THE ANALYTIC PERSPECTIVE

by

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In analytic perspective the image of the object is produced by computation according to the laws of the perspective applying the method of analytic geometry. The characteristic geometrical data of object are given in a spatial co-ordinate system, and these characteristics are projected from a suitably chosen origin to a given image plane.

Let us see first the computation for one point. Be point $P(x, y, z)$ given in co-ordinate system $C(x, y, z)$, with the origin point $C^\circ(0, 0, -d)$, and be the image plane the co-ordinate plane (x, y) in which the image points of the object are oriented to co-ordinate system $0(x', y')$. Axes x' and y' are identical with x and y.

According to Fig. 1 the transformation formulae are:

$$x' = \frac{d \cdot x}{d + z} \qquad y' = \frac{d \cdot y}{d + z} .$$

In the following, perspective images of simple geometrical configurations will be analytically determined.

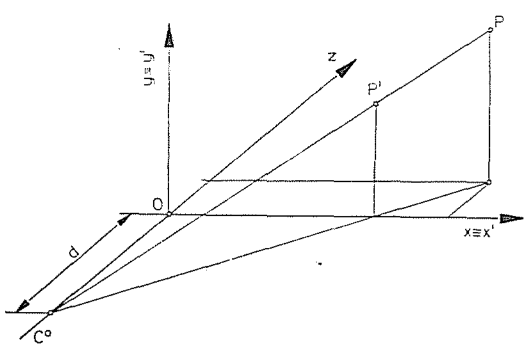

Fig. 1

1. The perspective of a cube

a) *The cube of the simplest position, first plotted in frontal perspective*

Co-ordinates of the corners of a cube placed as in Fig. 2 are:

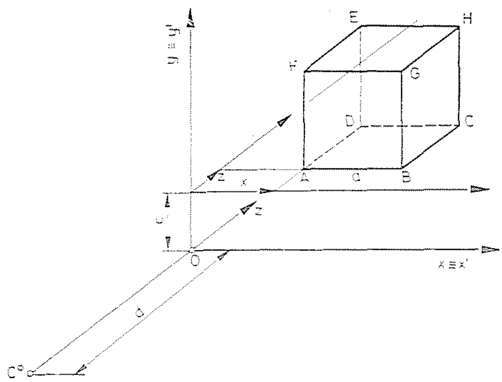

Fig. 2

$A(x, c, z)$ $\qquad$ $E(x, c + a, z + a)$
$B(x + a, c, z)$ $\qquad$ $F(x, c + a, z)$
$C(x + a, c, z + a)$ $\qquad$ $G(x + a, c + a, z)$
$D(x, c, z + a)$ $\qquad$ $H(x + a, c + a, z + a)$

Be

$d = 100$ mm
$c = 60$ mm
$a = 80$ mm
$x = 50$ mm
$z = 40$ mm .

In this case the co-ordinates of the cube corners are:

$A(50, 60, 40)$
$B(130, 60, 40)$
$C(130, 60, 120)$
$D(50, 60, 120)$
$E(59, 140, 120)$
$F(50, 140, 40)$
$G(130, 140, 40)$
$H(130, 140, 120)$.

Co-ordinates of the image points determined by the transformation formulae are:

$A'(35{,}7;\ 42{,}9)$	$E'(22{,}7;\ 63{,}6)$
$B'(92{,}9;\ 42{,}9)$	$F'(35{,}7;\ 100)$
$C'(59;\ 27{,}3)$	$G'\ (92{,}9;\ 100)$
$D'(22{,}7;\ 27{,}3)$	$H'(59;\ 63{,}6)$.

