

Fig. 5

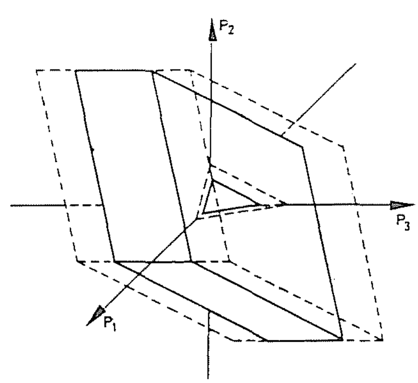

Fig. 6

shown by Fig. 7 in the space of angular rotations $\varphi_1, \varphi_2, \varphi_3$. Typically, the matrix composed of column vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ is the transposed of the inverse of the equilibrium matrix **A**:

$$\mathbf{A}^{-1*} = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] \, .$$

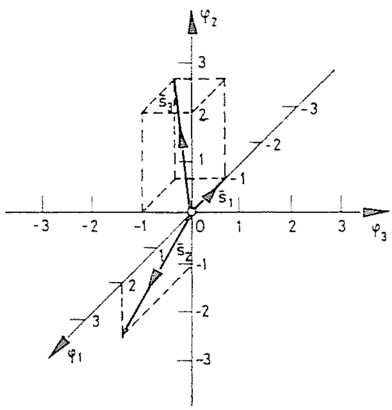

Fig. 7

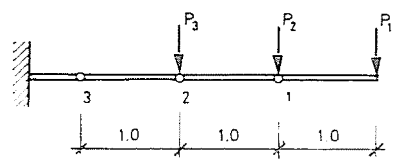

Fig. 8

Let us change now the original. statically determinate model structure to a hyperstatic one by inserting a support to prevent the vertical displacement of the application point of force P_1 (Fig. 8). Thereby, from among the characteristic displacement vectors, $\mathbf{s}_1$ will have zero as coefficient, and vanish from permissible deformation vectors. Hence, angular rotations producible as linear combinations of the "residual" characteristic displacement vectors are possible:

$$\mathbf{t} = f_2 \, \mathbf{s}_2 + f_3 \, \mathbf{s}_3$$

where parameters f_2 and f_3 are vertical displacements of points not prevented from displacement (application points of forces P_2 and P_3).

Restrictions of elastic hinge moments enforce restrictions of relative rotations $\varphi_1, \varphi_2, \varphi_3$:

$$|\varphi_1| \leq \varphi_{1H} \, , \; \varphi_2 \leq |\varphi_{2H}| \, , \; \varphi_3 \leq |\varphi_{3H}| \, .$$

These restrictions define a rectangular solid in the space of angular rotations. A cube corresponding to the case $\varphi_{1H} = \varphi_{2H} = \varphi_{3H} = 1$ is seen in dashed line, and a prism corresponding to values $\varphi_{1H} = \varphi_{2H} = 1$, $\varphi_{3H} = 0.75$ in continuous line in Fig. 9, showing at the same time the plane passing through the origin and defined by characteristic displacement vectors $\mathbf{s}_2$ and $\mathbf{s}_3$. For the actual hyperstatic structure, this plane is the subspace of permissible deformations. Its position in the space of angular rotations is defined exclusively by the basic layout of the structure, the position of elastic hinges. Strength characteristics, in particular the limit rotations φ_{1H}, φ_{2H}, φ_{3H} are irrelevant to the position of the plane of permissible deformations. At the same time it is perceived that for different limit rotations, the rectangular solid of restrictions cuts out ever different configurations from the plane of permissible deformations. (In fact, the plane of permissible deformations has been represented by means of these configurations.) ($\varphi_{1H} = \varphi_{2H} = \varphi_{3H} = 1$ produces a rhomboid, while reducing φ_{3H} to 0.75 results in a hexagon with parallel opposite edge pairs.)

