periodica polytechnica

Architecture
38/1 (2007) 11–16
doi: 10.3311/pp.ar.2007-1.02
web: http://www.pp.bme.hu/ar
© *Periodica Polytechnica 2007*

RESEARCH ARTICLE

Analysis of continuous reinforced concrete beams in serviceability limit state

Katalin Rákóczy / *György* Deák

Received 2006-09-14

Abstract

The paper investigates to what extent the effect of actions and deformations of statically indeterminate reinforced concrete beams are modified if, instead of the elastic, uncracked model (which is simple but approximate), an elastic, cracked model is applied (which is more accurate) in the computation of the effect of actions at serviceability check. (Continuous beams were modelled with beams fixed at both ends.) The effects of different load levels and moment-redistribution at design were also analysed. It is assumed that the variable load reached its characteristic value prior to the time considered, thus, the location of cracked zones and the tension stiffening effect is determined for that load level. Instead of the interpolation between the two values of deformation, the tension stiffening effect was taken into account by an effective moment of inertia (I_{eff}), which is equivalent to the other method.

Keywords

reinforced concrete beam · rare combination of actions · cracked elastic state · non-linear analysis · flexural stiffness · deformation · combination of actions · deformation · serviceability

Acknowledgement

This research was funded by the Hungarian Scientific Research Fund grant No. T29269.

Katalin Rákóczy
Department of Mechanics, Materials and Structures, BME, H–1111 Budapest, Műegyetem rkp. 3., Hungary
e-mail: rakoczy@freemail.hu

György Deák
Department of Mechanics, Materials and Structures, BME, H–1111 Budapest, Műegyetem rkp. 3., Hungary

1 Introduction

In the past decades, the limitation of stresses, deformation and crack width of reinforced structures under serviceability conditions has gained considerable importance due to the following facts:

- Buildings tend to be taller, structural spans tend to be greater, and structures with hinged connections (e.g. precast concrete structures) are spreading.
- High-strength structural materials have been applied lately, while their elastic moduli have hardly increased (e.g. in case of concrete) or not increased at all (e.g. in case of steel).
- Social expectations concerning the aesthetics and service of structures have increased; the limitation of deformation, vibration and crack width is based on several requirements.
- Structures used to be analysed applying the method of allowable stresses, that is, under service loads, assuming elastic, cracked state. However, according to the method of limit states applied recently, structural dimensions are determined for a different load level, assuming a different stress state. Thus, the latter method does not involve an indirect check of serviceability.

In case of statically determinate structures, the same method is employed at the computation of the effects of actions both at design situations and at the analysis of serviceability limit states: the conditions of equilibrium must be satisfied. However, in case of statically indeterminate structures, the flexural stiffness of the structure influences the effects of actions considerably.

The draft European Standard prEN1992-1-1:2003 proposes the following four models for the computation of the effect of actions if a structure is designed for the ultimate limit state:

- A linear elastic model applying the stiffness of the uncracked gross cross-section. (This model is obviously inaccurate, but its application is simple.)
- The previous model, assuming a limited redistribution of the effects of actions subsequently. (The effects of actions may be modified, what may be favourable in certain respects.)

- A non-linear elastic model, taking the effect of cracking into account. (This model is applied mainly for the check of structures, since the reinforcement area at cracked cross-sections is assumed to be known.)
- A plastic approach, assuming very ductile structural elements, in which plastic hinges form at ultimate limit state.

There are two models for the analysis of the effect of actions at serviceability check:

- The linear elastic model described above, applying the stiffness of the uncracked gross cross-section. (This simple model is generally employed in engineering practice.)
- A non-linear elastic model, which takes the effect of cracking into consideration.

In this paper, the non-linear elastic model was applied for the analysis of statically indeterminate reinforced concrete elements in serviceability limit state. Sections 2 to 5 introduce the assumptions of the analysis. The proposed algorithm for the analysis is illustrated by Fig. 5 in Section 6. Section 7 presents the numerical analysis of a beam fixed at both ends, which is similar to one span of a continuous beam.