Plotting the image points in co-ordinate system (x', y') and connecting the corresponding ones, the perspective image of the cube becomes:

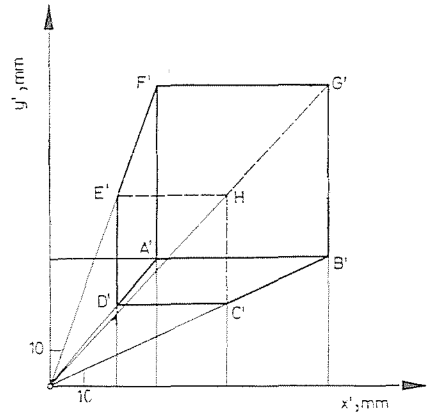

Fig. 3

b) *Representing a cube with two aiming points*

In the plane of equation $y = c$ be, with respect to base $ABCD$ (Fig. 4):

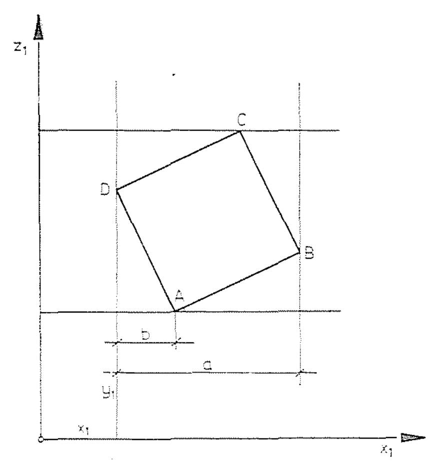

Fig. 4

$$
\begin{aligned}
a &= 100 \text{ mm} \\
b &= 20 \text{ mm} \\
x_1 &= 60 \text{ mm} \\
z_1 &= 50 \text{ mm} \\
d &= 100 \text{ mm} \\
c &= 80 \text{ mm} \, .
\end{aligned}
$$

Co-ordinates of the cube corners are:

$A(80, 80, 50)$	$E(60, 180, 130)$
$B(160, 80, 70)$	$F(80, 180, 50)$
$C(140, 80, 150)$	$G(160, 180, 70)$
$D(60, 80, 130)$	$H(140, 180, 150)$.

Finally, the co-ordinates of the image points:

$A'(53{,}3;\ 53{,}3)$	$E'(26;\ 78{,}3)$
$B'(94{,}1;\ 47)$	$F'(53{,}3;\ 120)$
$C'(56;\ 32)$	$G\ (94{,}1;\ 105{,}9)$
$D'(26;\ 34{,}8)$	$H\ (56;\ 72)$.

Note. The base $ABCD$ is oriented to co-ordinate system (x_1, z_1) in its plane.)

The image of the cube with the computed image points is shown in Fig. 5.

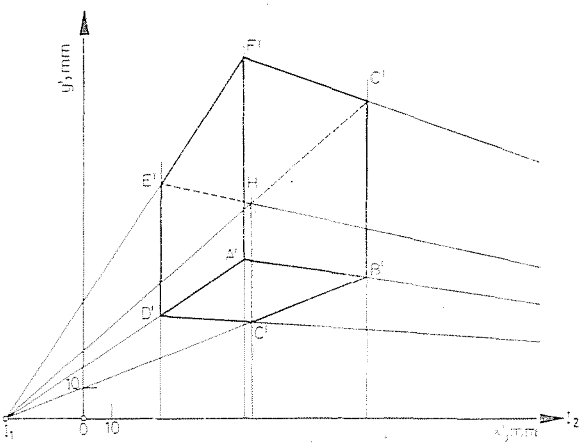

Fig. 5

2. Image of a dodecahedron

Fig. 6 shows the plan view of a dodecahedron in the plane of equation $y = -160$. Co-ordinates of the twenty corners of the dodecahedron with the given plane view, partly read off the graph, partly computed, are:

1(76; —160; 34)	7(79; —132; 17)	13(102,5; —116; 30)
2(96,5; —160; 57)	8(114; —132; 55,5)	14(108; —116; 82)
3(81; —160; 85)	9(88; —132; 100)	15(61; —116; 104)
4(50; —160; 78)	10(37; —132; 90)	16(64,5; —88; 85)
5(46,4; —160; 47)	11(26; —116; 66)	17(43; —88; 63)
6(31,5; —132; 38)	12(52; —116; 20)	18(59; —88; 36)
		19(90; —88; 42)
		20(93; —88; 74)

and $c = 160$ mm; $d = 100$ mm.

