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RESEARCH ARTICLE

A geometric construction for ensuring C^1 continuity of adjacent Bézier patches

Bálint Csabay

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Abstract

It is in many cases practical to compose a continuous surface out of some low-degree Bézier surface patches having C^1 *(first-order parametric) continuity between the adjacent patches. The paper presents a relatively simple geometric construction of the position of the control points in the neighbourhood of the common boundary curve to ensure the required* C^1 *continuity. The construction is based on the well-known criteria of the continuous joints of the Bézier surface patches and on a straightforward geometric similarity.*

Keywords

Bézier surfaces · Bézier patches · freeform surfaces

Bálint Csabay

Department of Architectural Representation, BME, H-1111 Budapest, Műegyetem rkp. 3., Hungary

e-mail: bcsabay@arch.bme.hu

Nowadays the freeform surfaces are already popular not just in industrial design but also among the computer-aided architectural design (CAAD) systems. For constructing freeform surfaces, the Bézier surfaces are preferred in many cases, due to their advantageous properties. The idea of a Bézier surface goes back to that of a Bézier curve.

The parametric vector of the Bézier curve with the given control points or knots

$$F_0, F_1, \dots, F_n$$

will be:

$$R(t) = \sum_{i=0}^{n} F_i B_{i,n}(t)$$

$B_{i,n}(t)$ at the right side is called the blending function, defined as follows:

$$B_{i,n}(t) = \binom{n}{i} t^i \, (1-t)^{n-i} \,,$$

where the binomial coefficient $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. Of course, the above vector equation can also be written separately for the parametric functions of the coordinates $x(t)$, $y(t)$ and $z(t)$ of the $R(t)$ vector by means of the scalar equations as follows:

$$x(t) = \sum_{i=0}^{n} x_i B_{i,n}(t),$$

$$y(t) = \sum_{i=0}^{n} y_i B_{i,n}(t),$$

$$z(t) = \sum_{i=0}^{n} z_i B_{i,n}(t),$$

where x_i, y_i, z_i are the corresponding coordinates of the F_i knots.

With the increased number of knots, the equation describing the curve becomes more complicated, consisting of more terms, the degree of the curve is also increased (in case of $n+1$ knots, the degree of polynomial describing the curve will be n), therefore, it is often preferable to describe more complicated shapes by connecting several Bézier curves of lower degree together, while using the property of the curve that the sides of its control polygon at the end points are tangential to the curve at its end points.

In case of Bézier curves and surfaces, it follows from the mathematical description with parameters that the simple geometric tangential continuity does not mean at the same time the continuity with respect to the parameters. In order to fulfil this latter requirement (that is, that the derivative with respect to t shall be continuous), the ratio of tangential sections ending at the common end point of the two sections shall be 1; that is, they shall be of equal length. Hereinafter, the continuity will also mean continuity with respect to the parameters.

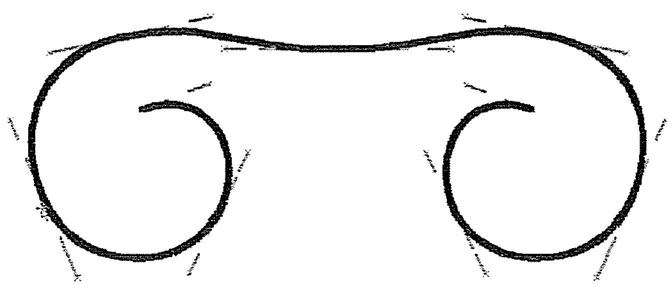

Fig. 1. Arc of Ion voluta – defined by means of several Bézier curves joined tangentially

The methods used in the case of freefrom curves can also be extended to surfaces. Thus, creating the Descartes product of two Bézier curves results in a Bézier surface. In mathematical terms, this results in the vector equation as follows:

$$R(t,u) = \sum_{i=0}^{n}\sum_{j=0}^{m} F_{i,j} \cdot B_{i,n}(t) \cdot B_{j,m}(u),$$

where the values of parameters t and u vary between 0 and 1, $F_{i,j}$ are the knots associated with the bidirectional set of curves, and $B_{i,n}(t)$ and $B_{j,m}(u)$ are separate blending functions associated with the parameters t and u, respectively, as described above with the Bézier curves.