2 The Influence of Flexural Stiffness on the Effect of Actions

Cross-sections of a structural member are considered to be in state 2 when we apply the non-linear elastic model. Flexural stiffness is not constant in different sections of the element, what influences the effect of actions in case of statically indeterminate elements. However, the variation of stiffness has different effects on specific flexural elements:

- Beams normally have quite short uncracked segments (i.e. cross-sections in state 1, I_1), which do not cause great changes in the behaviour of the structural element, since the moment at these parts is small. In case the beam was designed applying the linear elastic analysis, the ratio of the maximum and minimum bending moment (and, correspondingly, the reinforcement ratio) is approximately 2. In this case, the effects of actions (i.e. bending moments) computed according to a homogeneous cross-section are assumed to be only slightly modified if the variation of flexural stiffness is taken into account.
- In case of beams designed for plastic distribution of moments or redistribution of moments (when the moment-diagram is shifted either upwards or downwards) (Fig. 6), values of the moment of inertia are different compared to those in the previous case. This results in a shift in the moment-diagram in service state. (See Section 7, Numerical Analysis, for data.) Unlike in the previous case, effects of actions at design and in service state are not proportional, what suggests that highly stressed locations should be checked to make sure that steel stress does not exceed $0.8 f_{yk}$. (f_{yk} is the characteristic yield stress.) (Under serviceability conditions, steel stresses that would lead to inelastic deformation must be avoided, since that would involve excessive deformations and crack width (prEN 1992-1-1:2003, 7.2(4),(5)).)
- Flat slabs normally have both cracked and uncracked zones, and there is significant difference between reinforcement ratios of certain locations (the difference of reinforcement ratios of certain locations may be fivefold, corresponding to moments). Therefore, flat slabs show great variation in stiffness, what is supposed to lead to considerable changes in the effect of actions in service state, as compared to the effect of actions of a homogeneous, elastic slab.

3 The Effect of Cracking on Flexural Stiffness

For the computation of flexural stiffness values, which are necessary for the calculation of moments and deformations, cracked and uncracked zones (i.e. the crack pattern) have to be at first determined. During the service period of a building, the probability that the rare combination of actions (in case of variable actions of only one type, $F_r = G_k + Q_k$) occurs is 40 to 50 percent. Therefore it is advisable to assume that cracks induced by the rare combination of actions form prior to the time being considered (since serviceability requirements have to be fulfilled following crack formation, too) (Fig. 1).

According to provision 7.1(2) of prEN 1992-1-1:2003, a concrete section has to be assumed cracked if the concrete stress exceeds the mean value of the tensile strength of concrete ($\sigma_c > f_{ctm}$) under the rare combination of actions. However, for serviceability (irreversible) limit state, Annex A of ENV 1991-1 specifies a target reliability index (ß) of 1.5 for the design working life, i.e. the failure probability (P_1) is $0.7 \cdot 10^{-1}$. It is advisable to apply this reliability level also for reversible limit states. (The difference between reversible and irreversible limit states lies in the combination of actions to be taken into account (Deák, 2000)) [1]. Applying the characteristic values of material properties (E_{ck}, f_{ctk}) is a good method of obtaining this reliability level (DEÁK et al. 1998 [3]).

Following the determination of the sectional properties (flexural stiffness) of the structural element under the rare combination of actions, deformations shall be computed usually for the quasi-permanent ($F_{qp} = G_k + \psi_2 Q_k$) or frequent combination of actions ($F_f = G_k + \psi_1 Q_k$). Values of moment of inertia obtained in the previous step can be considered constant, since the tension stiffening effect (i.e. the stiffening effect of the cracked concrete in tension exerted on the reinforcement) determined for the highest expected load in service state will not change if the load decreases (Fig. 1). (It is disregarded that the concrete may prevent the steel reinforcement from shortening.)