The computed image points are:

1′(56,7; —119)	8′(73,3; —84,9)	15′(30; —57)
2′(62; —112)	9′(44; —66)	16′(34,6; —46,6)
3′(43,8; —86,4)	10′(19,4; —69,4)	17′(26,5; —54)
4′(28; —89,7)	11′(15,6; —69)	18′(43,3; —64,5)
5′(31,6; —108,8)	12′(43,3; —96,6)	19′(63,3; —62)
6′(22,8; —95,6)	13′(78,8; —89)	20′(53,4; —50)
7′(67,4; —112,8)	14′(59,3; —63,6)	

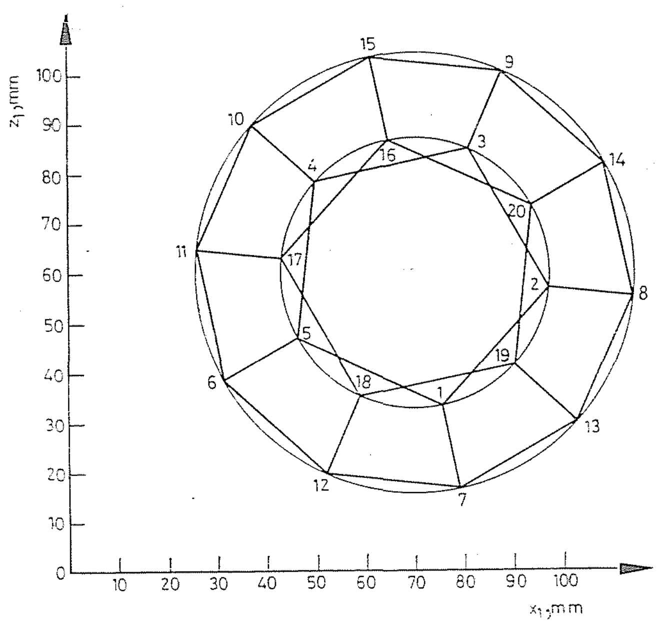

Fig. 6

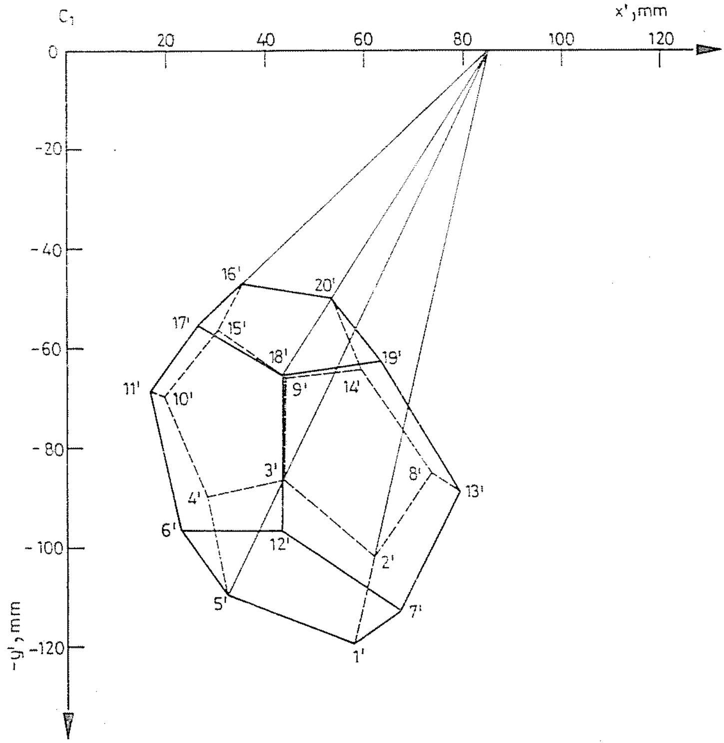

Fig. 7

Perspective image of the dodecahedron based on the known image points is shown in Fig. 7.