For the geometric interpretation, the diagram below provides assistance (Fig. 2).

As shown in the diagram, the surface is specified by means of a 4×4 control net (4 knots in each direction giving 16 knots). In case of surfaces, it especially holds that the increase in the number of their knots may render their handling very complicated; therefore, it is preferable to describe more complicated surfaces by joining several simpler surface patches together. The present paper aims at presenting a geometric approach suitable for this purpose.

However, the continuity criterion described in case of the Bézier curves is valid only with some addition. Two Bézier surfaces are of zero order continuity if the control points of the boundary curves of the neighbouring surface patches coincide; in fact, it follows from the above that, in such cases, the common boundary curves specified by the given control points are the same; but, although the two surfaces are joined together; there is obviously a break concerning the tangents along the joint. Namely, in a control point belonging to the common boundary curve of the two surface patches meet in general four edges of the common control net. Two of them run along the common boundary curve ensuring the zero order continuity, while the other two edges going inside the control nets of the two originally separate patches, are in general not collinear, resulting in the break along the joint.

For the C^1 continuity (which, as mentioned, also means the continuity with respect to the parameters t and u) it is not sufficient that, in case of joint surfaces, the edges going to a control point inside one or the other of the two patches from a control point on the common boundary lie along a straight line. Although is a necessary condition; for the required first order continuity, the ratios of the lengths of these edge pairs shall also be equal.

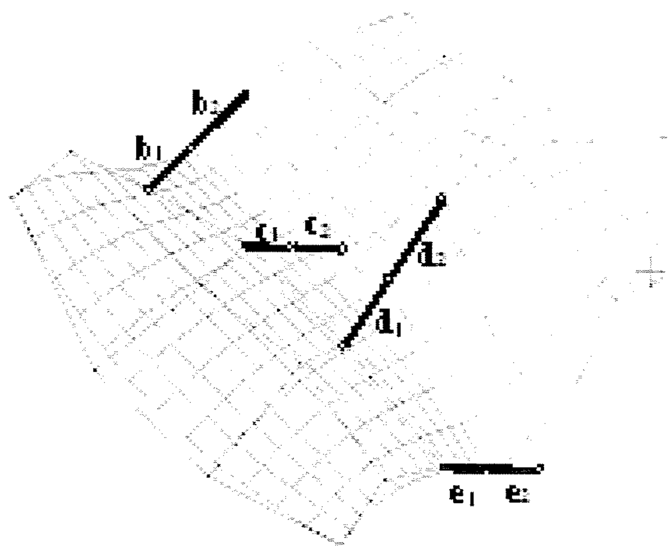

Fig. 3. The edge pairs of two continual adjacent Bézier patchses

Thus, $b_1/b_2 = c_1/c_2 = d_1/d_2 = e_1/e_2$ is also required for the C^1 continuity as specified above.

As opposed to curves, surface joints may be multi-directional; in fact, further surface elements can be joined in both lateral directions. In such cases, a continuous surface joint without a break line, that is, of tangential continuity would require that the above condition of C^1 continuity is fulfilled in both directions. This, however, sets further restrictions in case of four surface elements joined at their corner.

In the following, the joint of four Bézier-surfaces at a corner point is presented, with the condition that first order continuity is fulfilled. As shown in Fig. 4 below, let A_F, B_F, C_F and D_F the four surfaces be joined at the corner point S, and their corner knots lie along the lines a, b and c in one direction and p, q, and r in the other direction. On the other hand, the knot B of surface B_F lies at the intersection of lines a and p; the common knot E of surfaces A_F and B_F at the intersection of a and q; the knot A of the surface A_F at the intersection of a and r; the common knot F of surfaces B_F and C_F at the intersection of b and p; the common knot S of the four surfaces at the intersection of b and q; the common knot H of surfaces A_F and D_F at the inter-