Now, the load space will be considered to see what loads are supported by our structure. All strength data of the structure are needed. Be then $\varphi_{1H} = \varphi_{2H} = 1$ and $\varphi_3 = 0.75$ (continuous line in Fig. 9) and $M_{1H} = M_{2H} = 1$ and $M_{3H} = 2$ (dashed line in Fig. 6). Since limit moments and limit rotations describe one and the same condition of the structure (from two different aspects), the rectangular solid in continuous line of Fig. 9 corresponds to the oblique parallelepiped in dashed line of Fig. 6. Tracing in this latter the hexagon cut out of the plane of permissible deformations to the scale of Fig. 9 results in the load-capacity polyhedron, now a plane figure (Fig. 10). By other words, assigning unit vectors φ_1, φ_2, and φ_3 from the space of angular rotations to vectors $(M_{1H}/\varphi_{1H})\mathbf{q}_1 = \mathbf{q}_1$, $(M_{2H}/\varphi_{2H})\mathbf{q}_2 = \mathbf{q}_2$, and $(M_{3H}/\varphi_{3H})\mathbf{q}_3 = 2.6\mathbf{q}_3$, respectively, means to pass to the space of loads P_1, P_2, P_3 by linear transformation.

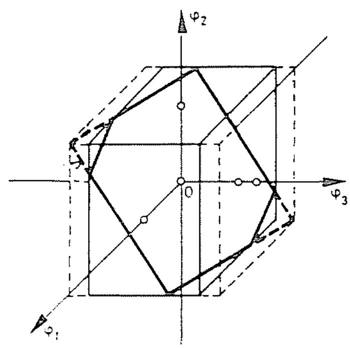

Fig. 9

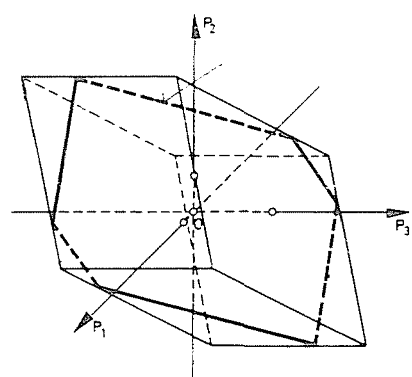

Fig. 10

The plane figure in Fig. 10 as load-capacity range expresses the requirement that only one, strictly defined load P_1 can belong to any values P_2 and P_3, obviously since the force P_1 is not an active load but a supporting force. If, rather than to have a closer look at the supporting forces, one is interested in the load capacity of our structure for effective (active) loads, it is the best to project the effective load capacity range onto the subspace of active loads, in our case in the plane P_2P_3. The hexagonal load capacity range for active loads is seen in dashed line in Fig. 11, where the load capacity range is seen, for the sake of comparison, also for the case where the elastic hinge 1 is replaced by a real hinge (allowing illimited angular rotation, bearing no moment, corresponding to a cantilever containing two elastic hinges).

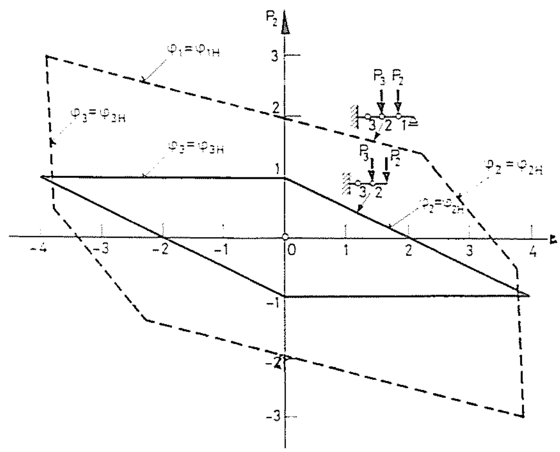

Fig. 11

The previous statements lead to certain conclusions on how certain strength characteristics (for the tested model structure, the limit angular rotations φ_{1H}, φ_{2H}, φ_{3H} and limit moments M_{1H}, M_{2H}, M_{3H}) affect the load capacity of the structure. Statically determinate structures are primarily affected by limit moments M_{1H}, M_{2H}, M_{3H}. "Direction" of the sides of the load capacity range is defined, change of some limit moment shifts the corresponding side parallel to itself (see Fig. 6). These considerations involve the well-known fact that cross sectional dimensions of statically determinate structures are easy to design by simply projecting the range of design loads on the normal of each side, and equalizing the limit moments to these projections, the limit moments, i.e. the cross sections can be designed.