3. Perspective of a part of a staircase

The object is given with its two ordered projections (Fig. 8) in the reference system $0(x, y, z)$. Readings of the corners are:

1(100; —30; 100)	12(100; 35; 0)	23(230; 30; 75)
2(110; —10; 0)	13(110; 35; 0)	24(230; 50; 75)
3(110; —10; 25)	14(100; 90; 75)	25(230; 50; 100)
4(110; 10; 25)	15(110; 90; 75)	26(240; —30; 0)
5(110; 10; 50)	16(100; 90; 100)	27(230; 35; 0)
6(110; 30; 50)	17(240; —30; 100)	28(240; 35; 0)
7(110; 30; 75)	18(230; —10; 0)	29(230; 90; 75)
8(110; 50; 75)	19(230; —10; 25)	30(240; 90; 75)
9(110; 50; 100)	20(230; 10; 25)	31(230; 90; 100)
10(110; 90; 100)	21(230; 10; 50)	32(240; 90; 100)
11(100; —30; 0)	22(230; 30; 50)	

and $d = 100$ mm.

Co-ordinates of the image points given by transformation formulae are:

1'(50; —15)	12'(100; 35)	23'(131,4; 17)
2'(110; —10)	13'(110; 35)	24'(131,4; 28,5)
3'(88; —8)	14'(57; 51,4)	25'(115; 25)
4'(88; 8)	15'(63; 51,4)	26'(240; —30)
5'(73; 6,6)	16'(50; 45)	27'(230; 35)
6'(73; 20)	17'(120; —15)	28'(240; 35)
7'(63; 17)	18'(230; —10)	29'(131,4; 51,4)
8'(63; 28,5)	19'(184; —8)	30'(137; 51,4)
9'(55; 25)	20'(184; 8)	31'(115; 45)
10'(55; 45)	21'(153,5; 6,6)	32'(120; 45)
11'(100; —30)	22'(153,5; 20)	