Provision 7.4.3(3) of prEN 1992-1-1:2003 takes the tension stiffening effect into account by applying a factor (ζ) to interpolate between values of deformation computed according to cross-sections in state 1 and state 2:

$$a = \varsigma a_2 + (1 - \varsigma)\, a_1 \tag{1}$$

where

a is the deformation (curvature, rotation, deflection etc.),
a_1 is the deformation assuming state 1,
a_2 is the deformation assuming state 2, and

$$\varsigma = 1 - \beta\, (M_{cr}/M)^2 \tag{2}$$

where

ζ is the factor of interpolation between I_1 and I_2 due to the tension stiffening effect,
β is a factor taking into account the type and duration of load,
M_{cr} is the cracking moment, and
M is the actual bending moment.

Assuming that $\kappa = M/E_c I_{\text{eff}}$, this procedure may be simplified by introducing an effective moment of inertia derived from the above formula:

$$I_{\text{eff}} = \frac{I_1 \cdot I_2}{\varsigma \cdot I_1 + (1 - \varsigma)\, I_2} \tag{3}$$

where

κ is the curvature,
E_c is the modulus of elasticity of concrete, and
I_1 and I_2 are moments of inertia of the transformed section in state 1 and state 2, respectively.

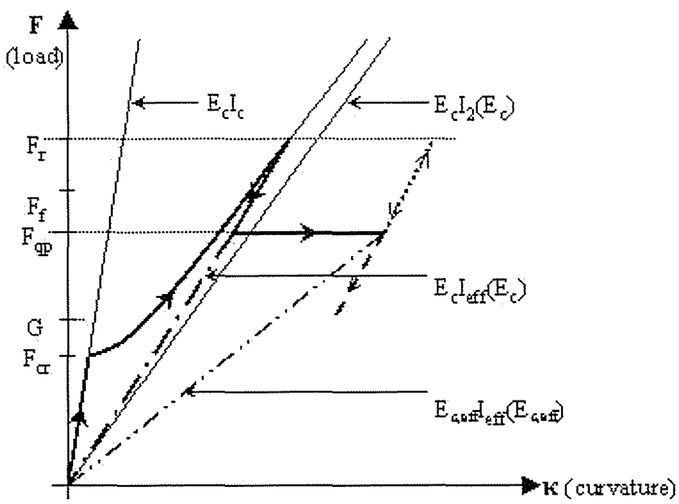


Fig. 1. Loading history and change of flexural stiffness of a section of a RC element, in case the load reaches the rare combination of actions prior to creep

4 The Effects of Creep

4.1 Effective Modulus of Elasticity of Concrete

At the computation of deformations, prEN 1992-1-1:2003 allows to take into account the concrete creep by replacing the modulus of elasticity of concrete, E_{cm}, by an effective modulus of elasticity, $E_{\text{c,eff}}$.

According to Expression (7.20) in section 7.4.3(5),

$$E_{\text{c,eff}} = \frac{E_{cm}}{1 + \varphi} \tag{4}$$

where ϕ is the final creep coefficient of concrete.

The final creep coefficient of concrete, $\phi(\infty, t_0)$, is specified in Fig. 3 in prEN 1992-1-1:2003. The notional size of the cross-section, the age at loading and atmospheric conditions are taken into consideration here. According to prEN 1992-1-1:2003, 2.3.2.2(3), the effects of creep should be evaluated under the quasi-permanent combination of actions. It implies that $\psi_2 Q$ should be considered as a long-term action.

Loads induce the greater concrete creep in a structure the earlier they are applied after concrete casting and the longer they act. If no data are provided, we may assume that the dead load (G) starts to act when the concrete strength reaches its design value and it acts permanently afterwards. The variable part of the quasi-permanent combination of actions ($\psi_2 Q$) also starts to act at an early age, and it acts in approx. 50 percent of the working life. Therefore, the total value of these two actions has been taken into account at the computation of long-term deformations due to creep. Since the variable part of the frequent and that of the rare combinations of actions are assumed to start to act later and for a shorter period of time, they were not considered as actions that induce creep. A k factor is introduced to obtain the creep coefficient of concrete for the quasi-permanent combination of actions:

$$E_{\text{c,eff}} = \frac{E_{ck}}{1 + k\varphi} \tag{5}$$

where
E_{ck} is the characteristic value of the modulus of elasticity of concrete, and

$$k = \frac{G_k + \psi_2 Q_k}{G_k + Q_k} \tag{6}$$

where

G_k is the characteristic value of the permanent action (self weight),
Q_k is the characteristic value of the variable action, and
ψ_2 is a multiplying factor for variable actions at quasi-permanent combination of actions.

