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RESEARCH ARTICLE

Spherical segment approximation of quadripartite vaults

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Abstract

The paper proposes spherical segment approximations of medieval quadripartite vaults having pointed arches, with the idea of finding simplified, yet representative geometric models for them. On the supposition that the bounding curves of every vault cell can be reckoned as vertical sections of spherical surfaces, it results in severe, yet expressive models of a wide range of vault shapes, revealing feasible computer representations of several vault types appearing in the books of architectural history either in textual or non-textual forms.

Keywords

quadripartite vault · spherical segment · geometry

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1 Basics

Though our subject is completely geometrical, its bearing is mainly on facilitating the understanding and the computer representation of the underlying architectural forms, therefore architectural terms will be used parallel with geometrical ones in quite a number of cases. The diagrammatic figures below show only the vaulting bays themselves (above the level of the imposts), disregarding the vault and rib thicknesses.

1.1 The Evolution of Rib Vaults

In the course of the development of the medieval architectural forms three fundamental steps preceded the appearance of the Gothic rib vault [1]. The first of these steps was the adoption of the transverse rib. It not only stiffened the vault by helping to maintain its shape, but it also served as a cover joint, thus permitted the vault to be erected one bay at a time. The second step was the change of the geometry of the diagonal arch of the cross vault from its traditional Roman semi-elliptical form (resulting from the right-angle intersection of two semi-cylindrical barrel vaults of equal span) to a more economical semicircular shape. Initially these diagonal arches were mere groins, causing complicated geometrical problems during the cutting of stones at the curves of intersections. Hence, the logical final step was the implementation of the diagonal rib. Although there are arguments about the structural role of ribs, the diagonal rib surely had several advantages, similar to those of the transversal rib. As a cover joint, it simplified the geometry of the cut stones located at the groin by freeing the stone courses of adjacent cells from each other. At the same time, it also helped to avoid certain difficulties during the erection of vault cells, since – along with the transversal and longitudinal boundary arches – it determined the shape of the infilling by determining the curve of its boundaries.

Since this way every boundary curve of the cells of the vault proper can be reckoned as an arc, it seemed reasonable to approximate these surfaces with spherical segments (disregarding obviously the possibility of stilted arches).

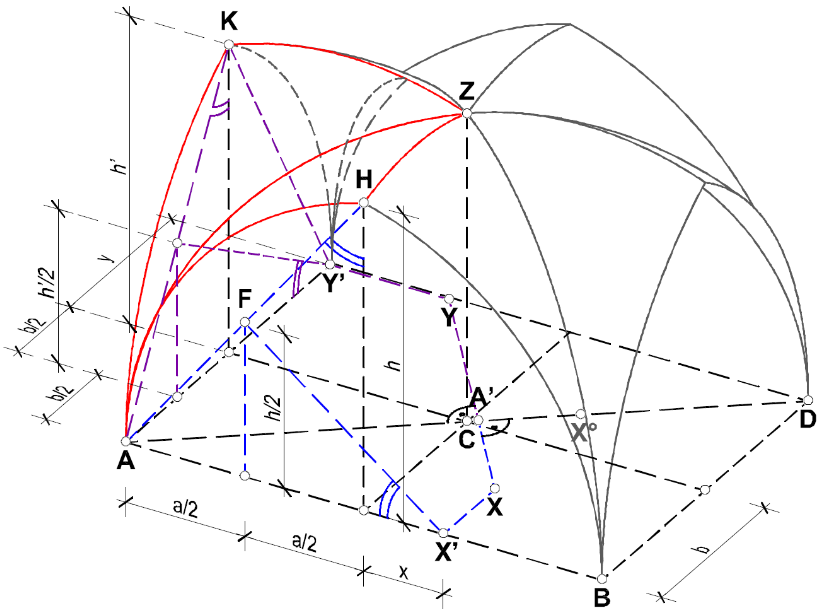

Fig. 1. Schematic layout of a quadripartite vaulting bay

1.2 Principles of the Geometrical Construction

First let's examine the general rules of the geometrical constructions of quadripartite pointed vaults.