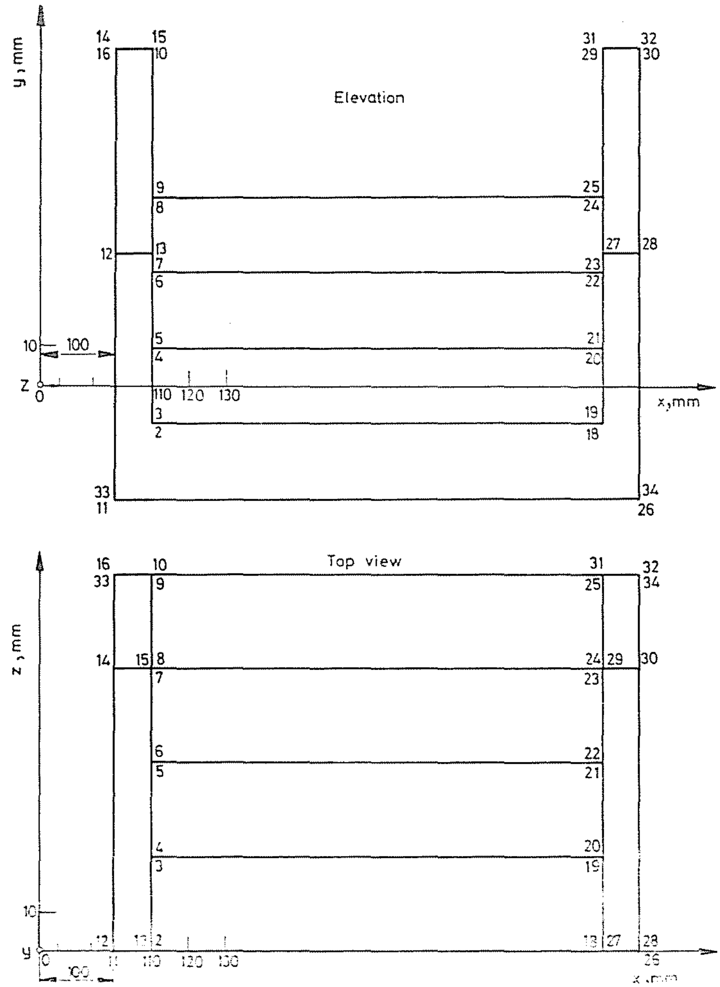

Fig. 8

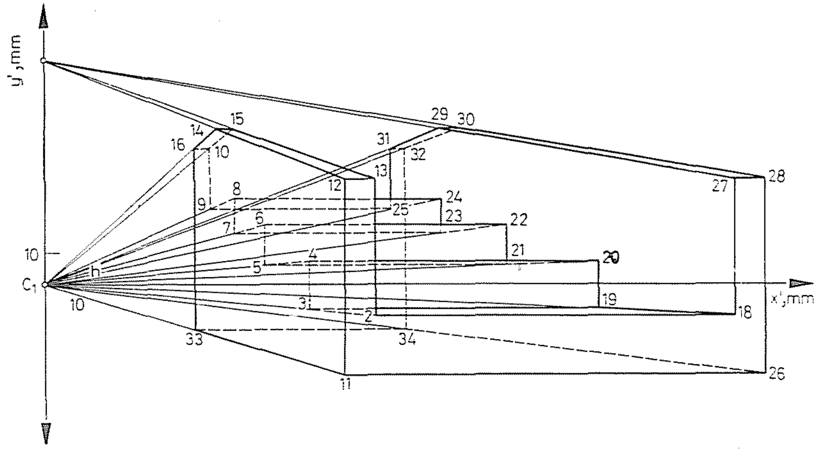

Fig. 9

Fig. 9 is the perspective picture of the staircase plotted from the image points.

4. Representing a cylinder

Let us consider a cylinder with a base circle in the plane $y = -100$ of the co-ordinate system $0(x, y, z)$, and be co-ordinates of its centre: $M(100; -100; 50)$; cylinder radius $r = 40$ mm and height $m = 180$ mm.

In Fig. 10 the drawing paper plane of the equation $y = -100$ contains the base circle of the cylinder. In this plane the base circle is oriented to co-ordinate system (x_1, y_1). The base face intersects the co-ordinate planes (xz) and (yz), at x_1 and at z_1, respectively.

It being a case of central projection, there exists a central collinear relation between the base circle and its image. The image of the circle is a conic section, and in the present case, it will prove to be an ellipse. The images of the points of the straight line q in the base circle plane are in the infinity. Therefore the opposed axis q is parallel to axis x_1 and at a distance d from x_1. The base circle face is not intersecting straight line q, therefore its image is an ellipse. Drawing tangents from point R of the opposed axis q to the base circle, contact points A and B will be central projections of the end points of one diameter. Thus the image of pole K of straight line q is the centre point of the image of the circle. (It is to be noted that accordingly, the image of the centre of the circle will not be the centre of the image.) The conjugate of diameter $A'B'$ becomes the image of the circle diameter DE. Therefore it is enough to compute only the image data of points A, B, D and K for determining the base

