CONDITIONS OF EQUILIBRIUM IN A SIX-FORCE SYSTEM

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Let us consider a rigid body subject to a general spatial load. The body is supported by a six-force system along given influence lines. Some geometrical situations of supporting forces are known not to permit equilibrium.

Any modern book on statics mentions such more or less special situations as examples, but as a general method, numerical analysis using available equations of equilibrium is recommended ([1], [2], [3], [4], [5], [7]).

The present paper proves only five special situations of the six influence lines to exist, where there is no possibility of equilibrium. These situations can be exactly defined. The followings are *a priori* valid for a six-force system, because then there is zero load on the rigid body.

Necessary and sufficient conditions of the possibility of equilibrium in a general six-force system:

1. there are not two of the influence lines such as to be collinear;
2. there are not three of them, if lying in one plane, to intersect in one point;
3. no four of them intersect in one point or lie in the same plane;
4. no five of them intersecting a straight do intersect another straight or run parallel to a plane;
5. no six of them intersect a straight or run parallel to a plane.

N.B. The parallel straights are regarded as intersecting at a point in infinity.

As a proof, let us consider a general force system with a force $\mathbf{F}_0$ and a moment $\mathbf{M}_0$. Let us have six influence lines with unit vectors $\mathbf{f}_i$ and position vectors $\mathbf{r}_i$.

Now the moment vector of a unit vector is

$$\mathbf{m}_i = \mathbf{r}_i \times \mathbf{f}_i . \quad (i = 1, 2, \ldots, 6)$$

as shown in Fig. 1.

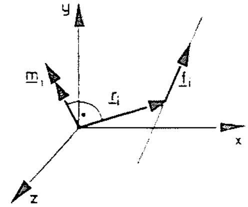

Fig. 1

Six forces act along the influence lines

$$\mathbf{F}_i = F_i \cdot \mathbf{f}_i .$$

where the values of scalars F_i are unknown. If the whole system $\mathbf{F}_0$, $\mathbf{M}_0$, $\mathbf{F}_i$ is in equilibrium, then:

$$\mathbf{F}_i + \mathbf{F}_0 = 0 ,$$

$$\mathbf{M}_i + \mathbf{M}_0 = 0 .$$

These six linear equations contain six unknowns: F_1, F_2, ... F_6. Or in a detailed form, we have

$$\begin{bmatrix} f_{1x} & f_{2x} & \dots & f_{6x} \\ f_{1y} & f_{2y} & & f_{6y} \\ f_{1z} & f_{2z} & & f_{6z} \\ m_{1x} & m_{2x} & & m_{6x} \\ m_{1y} & m_{2y} & & m_{6y} \\ m_{1z} & m_{2z} & & m_{6z} \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} + \begin{bmatrix} F_{0x} \\ F_{0y} \\ F_{0z} \\ M_{0x} \\ M_{0y} \\ M_{0z} \end{bmatrix} = 0 .$$

This system of linear equations has a solution only if it is not a singular one, i.e. the columns (or the rows) of the coefficient matrix are linearly dependent.

Let us investigate the five possible cases of linear combinations of the six columns. Every case corresponds to a special (undesirable) geometrical situation of the influence lines.

Case 1: one force equals the product of another force by a scalar:

$$\mathbf{f}_1 = a \cdot \mathbf{f}_2 ,$$

$$\mathbf{m}_1 = a \cdot \mathbf{m}_2 .$$

Thus, the two influence lines, i.e. the *two forces are collinear.*

Case 2: one force is a linear combination of two other forces:

$$\mathbf{f}_1 = a \cdot \mathbf{f}_2 + b \cdot \mathbf{f}_4 ,$$

$$\mathbf{m}_1 = a \cdot \mathbf{m}_2 + b \cdot \mathbf{m}_4 .$$

Two forces and their resultant are known *to lie in one plane and to intersect each other in one point.* (The parallels are intersecting in one point as well.)

Case 3: one force is a linear combination of three other forces:

$$\mathbf{f}_1 = a \cdot \mathbf{f}_2 + b \cdot \mathbf{f}_3 + c \cdot \mathbf{f}_4 ,$$

$$\mathbf{m}_1 = a \cdot \mathbf{m}_2 + b \cdot \mathbf{m}_3 + c \cdot \mathbf{m}_4 .$$

Resultant of any two forces is a force (not a wrench) because these forces and the third one have a force resultant equivalent to the fourth one. It means any two forces are crossing each other. Hence all the four *are lying* *in the same plane* or all the four *are crossing at the same point* (may be at a point in infinity).

Case 4: One force is a linear combination of four other forces

$$\mathbf{f}_1 = a \cdot \mathbf{f}_2 + b \cdot \mathbf{f}_3 + c \cdot \mathbf{f}_4 + d \cdot \mathbf{f}_5 ,$$
$$\mathbf{m}_1 = a \cdot \mathbf{m}_2 + b \cdot \mathbf{m}_3 + c \cdot \mathbf{m}_4 + d \cdot \mathbf{m}_5 .$$

Now, let us consider the principle of zero virtual work. The force system acts on an absolutely rigid body. The rigid body could make six independent motions: three axial displacements and three rotations. If during one of the motions the work of the balancing force system is zero then no equilibrium is possible since some work is done by the active forces during any of the six motions. If a force system does some work during any motion then it hinders that motion; if it does zero work then it allows it. One among the forces allows five motions.

Let us consider these five forces. If one of the five forces is a linear combination of the four other ones, then the five forces hinder only four motion components, thus they allow two free motions.

(a) Two axial displacements are allowed by the five forces if they are parallel to two planes, that is, all of them run parallel to the line of intersection of these planes (see *case 2!*);

(b) two rotations are allowed by the five forces if *all of them intersect two different straights;*

(c) one axial displacement and one rotation are allowed by the five forces if *all of them intersect one straight and run parallel to one plane.*

Case 5: one force is a linear combination of five others:

$$\mathbf{f}_1 = a \cdot \mathbf{f}_2 + b \cdot \mathbf{f}_3 + c \cdot \mathbf{f}_4 + d \cdot \mathbf{f}_5 + e \cdot \mathbf{f}_6 ,$$
$$\mathbf{m}_1 = a \cdot \mathbf{m}_2 + b \cdot \mathbf{m}_3 + c \cdot \mathbf{m}_4 + d \cdot \mathbf{m}_5 + e \cdot \mathbf{m}_6 .$$

Now, only five motion components are hindered by the six forces and one motion is allowed:

(a) one rotation is allowed by the six forces if *all of them intersect one straight;*

(b) one axial displacement is allowed by the six forces if *all of them run parallel to one plane*. The above theorem has been expounded in detail in [6].

Summary

The paper demonstrates the necessary and sufficient conditions of the possibility of equilibrium in a six-force system:

(1) there are not two of the influence lines being collinear;

(2) there are not three of them, if lying in one plane, to intersect in one point;

(3) no four of them intersect in one point (or lie in the same plane);

(4) no five of them intersecting a straight do intersect another straight (or run parallel to a plane);

(5) no six of them intersect a straight (or run parallel to a plane).

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*In Hungarian.