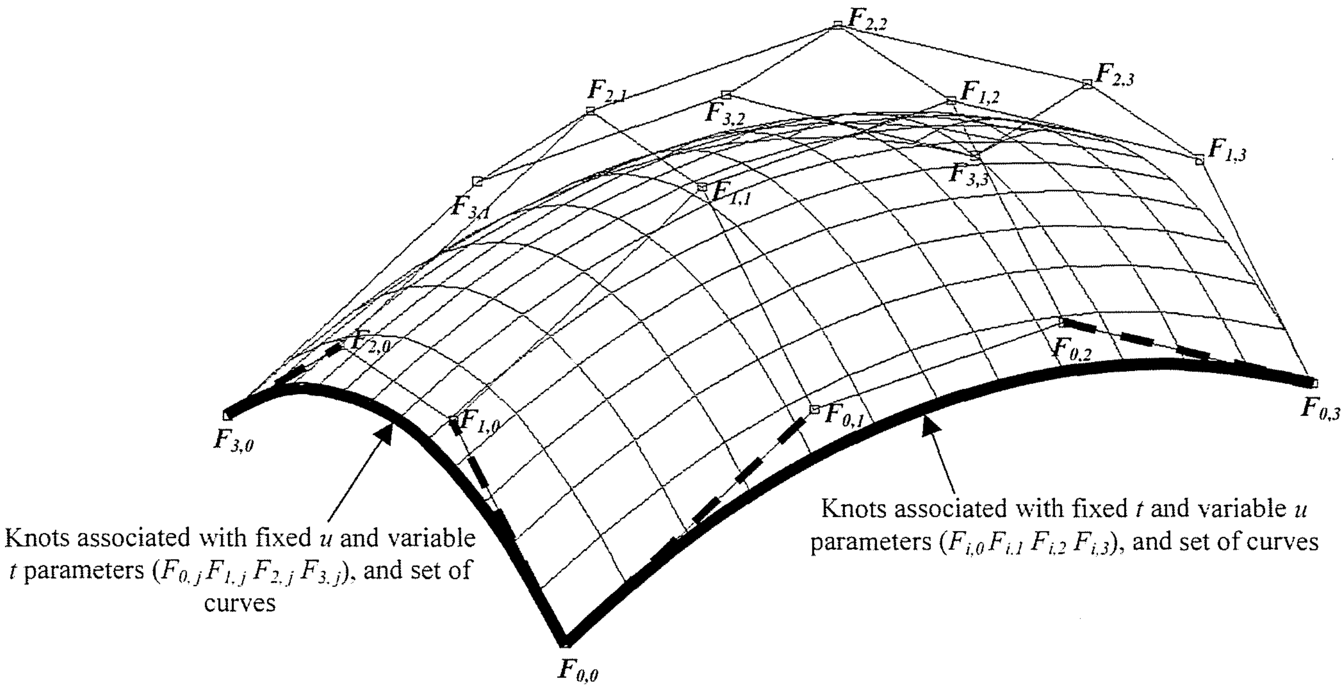

Fig. 2. Bézier-surface with an $(n+1) \times (m+1)$ control net

section of b and r; the knot C of surface C_F at the intersection of c and p; the common knot G of surfaces C_F and D_F at the intersection of c and q; and, finally, the knot D of surface D_F at the intersection of c and r; furthermore, for the sections a_1, a_2 of line a, sections b_1, b_2, of line b, sections c_1, c_2 of line c; sections p_1, p_2, of line p; sections q_1, q_2 of line q and sections r_1, r_2 of line r, the ratio described above is fulfilled, that is:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \text{ and } \frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}.$$

