AXIALLY SYMMETRIC SHELLS OF REVOLUTION, AN INVERSE PROBLEM*

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Received January 22, 1975

Presented by Prof. Dr. Gy. DEÁK

Introduction

Traditionally, shell design and analysis proceeds from an arbitrary, assumed shape to obtain internal stresses and displacements by means of the field equations, for given loading and boundary conditions. In many practical problems this procedure might lead to undesirable results in terms of stress resultants or displacements. For this reason it may be preferable to look for an appropriate shape, starting from given loading and boundary conditions and a specified stress-resultant field, as governed by the properties of the material to be used in the structure, or various particular support and load effects as dictated by some general requirements for the shape itself. Naturally, the number of arbitrary specifications for a proposed structure is not arbitrary. Further research is needed in this aspect of the problem, which, however, does not belong to the scope of this paper.

The so-called "inverse" problem and the question of optimum shell shapes was discussed during the last two decades by a number of authors [1, 2, 3, 4, 5, 6] and was recently extended to special shapes or loads using, as a rule, a numerical solution technique [1, 4, 7]. The aim of the present investigations was to deal with the problem from a more general viewpoint.

In this paper a general method for finding "optimum" shapes for membrane shells of revolution with complete axial symmetry is described. The usual membrane equilibrium equations are rewritten by means of geometric and trigonometric transformations in a form suitable for the solution of such inverse problems. A shape function is developed in terms of the force functions and a condition is established, which governs the relationship between the force functions, i.e. the stress resultant distribution and the external load functions. The use of the shape function is demonstrated on two examples, for which the force-functions are known:

1. The spherical dome subjected to snow load,
2. The paraboloidal shell under dead load.

* The results presented here were obtained during research work supported by research grants for P. G. Glockner, M. Sc., Ph. D., Prof. of Civil Eng., The Univ. of Calgary, Calgary, Canada. Lecture presented at the IASS Symposium 1973 in *Kielce*, Poland.

Definitions

1. *Geometry*

The geometry of thin shells of revolution with axial symmetry is taken as in Fig. 1a, and used to establish the membrane equilibrium equations [1].

Meridians and circles of latitude are, as usual, used as lines of curvature co-ordinates and designated by φ and θ, respectively. R_1 and r are the corresponding radii of curvature.

The shape of the shell will be represented by the meridian as a plane curve (Fig. 2a). The character of the shape in dimensionless form is described by the shape function $\varrho(\zeta)$ for

$$r = R_0\varrho(\zeta)$$

where R_0 is constant.

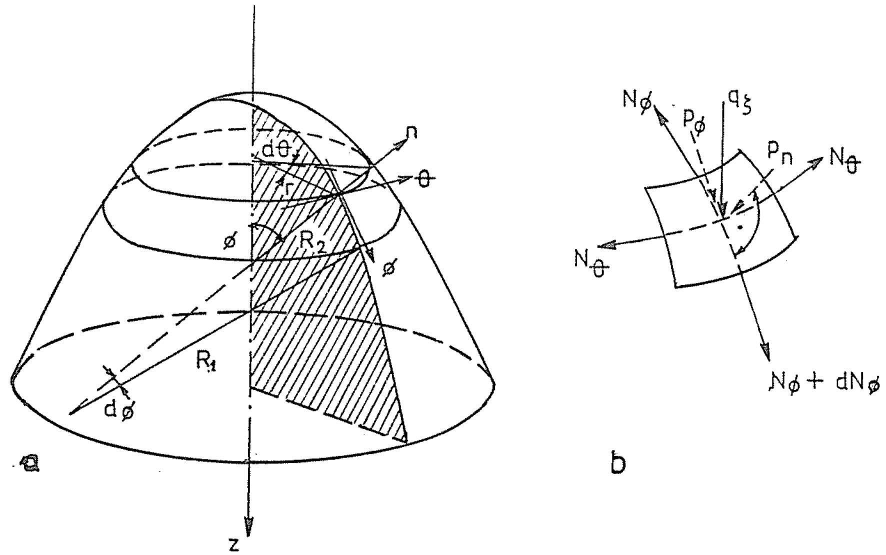

Fig. 1

2. *External and internal forces*

The force functions are also defined as functions of ζ (Fig. 2b) denoted by

$$N_\varphi = C_\varphi n_\varphi(\zeta)$$
$$N_\theta = C_\theta n_\theta(\zeta)$$
$$q_\zeta = A_q q_s(\zeta)$$

where N_φ and N_θ are the stress-resultants, q_ζ is the external load per unit surface area acting in the vertical direction (Fig. 1b), n_φ, n_θ and q_s are dimensionless functions, C_φ, C_θ and A_q are constants defined from some given or arbitrary conditions.