The shapes of its main arches are characteristic criteria of a vault shape, so usually – in addition to the bay plan – these arches serve as a logical basis for the geometrical construction. Let the base shape of the vault be a rectangle having sides $2a$ and $2b$, and consider that the centres of the transversal, longitudinal and diagonal arches are given as X', Y', and A' (see Fig. 1). Even if these points are not known, it is easy to find them from the spans and rises of the main arches (as piercing points that the halving perpendicular lines of line sections AH, AK and AZ produce at base plane ABC).

Let X and Y be the two centre-points of the two spherical segments, and let x and y be their distances from the two symmetry-planes, respectively. Naturally, X and Y cannot be chosen independently of each other. Given that diagonal arch AZ is the common circular section of adjacent spherical segments of the vault, line XY must be perpendicular to diagonal AD – or in other words: the perpendicular lines erected from X and Y to diagonal AD must intersect it at the same point, i.e. at centre-point A' of the diagonal arch.

Based on similar reasoning (as the boundary arches are also circular sections of spherical segments) the lines connecting the centres of the spherical segments with the centres of their respective boundary arches (e.g. X'X, and Y'Y) must also be perpendicular to the sides of the base shape.

According to these considerations the geometrical construction of quadripartite vaults in general requires finding three lines: after locating X', Y' and A', the spherical segments' centre-points X and Y can be found as the intersections of the lines perpendicular to the two sides and to the diagonal.

1.2.1 Equal Apex-heights

To construct a vault having equal rises of boundary arches, it is enough to set apex points H and K at identical heights ($h = h'$), and then taking X' and Y' at the points where the halving perpendicular lines of line sections AH and AK pierce plane ABC of the springing. In this way the locations of X'X and Y'Y are set, however, the number of possible solutions is still infinite, since the location of A' can be selected arbitrarily within certain limits. Evidently, the location of A' on diagonal AD cannot be closer to A than C is, because in that case diagonal arch AZD would be pointing downward. On the other hand, neither X nor Y can lie beyond diagonal AD because in that case the corresponding spherical segment could not prop against the diagonal arch in the usual way, and for this reason A' cannot be farther from C than the intersection of lines AD and X'X or AD and Y'Y, whichever is closer to C. Hence, in the case depicted in Fig. 1, A' must reside on line section CX°.

Let us examine algebraically the premise of having equal rises of boundary arches. Since the line connecting points A and H is the chord of transverse arch AH, the line erected perpendicularly to line section AH from centre-point X' of arc AH will intersect it at its midpoint F. This way the height of this point F is exactly the half of that of point H, i.e. ($\frac{h}{2}$). As the angles marked with two arcs (at H, and X') have perpendicular arms:

$$\frac{h}{a} = \frac{x + \frac{a}{2}}{\frac{h}{2}} \Longrightarrow h^2 = 2a \cdot \left(x + \frac{a}{2}\right) \Longrightarrow x = \frac{h^2 - a^2}{2a} \quad (1)$$

In case of the longitudinal arch (based on the angles marked with two arcs at K, and at Y') the reasoning is the same as above:

$$\frac{h'}{b} = \frac{y + \frac{b}{2}}{\frac{h'}{2}} \Longrightarrow h'^2 = 2b \cdot \left(y + \frac{b}{2}\right) \Longrightarrow y = \frac{h'^2 - b^2}{2b} \quad (2)$$

Assuming equal rises of the boundary arches:

$$h^2 = h'^2 = 2a \cdot \left(x + \frac{a}{2}\right) = 2b \cdot \left(y + \frac{b}{2}\right) \implies \frac{a}{b} = \frac{y + \frac{b}{2}}{x + \frac{a}{2}} \quad (3)$$

It is necessary then to comply with Eq. 3 in order to have equal rises of longitudinal and transversal arches.