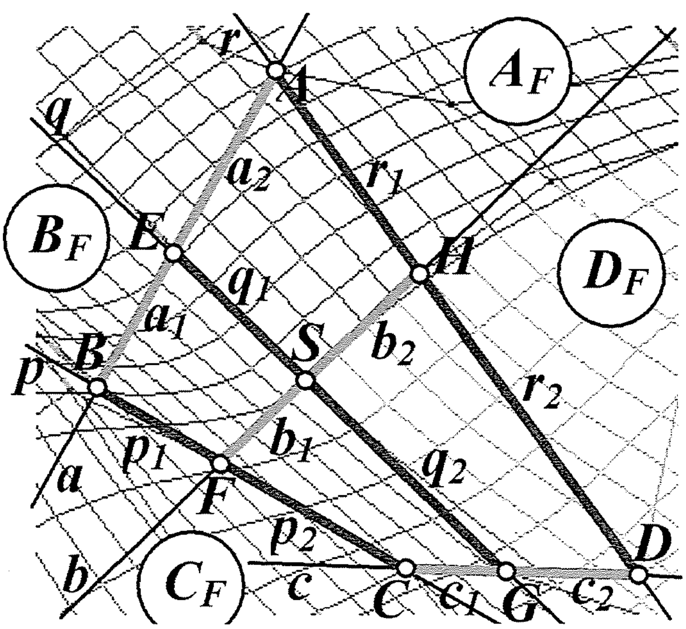

Fig. 4. The edge pairs of four C^1 continual adjacent Bézier patches

It will be shown that, for the plane determined by lines b and q, a parallel projection direction can always be found where the image points A', B', C' and D' of A, B, C and D knots projected on that plane form a trapezoid, the elongated non-parallel sides of which and one of the lines joining the knots passing through the point S (q in this case) meet at the point P while the other line joining the knots that passes through the point S (b in this case) lies parallel to the parallel sides of the trapezoid (of course, in extreme case, the intersection P may be in infinity; the projection will take the shape of a parallelogram).

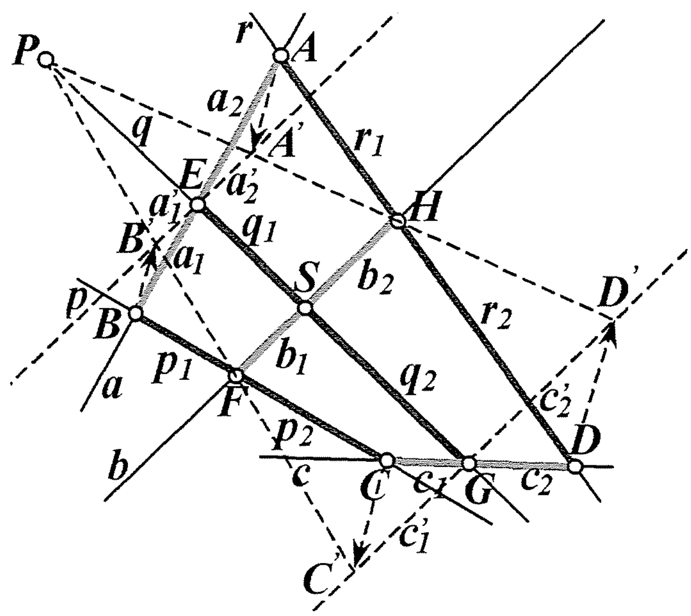

Fig. 5. The projection of ABCD knots is the A'B'C'D' trapezoid

Mark out an arbitrary point P on the line q and connect it to the points F and H. Draw straight lines through the points E

and G parallel to the line b. According to the theorem of parallel sectors, the length of section $B'E$ in the trapezoid $A'B'C'D'$ (the projection of a_1, that is a_1') is to the length of section EA' (the projection of a_2, that is a_2')) as b_1 is to b_2. Considering that $a_1/a_2 = b_1/b_2$, therefore $a_1'/a_2' = a_1/a_2$ also holds. This results in the similarity of triangles $BB'E$ and $AA'E$, with the consequence that AA' is parallel to BB'. It can also be seen that AA' is parallel to DD', BB' to CC' and CC' is parallel to DD'; that is, the directions of projection are parallel to each other.

The above considerations are important in setting the positions of the knots necessary for joining the Bézier-surfaces; in fact, the theorem is also true in reverse; starting from any trapezoid (or parallelogram in a special case), the knots of the Bézier-surfaces can be created by means of parallel projection of the corner points in an arbitrary direction – while preserving the proportions specified above, which ensures the C^1 continuity of the complex surface in both directions.

The pictures below (Fig. 6–11) show the constraints by means of which further three surfaces can be joined to a specific initial A_F Bézier-surface, with the C^1 continuity preserved.

Of course, the proportion valid for the given direction of joint shall also be ensured in the case of other neighbouring lines joining the knots – marked with dotted lines in the last figure - of the joint connecting surfaces.

