LOAD CAPACITY OF ELASTO-PLASTIC BARS WITH NO TENSILE STRENGTH*

by

T. Kőrössi

Department of Strength of Materials and Structures, Technical University, Budapest

Received: February 8, 1978
Presented by Dr. József Peredy

According to Hungarian design standards [2, 3], masonry and concrete structures can be considered as made of ideally elasto-plastic material of no tensile strength. The design value of eccentricity including random eccentricity and that due to the load are specified as a function of slenderness. A practically applicable method will be presented for determining load-induced displacements better approximating the ultimate real load capacity than the standard one.

1. Initial assumptions

The examined bar is made of a material of limited compressibility, ideally elasto-plastic and with no tensile strength. The deformation coefficient involving the creep is:

$$E_f = \beta_t \cdot \sigma_H \ ,$$

the compression due to ultimate stress:

$$\varepsilon_r = \frac{\sigma_H}{E_f} = \frac{1}{\beta_t},$$

the ultimate value of strain at failure being ε_H. β_t, σ_H, and ε_H values for different building materials are specified in design codes.

The bar is of rectangular cross section, both ends hinged (Fig. 1). Before buckling, the force is of constant eccentricity. Deflection occurs in the bending

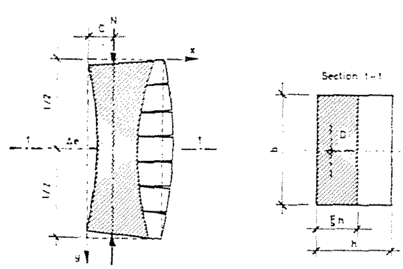

Fig. 1

* Abridged text of the Doctor Techn. Thesis by the Author.

plane, and during deformation, cross sections remain plane. Maximum displacement is determined by assuming the deflected bar to be sinusoidal and using the stress diagram of the mid-section.

The deflected bar is of the form:

$$y = \Delta e \sin \frac{\pi}{l} x .$$

with a curvature:

$$\frac{1}{\varrho} = \frac{y''}{(1 + y'^2)^{3/2}} .$$

With the usual approximation $(1 + y'^2)^{3/2} \approx 1$, the mid-bar curvature is:

$$\frac{1}{\varrho} = \Delta e \frac{\pi^2}{l^2} .$$

The central cross section may be in elastic or in plastic stress state depending on the bar slenderness.

Either the entire cross section or only a part of it is active, depending on the position of the load. Thus the possible stress diagrams are:

a) *Elastic deflection, the entire cross section is active* (Fig. 2),
b) *Elastic deflection, part of the cross section is active* (Fig. 3),
c) *Plastic deflection, the entire cross section is active* (Fig. 4),
d) *Plastic deflection, part of the cross section is active* (Fig. 5).

Expressing the curvature in terms of the angular strain at mid-section:

$$\frac{1}{\varrho} = \frac{\varepsilon}{\xi h} .$$

Introducing the notation $\alpha = \varepsilon_r/\varepsilon$

$$\frac{1}{\varrho} = \frac{\varepsilon_r}{\alpha \xi h}$$

yields for the deflection at mid-section:

$$\Delta e = \frac{\varepsilon_r \cdot l^2}{\pi^2 \alpha \xi h} . \tag{1}$$

2. The force-displacement relationship

The force-displacement relationship is obtained by taking Eq. (1) and the equilibrium conditions of external and internal forces into consideration.

a) *Elastic buckling, the entire cross section is active* (Fig. 2)

From the force equilibrium

$$\frac{N}{bh\,\sigma_H} = \frac{2\xi - 1}{2\alpha\xi}\,. \tag{2a}$$

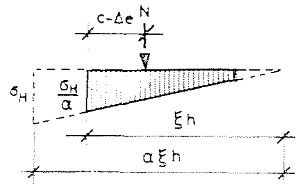

Fig. 2

The stress resultant and force N are on the same vertical:

$$c - \Delta e = \frac{h}{3}\,\frac{3\xi - 2}{2\xi - 1}\,. \tag{3a}$$

Utilizing (1) and arranging:

$$\left[6\frac{c}{h} - 3\right]\xi^2 - \left[3\frac{c}{h} - 2 + \frac{6\varepsilon_r}{\pi^2\alpha}\left(\frac{l}{h}\right)^2\right]\xi + \frac{3\varepsilon_r}{\pi^2\alpha}\left(\frac{l}{h}\right)^2 = 0\,. \tag{4a}$$

b) *Elastic buckling, part of the cross section is active* (Fig. 3)