2 Variations

Though the aesthetic and structural benefits (e.g. reduced lateral pressure) in themselves would have been enough to justify the development and wide use of pointed arches, the exact cause of their spreading is not known. Correspondingly, there are several theories trying to explain the basic construction principles of quadripartite vaults having pointed arches. As starting points, we quote statements from the textbooks used by the students of our Faculty.[1]

a. "The pointed boundary arches, which can rise to the same heights even over different spans, enabled the construction of cross-vaults in rectangular bays." [2], page 131.

b. "Pointed arches became higher and more steeply pointed over time. Initially, the centres of the arches were at the tierce-point of the span, then at the quarter-point, finally at the springing, or even outside of it" [2], page 134.

c. "The introduction of pointed arches had the benefit of constructing various arches with the same general appearance for various spans even if the identical springing line and crown height were being maintained, because the pointed arch is movable as a hinge at the crown" [3] page 442. This statement is contradictory (from an exact geometric point of view) in more than one respect. There is, however, an implicit statement in it, what can be relied upon in our investigations:

c. The introduction of pointed arches allowed using identical radii for the main arches of the vault.

d. Yet another mode of geometrical construction included in the mentioned literature, although not in textual form. Both sources [2] (page 8) [3] page 444 show a kind of axonometric drawing as an illustration of pointed quadripartite vaults where the crowns of the vault meet each other with horizontal tangents, without breaking above the centre of the ground plan.

Let us investigate the constructional and geometrical consequences of the principles listed in points **a** to **d** for assigning the main arches to each other. The general relationships described in point 1.2 will stand, so in each case we will discuss only the considerations and specifications of the given constructional variation.

[1] Note however, that we might be wrong when we tacitly assume that some kind of rules (knowingly or not) did exist for the setting out of the pointed vaults – it might as well be that there were no such rules, and the arches were set up independently from each other.

2.1 Isomorphic Main Arches

Perhaps the simplest way to form a pointed vault is to "get rid of" the central part of the semi-circular vault section (as it happens when we transmute barrel vault to pointed barrel vault). For example, if we omit one eighth of the central part of a spherical sail vault on both sides of the longitudinal and transverse axes of its bay, and attach the obtained truncated three eighths of the vault together, we can produce a pointed vault, the length and width of which are six eighths of the original. This way the centre-points of all main arches will be at two thirds of their span, hence the boundary arches will be isomorphic, i.e. the procedure will give a result corresponding to condition **b**.

As a result of the above truncation, the centre-points of the four spherical segments are shifted along the diagonals of the bay to the vertices of a rectangle having the same proportions as the bay itself (to point A' in case of spherical segment AHZK). This solution gives the possible maximum distance of sphere centre X from boundary arch AHB. If X would be shifted beyond diagonal AD, then a surface similar to a cloister vault would be produced, since the vault surface would break upward, not downward at diagonal arch AZD. Fig. 2 depicts the diagonal arch as a dashed line, because in this special case the surface does not break there.

2.1.1 Equal Apex-heights

According to the present construction method point A' will always be on diagonal AD, therefore the angles marked with three arcs (at A and at C) have the same alignment, and the following relationship can be found:

$$\frac{a}{b} = \frac{x}{y} \implies ay = bx \quad (4)$$

Substituting Eqs. 1 and 2 and assuming equal ($h = h'$) rises of boundary arches:

$$a \cdot \frac{h^2 - b^2}{2b} = b \cdot \frac{h^2 - a^2}{2a} \implies 2a^2h^2 = 2b^2h^2 \implies \frac{a}{b} = 1 \quad (5)$$

Consequently, this construction method can lead to identical apex heights of boundary arches only if the floor plan of the vault is a square ($a = b$), and the centres of the boundary arches are at equal distances from the bay axes ($x = y$).

2.2 Horizontal Central Tangents

To get crown-curves having horizontal tangents at point Z, over the centre of the bay (satisfying condition **d**), the centres of the spherical segments should obviously be on the axes of the bay. Because of this symmetrical placement, every two cells share the same centre-point, located in a corner of a diamond in plan (see Fig. 3).