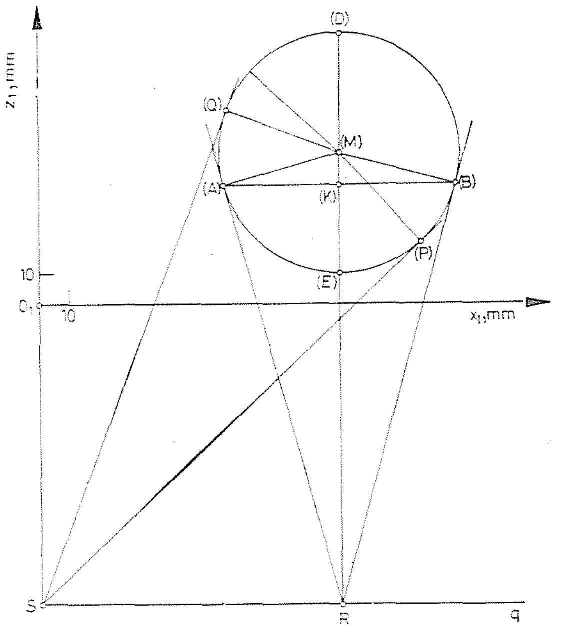

Fig. 10

circle image of the cylinder. (Considering that diameter $D'E'$ is the conjugate of diameter $A'B'$, only co-ordinates of points $A'D'$ and K' have to be computed.) For the sake of accuracy images B' and E' will be determined as well. The straight line parallel to the generatrices of the cylinder intersecting the projection centre C intersects the base circular face at S. Thus the images of contact points P and Q of circle tangents belonging to points S are contour points of the base circle image.

To determine the image of the cylinder top circle face, the points fitting generatrices passing through points A, B, E, P, Q are needed; these are: A^*, B^*, D^*, E^*, P^* and Q^*.

Their co-ordinates determining the image of the cylinder are:

$A(62; -100; 39)$	$A^*(62; 80; 39)$
$B(138; -100; 39)$	$B^*(138; 80; 39)$
$D(100; -100; 90)$	$D^*(100; 80; 90)$
E(100; —100 ;10)	$E^*(100; 80; 10)$
$K(100; -100; 39)$	$K^*(100; 80; 39)$
$P(127; -100; 20)$	$P^*(127; 80; 20)$
$Q(63; -100; 64)$	$Q^*(63; 80; 64)$.

Finally, co-ordinates of the image points are:

$A'(44{,}6;\ -72)$	$A^{*\prime}(44{,}6;\ 57{,}5)$
$B'(99{,}2;\ -72)$	$B^{*\prime}(99{,}2;\ 57{,}5)$
$D'(52{,}6;\ -52{,}6)$	$D^{*\prime}(52{,}6;\ 42)$
$E'(90{,}9;\ -90{,}9)$	$E^{*\prime}(90{,}9;\ 72{,}7)$
$K'(72;\ -72)$	$K^{*\prime}(72;\ 57{,}5)$
$P'(105{,}8;\ -83{,}3)$	$P^{*\prime}(105{,}8;\ 66{,}6)$
$Q'(38{,}5;\ -61)$	$Q^{*\prime}(38{,}5;\ 48{,}5)$.

Now, the perspective image of the cylinder can be drawn (Fig. 11).

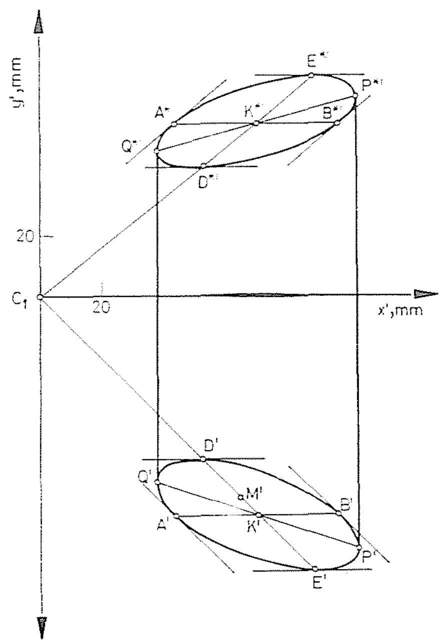

Fig. 11

5. Central image of a cone of revolution

Be the radius of the base circle of the cone of revolution $r = 30$ mm, its height $m = 60$ mm. The base circle of the cone fits the horizontal plane of equation $y = 20$ and the cone apex is over the base plane. Accordingly, the base plane intersects the co-ordinate plane (xy) along straight line t. In

Fig. 12 the drawing paper plane is co-ordinate plane (xy), into which the base face has been rotated around straight line t. The optionally given base circle face is drawn rotated, with centre point N chosen at (110; 20; 40).