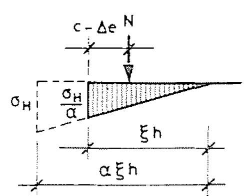

Fig. 3

Similarly as before:

$$\frac{N}{bh\sigma_H} = \frac{\xi}{2\alpha}, \tag{2b}$$

$$c - \Delta e = \frac{\xi h}{3}, \tag{3b}$$

$$\xi^2 - 3\frac{c}{h}\xi + \frac{3\varepsilon_r}{\pi^2\alpha}\left(\frac{l}{h}\right)^2 = 0\,. \tag{4b}$$

c) *Plastic buckling, the entire cross section is active* (Fig. 4)

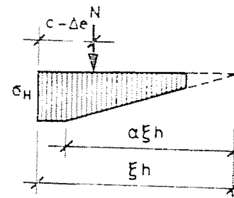

Fig. 4

From the force equilibrium:

$$\frac{N}{bh\sigma_H} = \frac{-(1-\alpha)^2 \cdot \xi^2 + 2\xi - 1}{2\alpha\xi} . \tag{2c}$$

The resultant of internal forces and the external force have a common influence line:

$$c - \Delta e = \frac{h}{3} \cdot \frac{-(1-\alpha)^3 \xi^3 + 3\xi - 2}{-(1-\alpha)^2 \xi^2 + 2\xi - 1} . \tag{3c}$$

Substituting the Δe value and arranging:

$$(1-\alpha)^3 \xi^4 - 3\frac{c}{h}(1-\alpha)^2 \xi^3 + \left[6\frac{c}{h} - 3 + \frac{3\varepsilon_r}{\pi^2\alpha}\left(\frac{l}{h}\right)^2 (1-\alpha)^2\right]\xi^2 -$$

$$-\left[3\frac{c}{h} - 2 + \frac{6\varepsilon_r}{\pi^2\alpha}\left(\frac{l}{h}\right)^2\right]\xi + \frac{3\varepsilon_r}{\pi^2\alpha}\left(\frac{l}{h}\right)^2 = 0 . \tag{4c}$$

d) *Plastic buckling, part of the cross section is active* (Fig. 5)

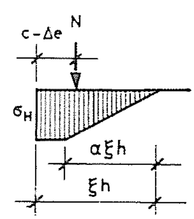

Fig. 5

From equilibrium equations:

$$\frac{N}{bh\sigma_H} = \xi\left(1 - \frac{\alpha}{2}\right) , \tag{2d}$$

$$c - \Delta e = h \frac{\frac{\alpha^2}{3} - \alpha + 1}{2 - \alpha} \xi . \tag{3d}$$

Substituting and arranging:

$$\frac{\frac{\alpha^2}{3} - \alpha + 1}{3 - \alpha}\, \xi^2 - \frac{c}{h}\, \xi + \frac{\varepsilon_r}{\pi^2 \alpha} \left(\frac{l}{h}\right)^2 = 0\,. \tag{4d}$$

Eqs (4) yield ξ values belonging to different α values, then (2) and (1) yield specific N and Δe values. Calculations have been made for prisms of different slendernesses, relative eccentricities and materials by means of a digital computer. Some force-displacement diagrams will be presented.

Figure 6 shows force-displacement diagrams of brick masonries of a slenderness $l/h = 30$ for different initial eccentricities; Fig. 7 refers to brick