Note that this type of construction is incompatible with semi-circular diagonal arches, just like the one described in 2.1. This limitation was obvious in the previous case, since that variation was derived from cutting out the middle parts of a surface having semicircular diagonal arches. Perhaps it's not so obvious in

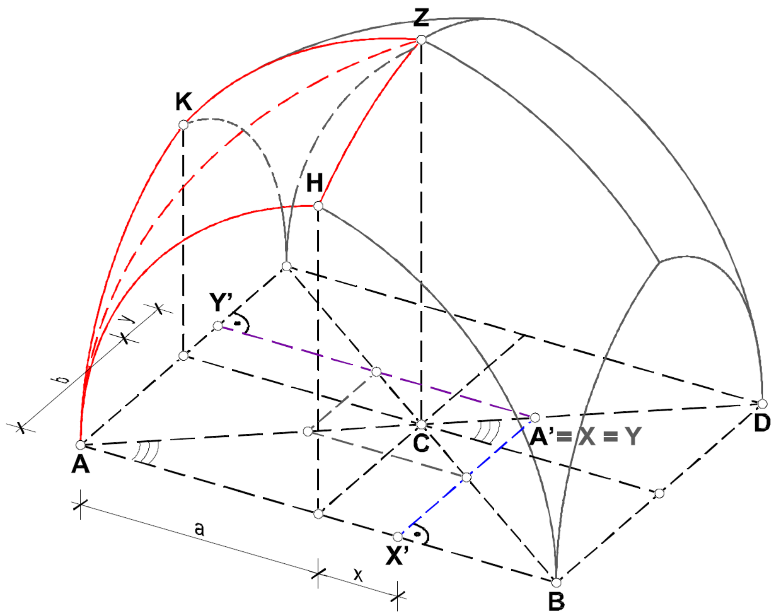

Fig. 2. Vault shape with isomorphic main arches

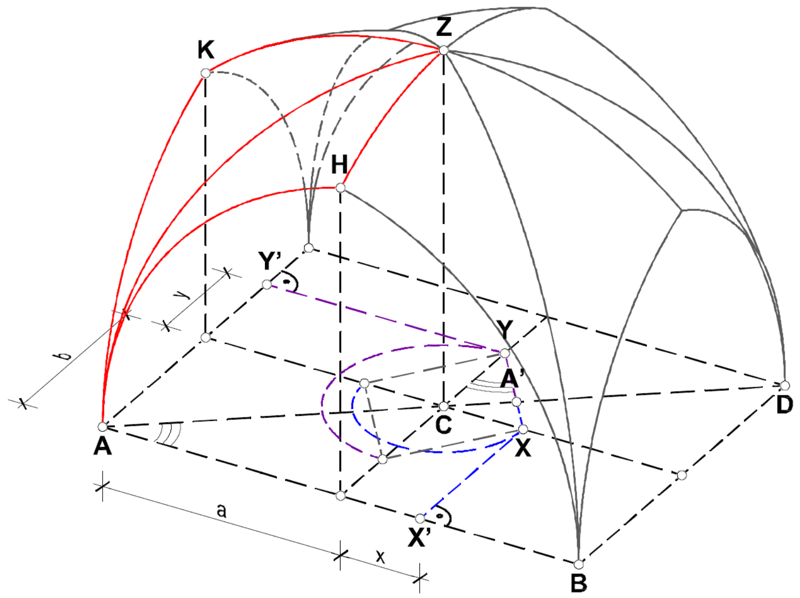

Fig. 3. Vault shape with horizontal central tangents

this case, but, having semicircular diagonal arches, this second type of construction gives the same result. Centre-points X and Y can be found as the intersections of the line erected from point A' perpendicularly to diagonal AD with the bay axes. Hence, as A' approaches C, the distance between X and Y becomes shorter and shorter, and when finally A' reaches C, X and Y will get there also – resulting a sail vault shape.

2.2.1 Equal Apex-heights

As the angles marked with three arcs (at A and at Y) have perpendicular arms, the following relationship can be found:

$$\frac{a}{b} = \frac{y}{x} \Longrightarrow ax = by \tag{6}$$

Substituting Eqs. 1 and 2 and assuming equal ($h = h'$) rises of boundary arches:

$$a \cdot \frac{h^2 - a^2}{2a} = b \cdot \frac{h^2 - b^2}{2b} \Longrightarrow h^2 - a^2 = h^2 - b^2 \Longrightarrow \frac{a}{b} = 1 \tag{7}$$

Consequently, this mode of geometrical construction can again lead to identical apex heights of boundary arches only if the floor plan of the vault is a square ($a = b$), and the centres of the boundary arches are at equal distances from the vault axes ($x = y$).