The base circle image has been determined just as for the cylinder. Thus the base plane image has for centre point the image of point K, whereas conjugated diameter pair in the image corresponds to chord AB and circle diameter DE.

In the following, the contour generatrices will be constructed. The contact generatrices of tangent planes passing the projection centre C° have to be determined. Straight line $C^\circ M$ intersects the base plane at T and circle tangents drawn from this point are already straight lines fitting the tangent planes. Let $C^\circ 1$ be the projection of the projection centre C° on the base plane and (C°_1) its rotated image.

Straight line $(C^\circ M)$ rotated in its vertical projection plane is drawn to $|C_1{}^\circ N|$. $(C^\circ M)$ cuts out point T. Points P and Q are points of the contour generatrices.

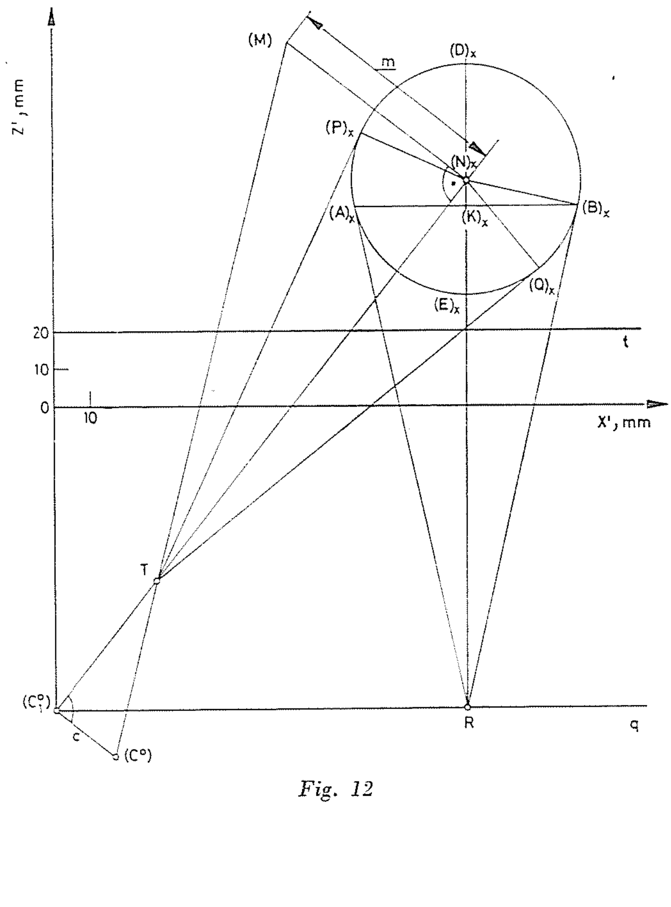

Fig. 12

Thus, for determining the cone image, the following points have to be transformed:

$A(80{,}5;\ 20;\ 33)$	their image points are:	A' $(60{,}5;\ 15)$
$B(139;\ 20;\ 33)$		B' $(104{,}5;\ 15)$
$D(110;\ 20;\ 70)$		D' $(64{,}7;\ 11{,}7)$
$E(110;\ 20;\ 10)$		E' $(100;\ 18)$
$K(110;\ 20;\ 33)$		K' $(82;\ 15)$
$P(83;\ 20;\ 52)$		P' $(54{,}6;\ 13{,}4)$
$Q(129;\ 20;\ 16)$		Q' $(111;\ 17{,}2)$
$M(110;\ 80;\ 40)$		$M'(71{,}4;\ 57)$

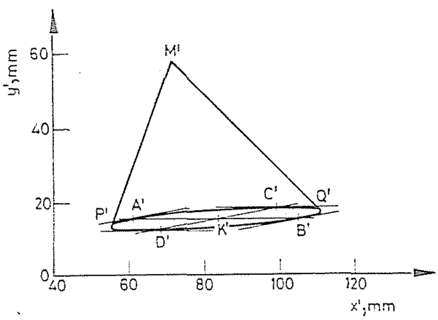

Fig. 13

Finally, the perspective image of the cone can be constructed on the basis of the image points, in co-ordinate system $(x'y')$ (Fig. 13).