4.2 Moment of Inertia

When determining the flexural stiffness of a concrete section after taking creep into account, it is not only the modulus of elasticity of concrete that changes but also the moment of inertia of the transformed concrete section. This change is due to the fact that a factor (α) is applied at the computation of the moment of inertia of the transformed section:

$$\alpha = E_s/E_{ck} \tag{7}$$

where E_s is the modulus of elasticity of reinforcing steel, not taking concrete creep into consideration. However, α will change if creep is considered:

$$\alpha = E_s / E_{c,\text{eff}}. \tag{8}$$

Due to this fact, the moment of inertia of the transformed section will increase. The degree of increase of the moment of inertia is different in case of different reinforcement ratios. Strain-diagrams of concrete sections in case of low and high reinforcement ratios can be seen in Fig. 2. Since the stress-diagram is proportional to the strain-diagram, point $\varepsilon = 0$ represents the neutral axis. The figure illustrates that, in case of different reinforcement ratios, the same degree of creep increases the height of the compression zone (and thus, the moment of inertia) to a different extent. (There is a slight change in the steel stress, and correspondingly in steel strain, due to creep, but this is ignored in the figure.)

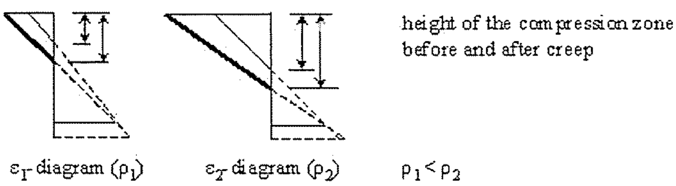


Fig. 2. Strain-diagrams of cross-sections with low and high reinforcement ratio, ρ, illustrating the effect of creep on strain

The increase of the moment of inertia is greater in case of smaller reinforcement ratios, therefore, the flexural stiffness decreases to a lesser extent. It is indicated in Fig. 3. According to Colonnetti's first theorem, the deformation of a statically indeterminate structure loaded by constant load will increase but the effect of actions will remain unchanged, if the creep law of each structural element is the same. Since the reinforcement ratio is not constant along the element, the creep law is different in each cross-section, what leads to a change in the effect of actions.

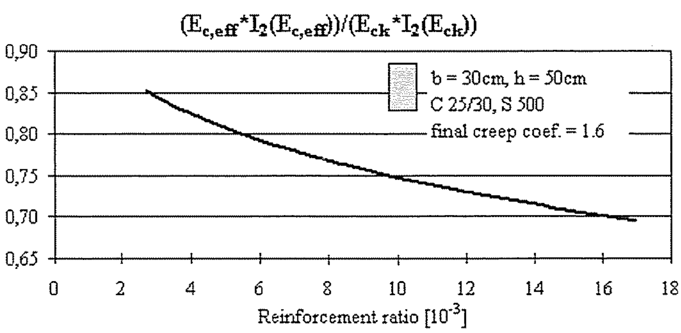


Fig. 3. The modification of flexural stiffness due to creep as a function of the reinforcement ratio

5 Loading History

Fig. 1 illustrates the loading history in which the total load reaches the load level of the rare combination of actions at an early age, what is followed by creep. However, concrete creep starts as early as loading (including self-weight), and the effects of creep are the most significant in the first year. The probability that the variable action reaches its characteristic value during the first year of service time is only 2 percent. Therefore, it is probable that the main part of creep has already taken place by the time the total load reaches the level of the rare combination of actions. This loading history is illustrated by Fig. 4.