2.3 Main Arches Having Equal Rises

If all transversal and diagonal arches of the nave have equal apex-heights, its general appearance will resemble that of the pointed barrel vault – in other words, this construction method

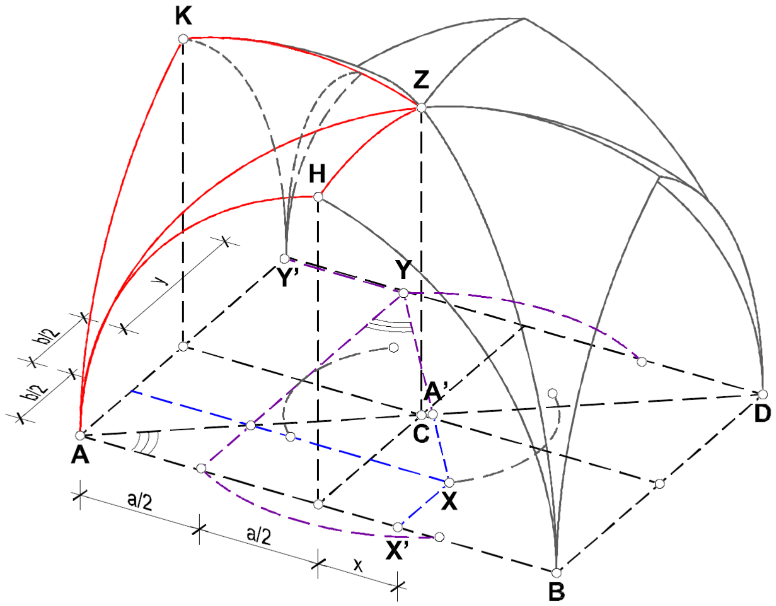

Fig. 4. Vault shape with main arches having equal rises

results in the minimum vertical undulation of the longitudinal crown of the vault.

According to condition **a**, only the boundary arches were expected to have equal rises (we examine this condition in each case separately), but now consider the diagonal arch having the same rise too (see Fig. 4).

To ensure that apex-points H and Z have equal heights, the centre-point of crown HZ – and thus centre-point X also – has to be halfway between them, on a transverse line having $\frac{b}{2}$ (one fourth of the length of the bay) offset from the frontal (AB) side of the bay.

Centre-point Y of the transversal cell can be found as the intersection of a longitudinal line having $\frac{a}{2}$ (one fourth of the width of the bay) offset from the side of the bay, with another line being erected from X perpendicularly to diagonal AD.

Since the centre-points are not in a symmetrical position, this type of construction results separate centre-points for all eight spherical triangles of the vault. However, all of these centre-points can be obtained by mirroring X and Y about the axes of the vault.

This type of construction is possible even if the diagonal arch is semicircular. Note also that the radius of the diagonal arch is smaller then the radii of either of the boundary arches, since while all main arches have equal rises, the diagonal arch has the greatest span.

2.3.1 Equal Apex-heights

As the angles marked with three arcs (at A and at Y) have perpendicular arms, the following relationship can be found:

$$\frac{a}{b} = \frac{y + \frac{b}{2}}{x + \frac{a}{2}} \tag{8}$$

As this equation is the same as Eq. 3, it is obvious that this setup ensures equal apex-heights for all boundary arches no matter how long a, b or h is, or in other words what proportions the bay or the boundary arches have.

2.4 Main Arches Having Equal Radii

In order to ensure that all main arches of the vault have equal radii (see statement **c**), the centres of the spherical segments (e.g. X) should be at equal distances from the respective side of the bay (e.g. AB), and from the diagonal (e.g. AD). This obviously means that X and Y must reside on angle bisectors AX and AY respectively (see Fig. 5).

Since (assuming a non-square plan) the resulting centre-points are not in a symmetrical position, all sphere segments will have different centre-points, which can be obtained by mirroring X and Y about the axes of the vault.

Note that this type of construction is also possible even if the diagonal arch is semicircular.