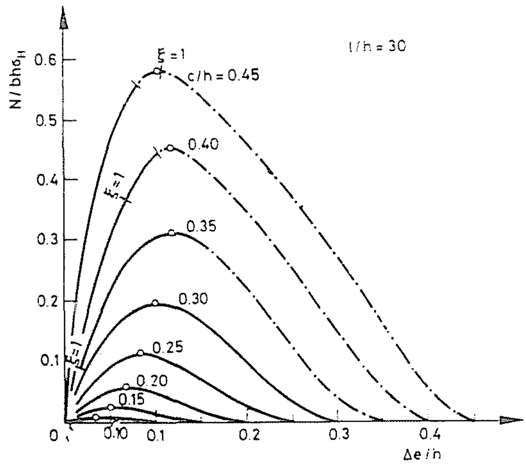

Fig. 6

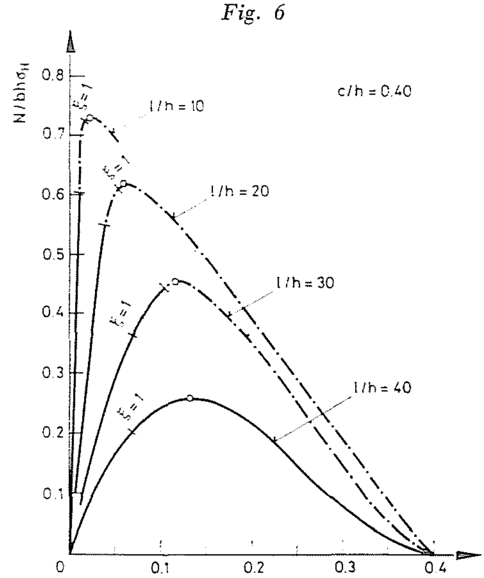

Fig. 7

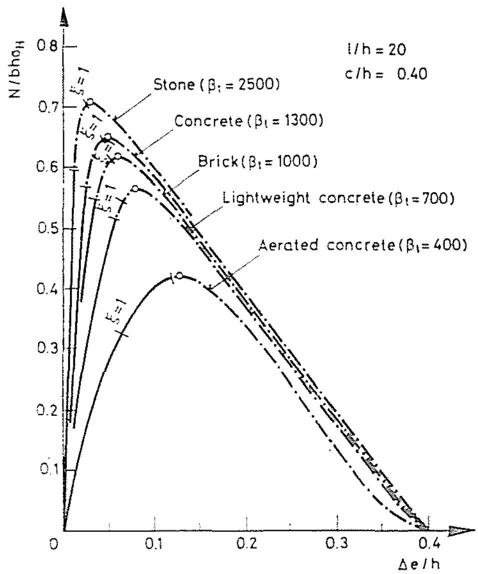

Fig. 8

masonries of different slendernesses, with intersection points at identical distances c/h; Fig. 8 represents diagrams for columns of identical slenderness and eccentricity, made of different materials.

Deformation characteristics have been assumed according to the standard [3]. Although only the rising branches of the diagrams are of importance, curves have been plotted to the α_H values where compression in the extreme fibre is at its ultimate value ε_H. The diagram for cross sections in the elastic stress state has been plotted in continuous line, while dash-and-dot lines refer to the plastic range. In case of $\xi = 1.0$ the neutral axis contacts the cross section. The circles refer to diagram peaks.

Load capacity of the prism is at the curve maximum as a rule. Specific N values for α_H may be on the rising branch of the curve for prisms of very low slenderness, to be considered as the load capacity, since the extreme fibre compression cannot exceed the ultimate value.

Displacement of the elastically buckled central cross section is so great that the neutral axis belonging to the maximum intersects the cross section even for the least initial eccentricity. In case of plastic buckling — hence when the maximum is in the plastic range — either the entire cross section or only a part of it is active, depending on the initial eccentricity value.

3. Force-displacement function maxima

In case of elastic buckling, combined use of Eqs (1), (2b) and (3b) yields the function $N - \Delta e$, with a maximum at $\Delta e = c/3$ of a value:

$$N_{\max} = \frac{2b\sigma_H c^3 \pi^2}{3\varepsilon_r l^2} .$$

This value being independent of the dimension h of the cross section, it is expediently related to the product of the cross section area centric about the intersection point by the ultimate stress:

$$\nu = \frac{N_{\max}}{2bc\sigma_H} = \frac{\pi^2}{12\varepsilon_r \left(\frac{l}{2c}\right)^2} . \tag{5}$$

The relationship is valid until $\alpha \geq 1$, that is:

$$\frac{l}{2c} \geq \sqrt{\frac{\pi^2}{6\varepsilon_r}} .$$

In plastic buckling — when the neutral axis is in the cross section — function $N - \Delta e$ is obtained by combining Eqs (1), (2d) and (3d). The specific value of $N_{\max}$ is given by:

$$\nu(1-\nu)^3 = \frac{9}{4}\left[\frac{\varepsilon_r}{\pi^2}\left(\frac{l}{2c}\right)^2\right]^2 , \tag{6}$$

value and place of the maximum are related as:

$$\nu = 1 - 3\,\frac{\Delta e}{2c} .$$

Maxima of the different force-displacement diagrams — provided the neutral axis is in the cross section — are on straight lines in Fig. 9.