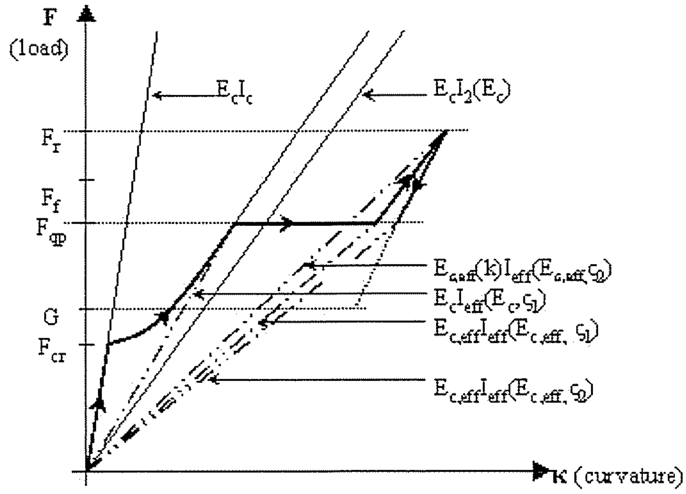
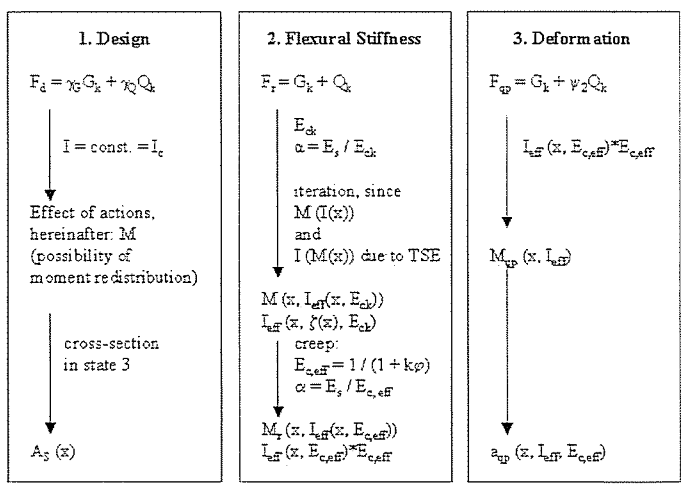


Fig. 4. Loading history and change of flexural stiffness of a section of a RC element, in case creep precedes cracking due to the rare combination of actions

6 Proposed Algorithm for Computation of Deformation

Taking into account all the above considerations and assuming the loading history illustrated by Fig. 1, the algorithm according to (Fig. 5) is proposed for the computation of deformation of flexural RC elements.



F_d, F_r and F_{qp}	are the design, rare and quasi-permanent combinations of actions, respectively,
γ_G and γ_Q	are partial safety factors for permanent and variable actions, respectively,
A_s	is the area of steel reinforcement,
M_{qp} and M_r	are bending moments at the quasi-permanent and rare combination of actions,
a_{qp}	is the deflection computed for the quasi-permanent combination of actions, and
x	is a coordinate along the span.

Fig. 5. Proposed algorithm for computation of deformation

7 Numerical Analysis

The effect of the factors described in the previous sections on the bending moments and deformations of a flexural RC element was studied in a numerical analysis. The analysis was carried out applying a computer program that divided the beam into finite segments along the span, which have different sectional properties (reinforcement ratio, moment of inertia, etc.)

The examined structural element was a beam of 8 m, fixed at both ends; its cross-section was 30/50 cm (Fig. 6). The permanent action (i.e. self-weight of the structure, g) was taken 20 kN/m. In order to have three different tension reinforcement ratios, the variable actions (q) were chosen to be 9, 18 and 24 kN/m. (The smallest live load is that of a hotel or hospital room, the next one is that of a school hall, and the greatest live load belongs to a department store.) The multiplying factor of the variable load: $\psi_2 = 0.5$. The total load at the quasi-permanent combination of actions ($p_{qp} = g + 0.5\ q$) was 24.5, 29 and 32 kN/m. S 500 steel and C 25/30 concrete was used ($E_{ck} = 2780$ kN/cm^2). The final creep coefficient of concrete, ϕ, was taken 1.6. Values of the reinforcement ratio varied between 3‰ and 17‰.