2.4.1 Equal Apex-heights

As in this case the radii of boundary arches (AH and AK) having their centre-points in X' and Y' are equal, it is evident, that the heights of their apexes can only be equal if these arcs – and hence their AH and AK chords also – have equal length:

$$\sqrt{a^2 + h^2} = \sqrt{b^2 + h^2} \Longrightarrow a^2 = b^2 \Longrightarrow \frac{a}{b} = 1 \tag{9}$$

Again, this mode of geometrical construction can lead to identical apex heights of boundary arches only if the floor plan of the vault is a square ($a = b$), and the centres of the boundary arches are at equal distances from the vault axes ($x = y$).

2.5 Highest Possible Boundary Arches

Considering the locations of points X and Y from figure to figure we can see that they get closer and closer to the sides of the bay, and all previous cases belonged to certain special characteristic locations of these points. There is, however, one more

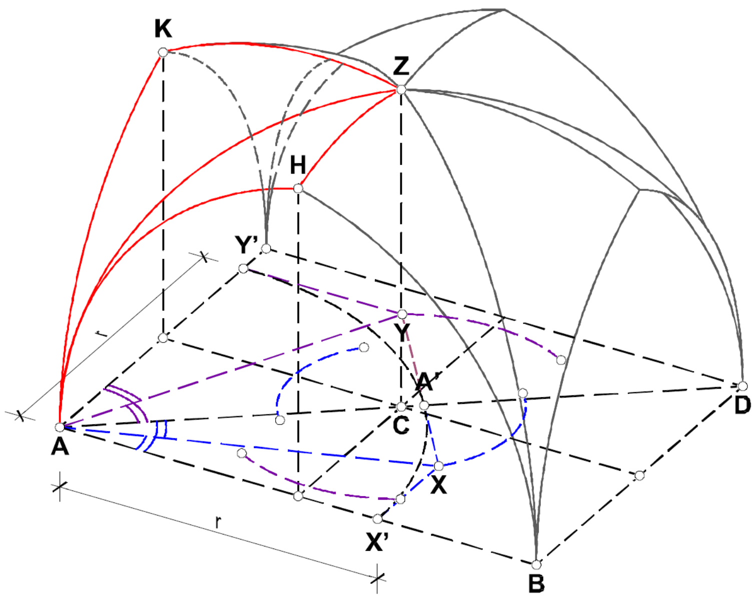

Fig. 5. Vault shape with main arches having equal radii

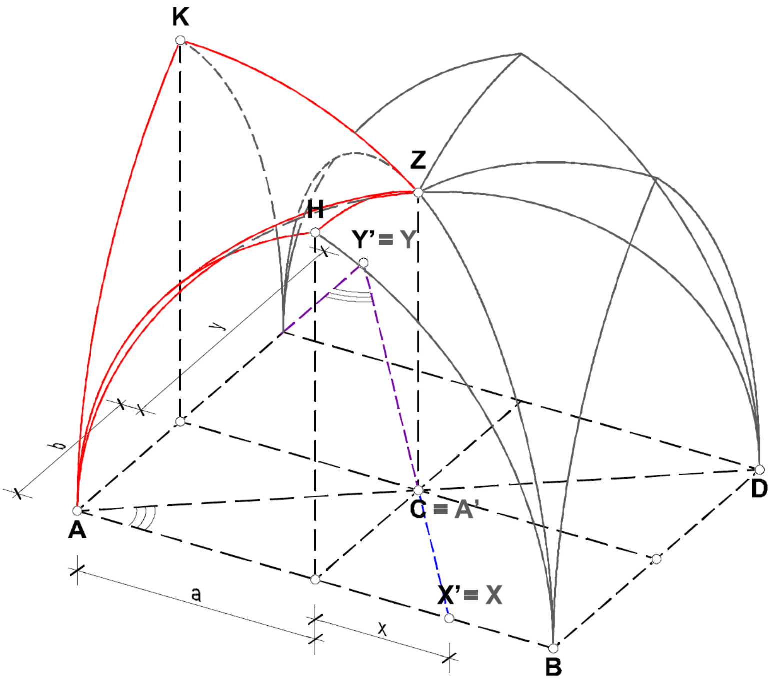

Fig. 6. Vault shape with the highest possible boundary arches

such characteristic location: namely the planes of the boundary arches. Let's construct the vault with this assumption (see Fig. 6).