In case of small slenderness and initial eccentricity, the entire cross section may be active (Fig. 4). Now, the force-displacement function maximum is advisably obtained by approximation.

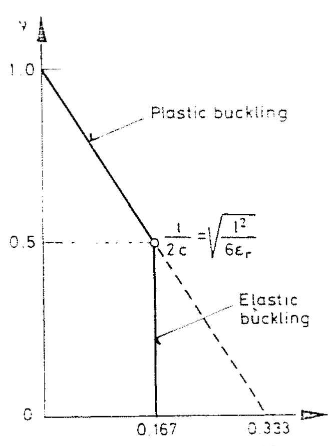

Fig. 9

4. Ultimate load bearing

In addition to initial eccentricity $e_0 = M/N$ and stress induced eccentricity increments, the determination of the ultimate compression of prisms has to take the standard deviation Δe_0 [2, 3] into account:

$$\Delta e_0 = 0.03h + 0.1 \left(\frac{l}{10h}\right)^2 h. \tag{7}$$

Before deformation, the distance between the section edge and the application point is given by:

$$c = \frac{h}{2} - (e_0 + \Delta e_0) .$$

Ultimate load of the prism without tensile strength:

$$N_H = \nu \cdot F_{ny} \cdot \sigma_H , \tag{8}$$

where $F_{ny} = 2bc$. ν values in the formula are given by (5) or (6). Fig. 10 shows diagrams ν for masonries of different materials. Load capacity for slendernesses below that for α_H have been determined by taking the ultimate strain value into consideration, these cases are, however, of little practical

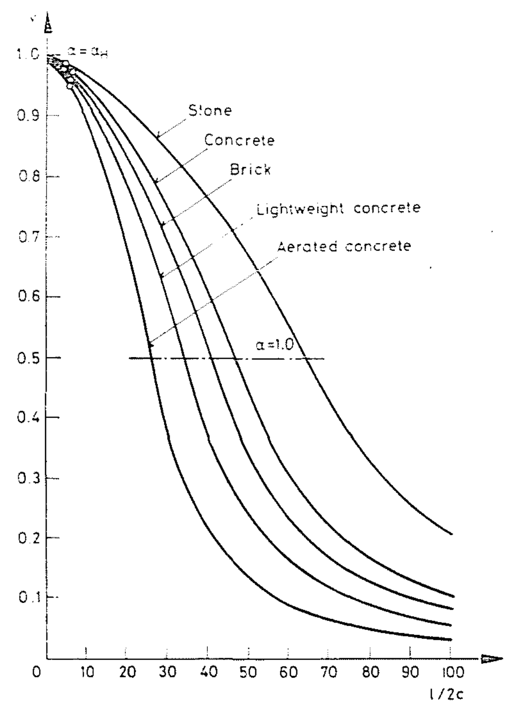

Fig. 10

importance. In the little frequent case where the entire cross section is active, the ultimate load given by Eq. (8) is an approximation. Determining maxima of force-displacement functions on a computer showed them to exceed those from Eq. (8), an error on the safe side. The errors are for stone masonries 1%, for brick masonries 1.7% and for aerated concrete masonries below 3%.

Let us compare the load capacity given by Eq. (8) — taking the eccentricity standard deviation according to Eq. (7) into consideration — to test results obtained at the Structural Clay Products Institute [4]. Ultimate forces of prisms with initial eccentricities $e_0 = h/6$ and $h/3$ referred to $bh\sigma_t$ are shown in Fig. 11 vs. the slenderness ratio l/h. σ_t means the ultimate compressive strength of the brick masonry, determined at SCPI by testing to failure short prisms of the same material. Test results for eccentricities $e_0=h/6$ and $h/3$ have been affected by marks + and ×, respectively. Load capacities calculated with a deformation coefficient $E_f = 1000\ \sigma_H$, taking also creep effect into consideration, and with a coefficient of elasticity $E_{f0} = 2500\ \sigma_H$ have been plotted in a continuous, and a dashed line, respectively, in Fig. 11.