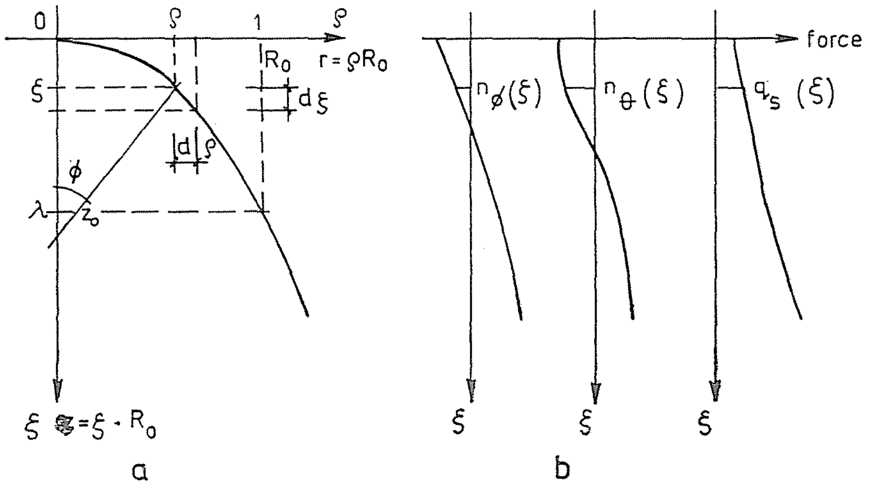

Fig. 2

The shape function

The derivations in this paper are based on the well-known equilibrium equations for membrane shells of revolution with complete axial symmetry:

$$\frac{1}{R_1}\frac{d}{rd\varphi}(rN_\varphi) - \frac{\cot\varphi}{R_2}N_\theta + p_\varphi = 0 \tag{1a}$$

$$\frac{N_\varphi}{R_1} + \frac{N_\theta}{R_2} - p_n = 0 \tag{1b}$$

rewritten, by means of relationships:

$$\frac{d}{d\varphi}(\quad) = R_1 \sin\varphi \frac{d}{dz}(\quad) = R_1 \sin\varphi (\quad)'$$

$$\frac{dr}{dz} = r' = \cot\varphi$$

$$R_2 = \frac{r}{\sin\varphi}$$

$$p_\varphi = q_\zeta \sin\varphi = q_\zeta \frac{1}{\sqrt{1 + r'^2}}$$

$$p_n = -q_\zeta \cos\varphi = -q_\zeta \frac{r'}{\sqrt{1 + r'^2}}$$

in the form

$$\frac{r'}{r} = \frac{N'_\varphi + q_\zeta}{N_\theta - N_\varphi} \tag{1c}$$

$$N_\varphi \frac{R_2}{R_1} = -(q_\zeta r' r + N_\theta) . \tag{1d}$$

Substitutions:

$$r = R_0 \varrho ,$$

$$r' = \frac{d\varrho}{dz/R_0} = \frac{d\varrho}{d\zeta} = \varrho' ,$$

$$r'' = \frac{1}{R_0} \varrho'' ,$$

$$\frac{A_q R_0}{C_\varphi} = a , \quad \text{and} \quad \frac{C_\theta}{C_\varphi} = b$$

yield the equilibrium equations in dimensionless form:

$$\frac{\varrho'}{\varrho} = \frac{n'_\varphi + aq_s}{bn_\theta - n_\varphi} \tag{2a}$$

$$-\frac{R_2}{R_1} = \frac{\varrho\varrho''}{1 + \varrho'^2} = \frac{aq_s}{n_\varphi} \varrho\varrho' + \frac{bn_\theta}{n_\varphi} . \tag{2b}$$

The first of the two differential equations can be solved for the shape:

$$\varrho = C_1 \exp \int F \, d\zeta \tag{3a}$$

or

$$\varrho = \int \varrho F \, d\zeta + C_2 \tag{3b}$$

where C_1 and C_2 are constants of integration,

$$F = \frac{n'_\varphi + aq_s}{bn_\theta - n_\varphi} . \tag{4}$$

From the shape function (Eqs 3a, b) follows that:

$$\varrho' = \varrho F \tag{5a}$$

$$\varrho'' = \varrho' F + F' \varrho = \varrho(F^2 + F') . \tag{5b}$$

By means of expressions (5), Eq. (2b) can be transformed into a fourth-degree algebraic equation in terms of (ϱF):

$$(\varrho F)^4 - \left[\frac{n_\varphi}{aq_s}\frac{F'}{F} - \frac{n'_\varphi}{aq_s} - 2\right](\varrho F)^2 + \frac{bn_\theta}{aq_s}F = 0\,. \tag{6}$$