As the sphere-centres get closer and closer to the sides, the apex-height of the diagonal arch becomes the lowest, and the apex-height of the boundary arch having smaller span (usually the longitudinal arch) becomes the highest. When they finally reach the sides, the highest points of the crowns will be on the boundary arches, and the crowns of adjacent vault bays will lie on a single arc. Hence, this configuration also has its own special attribute: the chord-lengths of the main arches (e.g. AK, AZ and AH distances) will always be equal. As point A is located on the line erected perpendicularly to the plane of arc HZ at its centre-point, line segments AH and AZ can be considered as two generators of a cone having its apex in point A, and part of its base circle in HZ.

Note that this construction method results in steeply pointed boundary arches. For example, over a square bay, the centres of the boundary arches cannot be inside the springing ($x \geq a$), even if the diagonal arch is semicircular.

2.5.1 Equal Apex-heights

As it can be seen on Fig. 6, the angles marked with three arcs (at A and at Y) have perpendicular arms, hence the following

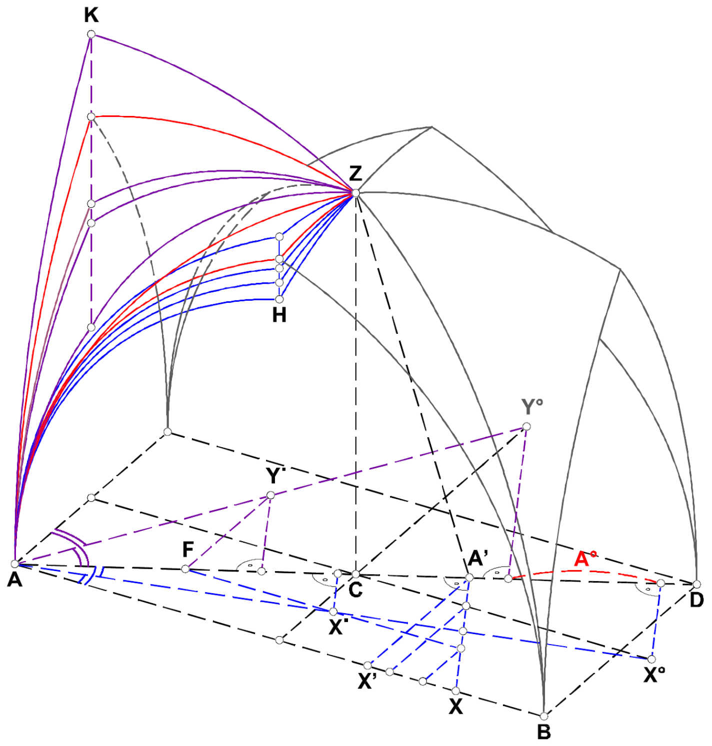

Fig. 7. Different quadripartite vault shapes using the same diagonal arch.

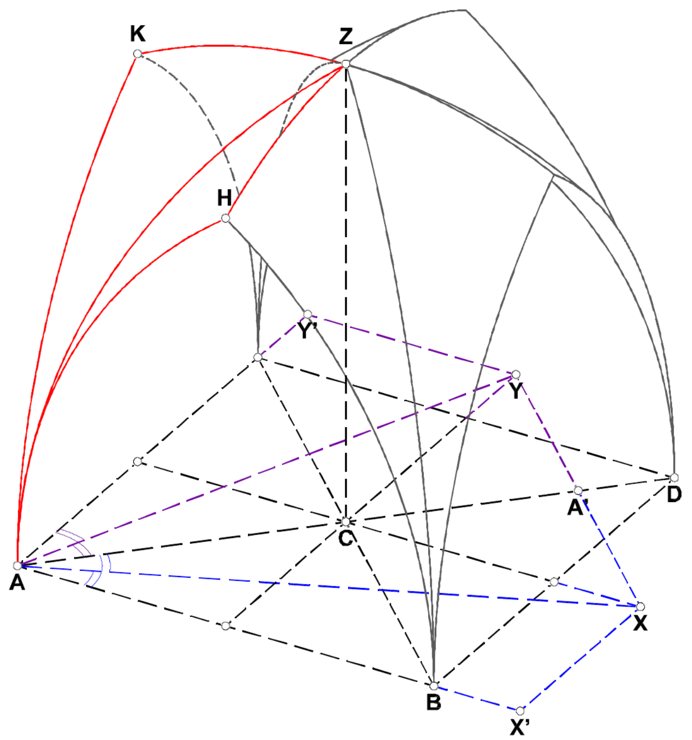

Fig. 8. Quadripartite vault whose main arches have equal radii and whose crowns have horizontal tangents over the centre of the bay.