The dashed-line load capacity diagrams ignoring the creep effect are seen in Fig. 11 to fairly approximate test results for prisms of low or medium slenderness. The usual excess of test results over calculated values is due to the fact that the presented method ignores the tensile strength of brick masonry, and that in laboratory tests the random eccentricity increment is less than that obtained from standard Eq. (7). The effective load capacity of very slender columns is the multiple of calculated values, namely at a low ultimate load capacity, the effect of tensile strength — left out of consideration — is of importance.

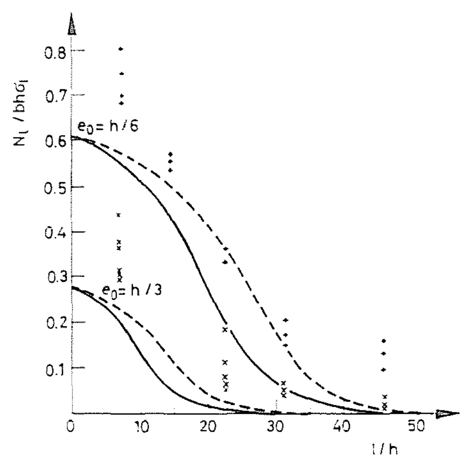

Fig. 11

Legend

b	cross section width;
c	distance between the influence line of the compressive load and the compressive face of the member at the support;
e_0	initial eccentricity;
Δe_0	random increment of eccentricity;
Δe	eccentricity increment due to stresses;
E_f	deformation coefficient taking creep effect into consideration;
E_{f0}	modulus of elasticity;
$F_{ny} = 2bc$	part of the cross section centric about the influence line;
h	cross sectional dimension in the bending plane;
l	prism buckling length;
Nl	external force axial to the bar;
N_H	ultimate force;
N_t	force at failure;
$\alpha = \varepsilon_r/\varepsilon$	coefficient of the cross section rotation ability;
$\alpha_H = \varepsilon_r/\varepsilon_H$	ultimate value of rotation ability;
$\beta_t = E_f/\sigma_H$	ratio of the deformation coefficient to the ultimate compression under permanent load;
ε	compressive strain in the extreme fibre;
ε_H	ultimate strain at failure;
$\varepsilon_r = \sigma_H/E_f$	ultimate value of the elastic strain;
ξ	relative distance of the neutral axis;
ν	ratio of ultimate load to the force, product of the surface part under central compression by the ultimate stress;
$1/\varrho$	prism axis curvature;
σ_H	ultimate compression;
σ_t	ultimate strength.

Summary

A method has been suggested for determining the compressive load capacity of prisms hinged both ends, of constant eccentricity, of rectangular cross section, made of an ideally elasto-plastic material, taking the eccentricity increment due to stresses into consideration. The deformed prism has been considered as of sinusoidal shape, the eccentricity increment due to stresses has been determined from the stress diagram of the central cross section.

References

1. Bölcskei-Dulácska: Manual of Structural Engineers.* Bp. Műszaki Könyvkiadó, 1974.
2. Hungarian Standard MSz 15022/3—72. Design of Load Bearing Structures. Concrete Structures.*
3. Hungarian Standard MSz 15022/3—72. Design of Load Bearing Structures. Masonries.*
4. Structural Clay Products Institute: Recommended Practice for Engineered Brick Masonry McLean, Va. Nov. 1969. Tables 5—21.
5. Szalai, K.: Load Capacity of Reinforced Concrete Beams in Compression.* Mélyép. Tud. Szle. 6 (1966) p. 241.
6. Yokel, F. Y.: Stability and Load Capacity of Members with no Tensile Strength. Journal of the Structural Division ASCE. Vol. 97. July 1971. pp. 1913—1926.

Dr. Tibor Kőrössi, H-1521 Budapest

*In Hungarian.