Evaluation

A plane-faced configuration to be imaged is given by its geometrical properties (e.g. in case 1*a* and 1*b* it is sufficient to supply one corner point of the cube).

In another group of plane-faced configurations, the base can be constructed and so can be the height of the apices outside the base from the base view; their co-ordinates can be computed.

Thus the co-ordinates of the configuration apices can be read and determined from the base (Example 2). The reading errors were generally found to be smaller than the construction errors.

The co-ordinates of plane face configuration apices given by ordinate projections are determined from readings. The number of computed apex images may be reduced if also the obtained aiming points are used in determining the image (Example 3). For instance, in case of the staircase part it is sufficient to compute data of 16 — rather than 32 — image points.

For curve-faced configurations bounded by circles, the centre of the circle image and its conjugate diameter pair are constructed by central collineation. The image data of the obtained points are computed. Eventual contour generatrices of the configuration are simple to determine even by construction, not interfering with accuracy. Examples 3 and 5 are illustrative of combined use of construction and computation.

Making use of the configuration geometry generally results in great many simplifications.

Rather than the results shown in the examples (though they are not to be depreciated), the main importance of the analytic perspective is the possibility of computerized determination of the perspective images.

Because of space shortage, no detailed description of the computer program will be given but only the chief phases of the programming.

Use of a computer in preparing the drawings means a progress in both speed and accuracy compared to construction. Drawing can be made by a plotter controlled by a computer-programmed punch-tape. All that has to be done is to punch on tape the geometrical characteristics of the object to be represented.

Input data are:

1 Number of characteristic points;
2 Co-ordinates of characteristic points;
3 A set of instructions on the movement of the pen;
4 Projection centre.

Data are advisably given in mm, whereas the plotter produces the plane co-ordinates with an accuracy of 0.01 mm.

Thereafter the program is punchad on the control tape of the drawing machine. We have a plotter type Digigraf 100s of this University at our disposal. Its control tape is built of so-called blocks separated by signals. Each block contains coherent information.

Drawing consists in connecting the points of programmed co-ordinates by a straight line or a curve. The curve may be a circular arc or another general curve, e.g. a cone section. The principle of drawing a general curve is as follows:

A fitting curve determined by a third-degree polynomial is constructed for a set of points assumed at close intervals and connected by straight sections.

Programming* is rather lengthy, however it is of practical importance

* The program was made and run by G. E. Kovács, second-year student in architecture at the Technical University, Budapest.

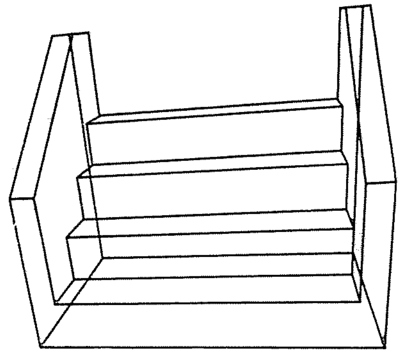

Fig. 14

by speedily drawing the once programmed object, e.g., the images of a building from different angles.

Fig. 14 shows the raw perspective image of the stair in Example 3, drawn by the programmed drawing machine.

Summary

Analytic perspective determines the central image of the object by computation, permitting to apply computer programming and peripheral plotter for drawing the central image. Its practical importance is to yield perspective pictures from different angles of the once programmed object.

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