relationship can be found:

$$\frac{a}{b} = \frac{y+b}{x+a} \Longrightarrow a \cdot (x+a) = b \cdot (y+b) \tag{10}$$

Substituting Eq. 1 and 2 and assuming equal ($h = h'$) rises of boundary arches:

$$a \cdot \left(\frac{h^2 - a^2}{2a} + \frac{2a^2}{2a} \right) = b \cdot \left(\frac{h^2 - b^2}{2b} + \frac{2b^2}{2b} \right)$$
$$\Longrightarrow h^2 + a^2 = h^2 + b^2 \Longrightarrow \frac{a}{b} = 1 \tag{11}$$

Once again, this mode of geometrical construction can lead to identical apex heights of boundary arches only if the floor plan of the vault is a square ($a = b$), and the centres of the boundary arches are at equal distances from the vault axis ($x = y$).

3 Combinations

Since only the diagonal arch connects the two adjacent cells, evidently even cells constructed with entirely different methods can be used in a single vaulting bay. Fig. 7 shows variations of vault cells constructed according to the methods described above, all derived from the same diagonal arch having its centre in point A'. The centre-points of all these cells reside on line A'X, erected from point A' perpendicularly to diagonal AD.

According to the examinations described above, it's obvious that there are only two ways for ensuring equal apex-heights for the boundary arches of a quadripartite vault. The result can be guaranteed by either having a square bay or (regardless to the proportions of the bay) by using the method described in 2.3.

The only unanswered question is whether it is possible to combine the above constructions – or rather their advantageous properties. As these properties result from restricting the positions of cell centres to certain lines, searching for the combination of the above features means searching for intersections of these construction lines.

In three cases the sphere centres were restricted to three parallel lines obviously not intersecting each other. Their intersections with diagonal AD also can be ignored, as they appear on the wrong side of diagonal AD – one in point A, the second in point C, the third in point F, halfway between them.

The first of their intersections with bisector AX$^\circ$ of angle DAB appears again in point A, so it also can be ruled out. The second one, intersection point X$^\cdot$ of bisector AX$^\cdot$ and line FX$^\cdot$ is again inappropriate, as the line erected from this point perpendicularly to diagonal AD meets it on its wrong side, and this way the resulting diagonal arch would be bent over, its apex being higher than point Z is – a highly unlikely solution in case of quadripartite vaults.

Therefore, the only remaining potential location is point X$^\circ$, the intersection of angle bisector AX$^\circ$ and vault axis CX$^\circ$. What it means is that it is possible to construct a vault having horizontal tangents in point Z, and all of its main arches still having equal radii. Yet, in a rectangular vaulting bay this is possible only in two of the four cells, as in the other two cells, the same construction method would give a different result (point Y$^\circ$). The method offers a correct solution for a square bay, however, due to its symmetry (see Fig. 8).

4 Conclusions

Gothic vaults are testimonies of one of the most unique eras of architecture. Up to the present time they were described and typified almost exclusively by the shapes and proportions of their bays and boundary arches – sometimes with quite ambiguous results. Nowadays, as the means of informatics and computer graphics are used more and more often in order to visualize and elucidate forms and shapes, it becomes more and more important to interpret the actual three-dimensional shapes of the vaults proper, instead of trying to imagine them based on their boundary curves or certain sections.

This paper proposes a geometrical approximation for depicting the shapes of Gothic quadripartite vaults having pointed arches. On the supposition that the bounding curves of every vault cell can be reckoned as vertical sections of spherical surfaces, it results in severe, yet expressive models of a wide range of vault shapes, revealing feasible computer representations of several vault types appearing in the books of architectural history either in textual or non-textual forms.

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