The load capacity of hyperstatic structures is necessarily affected by both limit angular rotations φ_{1H}, φ_{2H}, φ_{3H} and limit moments M_{1H}, M_{2H}, M_{3H}. Statements on the load capacity range make it obvious that simultaneously changing the ultimate angular rotation and the ultimate moment of an elastic hinge, keeping their ratio ($r_i = M_{iH}/\varphi_{iH}$) constant, shifts the corresponding confining side of the load capacity range parallel to itself, keeping the other sides inaffected, just as in the statically determinate case. Thus, also hyperstatic structures are accessible to a similarly simple design method as the statically determinate ones, provided limit rotations and moment bearings of the cross sections will be changed simultaneously and proportionately.

The situation is different if the limit moment and the limit rotation are changed disproportionally. Let us consider the case where the limit rotation is kept constant and only the limit moment varies. (Any other case can be reduced to this case and to the proportionate variation discussed in the previous paragraph.) The presented train of thought is of illustrative help in following the modification of the load capacity range. Variation of the load capacity range upon halving the limit moment of elastic hinge 1 has been examined in Fig. 12. Variation of an ultimate moment means in fact to apply a different multiplier to the characteristic load vector belonging to that elastic hinge. Thus, all vectors connecting the corners of the original (dashed) and the modified (continuous) load capacity ranges in Fig. 12 are parallel and correspond to the projection of characteristic load vector $\mathbf{q}_1$ on the plane P_2P_3. The size of vectors connecting the corners depends on the rotation value caused by the load corresponding to the given corner in the elastic hinge 1. Since $\varphi_1 = \varphi_{1H}$ all along the side corresponding to the restriction in elastic hinge 1, this side is shifted parallel to itself during a change from $M_{1H} = 1$ to $M_{1H} = 0.5$. The other sides of the load capacity range vary by both direction and distance from the origin, but typically, points of intersection between related "old" and "new" sides are on a straight line passing through the origin and parallel to the confining side corresponding to the elastic hinge 1. As a comparison, Fig. 12 shows in dotted line the case where φ_{1H} decreases in proportion to the reduction of M_{1H}. Now, only the confining side corresponding to elastic hinge 1 is shifted towards the origin, somewhat more than if φ_{1H} were inaffected.

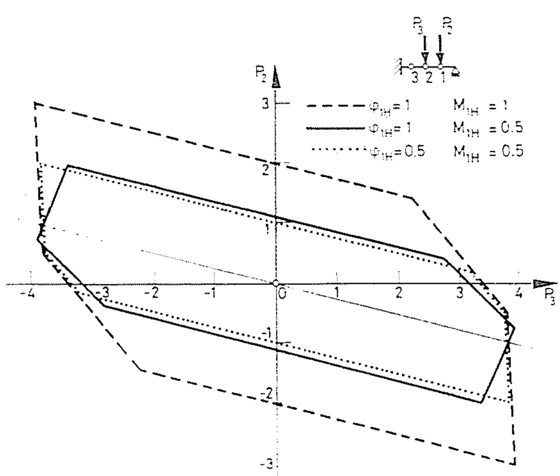

Fig. 12

Thus, dependence of the shape of the load capacity range on various strength parameters is felt to be comprehensible, followable, and likely of help for the practical engineering construction work. A more general, detailed exposure would be outside the scope of this paper, only intended to raise certain thoughts.

Summary

Some problems of frameworks under multiparametric loading conditions are investigated, based on a discrete-type model structure consisting of perfectly stiff bars and elastic hinges, taking use of the concepts of n-dimensional *Euclidean* spaces. Some basic properties of hyper-parallelepipeds and hyper-polyhedra describing the load capacities of statically determined and redundant structures are established. In an illustrative problem some ideas are presented on how to use the obtained results in the study of the influence of the cross sectional properties on the load capacity of the whole structure and in the practical engineering constructional work.

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