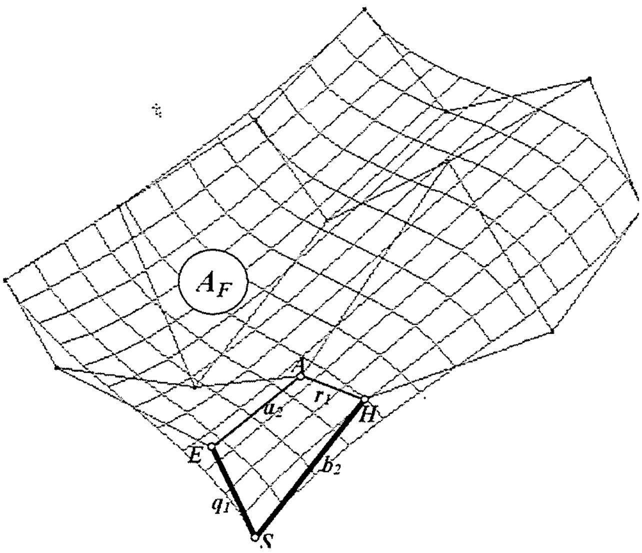

Fig. 6. Specific initial A_F Bézier-surface with knots S, E, A, and H around the corner and with lines joining the knots a_2, b_2, q_1, r_1.

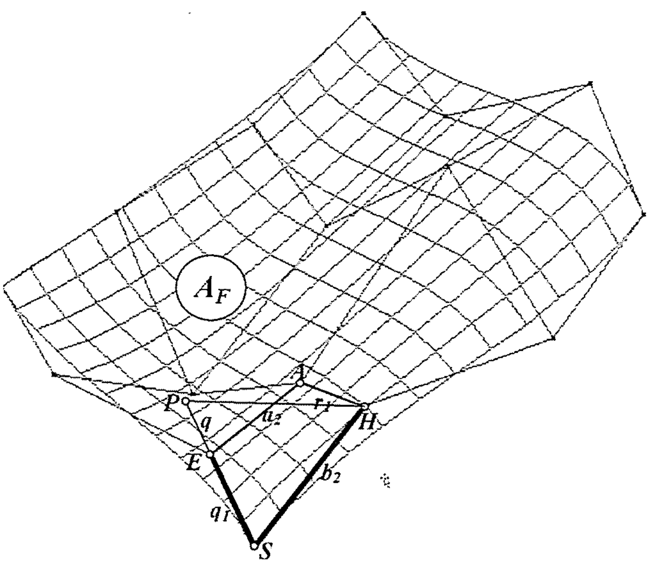

Fig. 7. Marking out a point P on the line q passing along the q_1 line joining the knots and connecting it with the knot H.

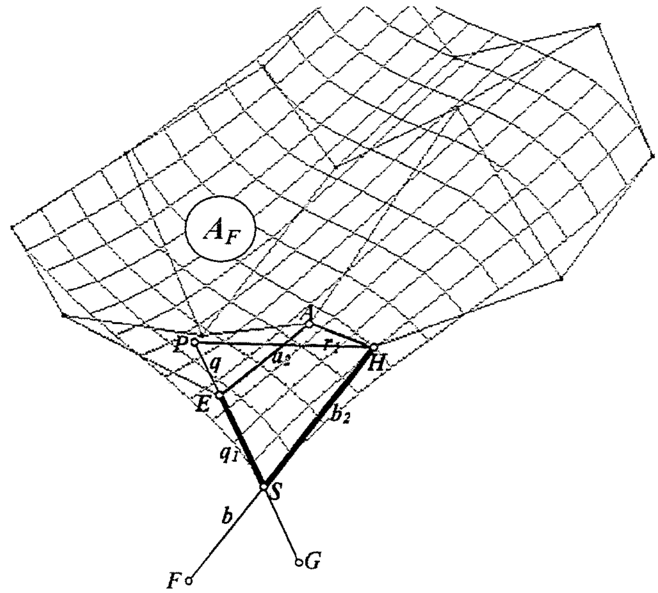

Fig. 8. Marking out the knots F and G neighbouring surfaces along the line b passing through the knots b_2 and along the line q.