Solving for $(\varrho F)^2$ and using (5a) and (3a):

$$[C_1 F \exp \int F d\zeta]^2 = \left[\frac{n_\varphi}{aq_s}\left(\ln\frac{F}{n_\varphi}\right)' - 1\right] \pm$$

$$\pm \sqrt{\left[\frac{n_\varphi}{aq_s}\left(\ln\sqrt{\frac{F}{n_\varphi}}\right)' - 1\right]^2 - \frac{bn_\theta}{aq_s}F}\,. \tag{7}$$

This means that if one defines a shape for a shell in the form of the function (3a), the external and internal force-distribution along the height of the shell should satisfy the requirements in Eq. (7).

Summarizing the former statements and derivations: Eqs (3) and (7) represent shells of revolution with complete axial symmetry, subjected to vertical, distributed load on the surface of the shell, in the state of membrane equilibrium.

This representation of the problem seems to be convenient to analyse the effects of shape, load and stress resultants on each other in several ways, under several conditions. Eq. (7), however, needs further investigations to find a stress function, if possible fit to express all the force functions in terms of the other two in a closed form.

Next, two examples are going to be shown to illustrate the use of the shape function, i.e. the solution of the first equilibrium equation (2a). Both examples are well-known problems from the literature [1, 8]. The known solutions were rewritten as functions of the co-ordinate ζ and were taken as requirements to which the shape was sought.

Examples

Example 1. The spherical dome loaded by snow

The case of a spherical dome subjected to snow-load q_v uniformly distributed over the plane [1], in the Cartesian co-ordinate system shown in Fig. 2a is as follows:

Geometry:

$$r = \sqrt{R_0^2 - (z - R_0)^2}\,.$$

Let

$$r = R_0 \varrho(\zeta)$$

thus

$$\varrho = \sqrt{1 - (\zeta - 1)^2} = \sqrt{2\zeta - \zeta^2} \tag{8}$$

further

$$\sin \varphi = \sqrt{2\zeta - \zeta^2}$$
$$\cos \varphi = 1 - \zeta \ .$$

Load distribution over the shell surface along the height:

$$q_\zeta = q_v \cos \varphi = q_v (1 - \zeta) . \tag{9}$$

The known solutions for the stress resultants:

$$N_\varphi = C_\varphi = \text{const.} \tag{10}$$
$$N_\theta = N_\varphi (2 \cos^2\varphi - 1)$$

or

$$N_\theta = C_\theta (2\zeta^2 - 4\zeta + 1) \tag{11}$$
$$N_\theta(\zeta = 0) = C_\theta = C_\varphi \ .$$

Substituting into (4) for

$$n_\varphi = 1$$
$$n_\theta = 2\zeta^2 - 4\zeta + 1$$
$$q_s = 1 - \zeta$$

with constants

$$a = \frac{q_v R_0}{C_\varphi}$$
$$b = 1$$

one gets

$$F = \frac{a(1-\zeta)}{2\zeta^2 - 4\zeta} = - \frac{a}{2} \frac{1-\zeta}{2\zeta - \zeta^2} \tag{12a}$$

or

$$F = - \frac{a}{2} (\ln\sqrt{2\zeta - \zeta^2})' . \tag{12b}$$

From equilibrium equation (2a):

$$(\ln \varrho)' = F$$

so

$$\ln \varrho = - \frac{a}{2} \ln C_1 \sqrt{2\zeta - \zeta^2} \ . \tag{13}$$

From Eq. (13):

$$\varrho = (C_1 \sqrt{2\zeta - \zeta^2})^{-a/2} \ . \tag{14}$$

Applying the second equilibrium equation in the form of static equilibrium condition at the boundary

$$\frac{q_v R_0^2 \varrho^2}{2C_\varphi R_0 \varrho} = -n_\varphi \frac{1}{\sqrt{1 + \varrho'^2}} \tag{15}$$

along with the geometrical condition arisen in Eq. (13)

$$\varrho_{(\zeta=1)} = 1 \tag{16}$$

it follows:

$$C_1 = 1 \tag{17a}$$

and (since $\varrho'_{(\zeta=1)} = 0$)

$$- \frac{q_v R_0}{2C_\varphi} = - \frac{a}{2} = 1 . \tag{17b}$$

Substituting these constants into Eq. (14) the sought shape function becomes completely defined in the form:

$$\varrho = \sqrt{2\zeta - \zeta^2}$$

analogous to Eq. (8).

Beside the shape, or rather for it, there is a necessary condition established in Eq. (17b) prescribing the relationship among three quantities, relationship bounded only by the material properties of the shell.

Example 2. The paraboloidal shell under dead load

The case of the paraboloidal shell subject to dead load alone [8] rewritten in the Cartesian system (Fig. 2a), as it was done with the former example, is the following:

Geometry:

$$r = \sqrt{C_0 z}$$

Let

$$r = C_0 \varrho(\zeta)$$

thus

$$\varrho = \sqrt{z/C_0} = \sqrt{\zeta} \tag{18}$$

further

$$\sin\varphi = \sqrt{\frac{4\zeta}{4\zeta + 1}}$$

$$\cos\varphi = \frac{1}{\sqrt{4\zeta + 1}} .$$

If uniform thickness is assumed, the load is constant:

$$q_\zeta = A_q\, q = q_d = \text{const.} \tag{19}$$

The known solution for the stress resultants:

$$N_\varphi = -\frac{q_d C_0}{6}\left(\frac{1}{\cos^2\varphi} - \cos\varphi\right)\frac{1}{\sin^2\varphi} \tag{20a}$$

$$N_\theta = -\frac{q_d C_0}{6}\left(2 - \frac{\cos^2\varphi}{1 + \cos\varphi}\right) \tag{20b}$$

or, as a distribution along the rise of the shell:

$$N_\varphi = C_\varphi \frac{\sqrt{4\zeta + 1}}{4\zeta}\left[\sqrt{(4\zeta + 1)^3} - 1\right] = C_\varphi\, n_\varphi(\zeta) \tag{21a}$$

$$N_\theta = C_\theta \frac{2\sqrt{4\zeta + 1}\left(4\zeta - \frac{1}{2}\right) + 1}{4\zeta\sqrt{4\zeta + 1}} = C_\theta\, n_\theta(\zeta) \tag{21b}$$

where

$$N_{\varphi(\zeta=0)} = N_{\theta(\zeta=0)} = C_\varphi = C_\theta = -\frac{q_d C_0}{6} .$$

Therefore the constants:

$$a = \frac{q_d C_0}{C_\varphi} = 6 ; \quad b = 1.$$

To build up the function F according to Eq. (4) it can be written:

$$b n_\theta - n_\varphi = \frac{1}{4\zeta\sqrt{1 + 4\zeta}}\left[4\zeta(1 - 4\zeta\sqrt{1 + 4\zeta}) + 2(1 - \sqrt{1 + 4\zeta})\right]$$

$$a q_s + n'_\varphi = \frac{2}{(4\zeta)^2\sqrt{1 + 4\zeta}}\left[4\zeta(1 - 4\zeta\sqrt{1 + 4\zeta}) + 2(1 - \sqrt{1 + 4\zeta})\right]$$

thus

$$F = \frac{1}{2\zeta} = (\ln \sqrt{\zeta})'. \tag{22}$$

Substituted into Eq. (3a) it gives for the shape of the meridian:

$$\varrho = \sqrt{\zeta}.$$

Conclusions

The formulation of the shell problem presented here seems to suit analysis of membrane shells of revolution with complete axial symmetry from different view-points. A flexible way of handling such problems is indicated with the purpose to discover the relationships among factors (geometry, load, internal forces) which in a certain togetherness result in such shell structures. The connection of these components was visualized in the first equilibrium equation (2a). The examples given were to show solutions for "inverse" problems, i.e. for the shape of a shell to meet load and internal force requirements. Further investigation is needed in this subject, more applications, to possibly clear the character of the related functions, their mutual effects and interdependency.

Summary

In the usual shell analysis problem, the designer is given or assumes a certain form and dimensions for the proposed structure and then proceeds to calculate internal stresses, deformations and deflections for a prescribed set of loading and boundary conditions.

This paper approaches the inverse problem, in which, given a particular loading condition, support conditions and the overall span and rise of a proposed structure, an internal stress-resultant field throughout the shell is assumed "a priori" and the corresponding shape is sought.

In the investigation, membrane stress-state and complete axial symmetry were assumed. The differential equations for membrane equilibrium are rewritten in a suitable form to analyse the interaction of stress-resultant field, loading conditions and shape.

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