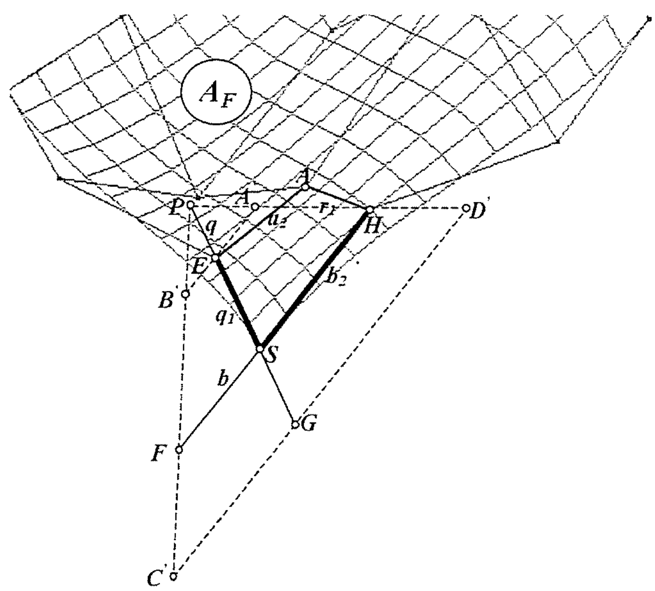

Fig. 9. Drawing the projected trapezoid with corners $A'B'C'D'$ where the parallel sides are parallel to line b and the other sides meet at the point P.

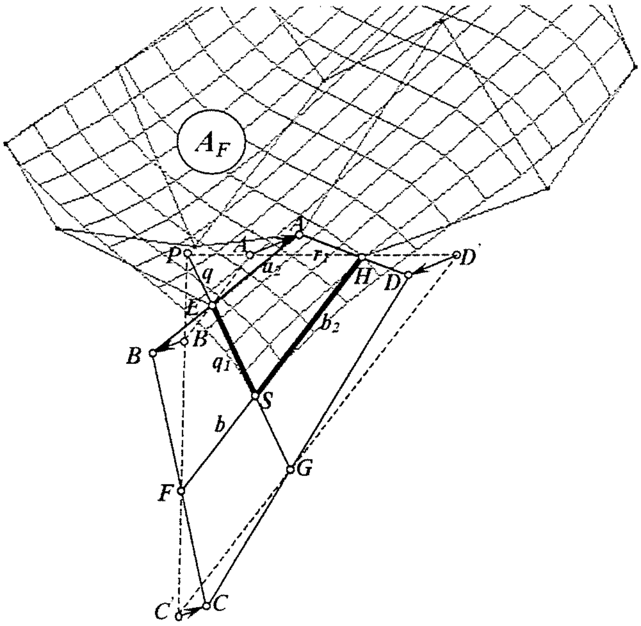

Fig. 10. Projecting the points B', C' and D' parallel to the direction of projection $A'A$ on the corresponding lines joining the knots (knots B, C and D).

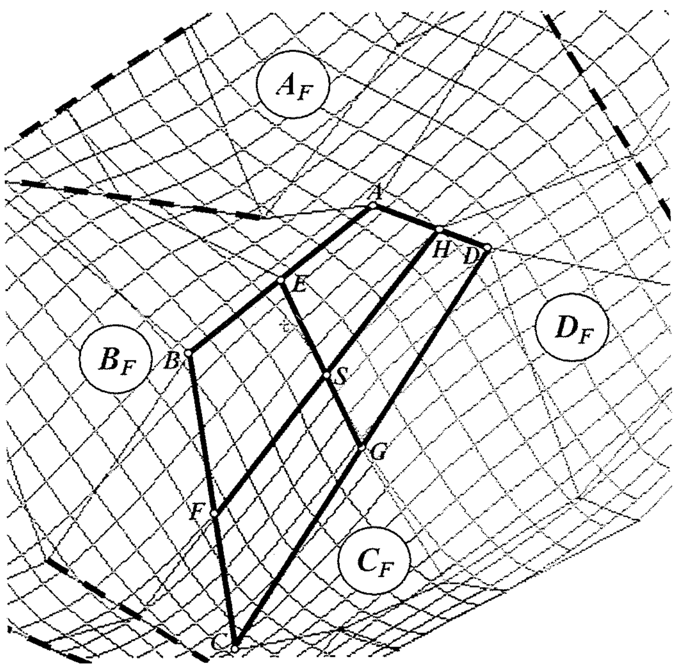

Fig. 11. Bézier-surfaces of continuous joint, specified by the knots drawn.

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