According to prEN 1992-1-1:2003, 5.5(4), the moments calculated using a linear, elastic analysis may be redistributed, provided that the ratio of the redistributed moment to the moment before redistribution is less than or equal to 0.7 (in case of high ductility steel). Therefore, at the design of the reinforcement, four cases were considered: 1) a bending moment diagram assuming homogeneous cross-section (I = const.), without redistribution, 2) and 3) moment diagrams shifted by 15 percent upwards and downwards at the supports, respectively, and 4) a moment diagram shifted downwards by 30 percent at the supports (Fig. 6). Applying such a redistribution, the stress limitations specified in prEN 1992-1-1:2003 may be assumed to be satisfied if the stress in the reinforcement is lower than $0.8{\cdot}f_{yk}$ under the rare combination of actions, i.e. the steel reinforcement will have no inelastic deformations under service loads (prEN 1992-1-1:2003, 7.2.(5)).

The numerical analysis was carried out for both types of loading history illustrated by Figs. 1 and 4 above. The same results were obtained both for bending moments and deflections; the different assumptions for loading history led to no practical changes in the final results.

Due to the effect of creep, the effective modulus of elasticity of concrete, $E_{c,\mathrm{eff}}$, decreased to 38 percent of E_{ck}, while the increase of the moment of inertia of cracked sections, $I_{\mathrm{eff}}(E_{c,\mathrm{eff}})$ / $I_{\mathrm{eff}}(E_{ck})$, was 180 to 210 percent. The value of the flexural stiffness taking creep into account, $E_{c,eff}\,I_{\mathrm{eff}}(E_{c,\mathrm{eff}})$, is approx. 70 to 80 percent of the initial value, $E_{ck}\,I_{\mathrm{eff}}(E_{ck})$.

At the analysis of moments at service state, it was found that the distribution of moments was only slightly modified as compared to the diagram obtained assuming constant stiffness ($-M_{\max} = p \cdot L^2/12$; $+M_{\max} = p \cdot L^2/24$, where p is the total load, $p = g + q$). However, the increased stiffness resulting from raising the moment diagram at design led to an approx. 10 percent increase of the moments at the support, and consequently to a decrease of the midspan moment. In case of lowering the moment diagram by 15 percent at the support, the increased stiffness of the midspan resulted in an approx. 10 percent increase of the midspan moment, and a decrease of the negative moments. The ratio of moments computed according to the proposed method (see the algorithm in Fig. 5) and according to the traditional method (assuming constant stiffness) is indicated in Fig. 7.

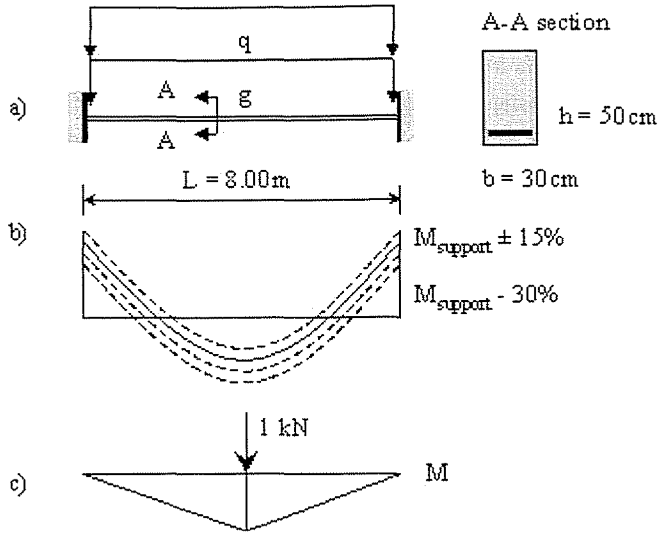


Fig. 6. a) The examined structural element;
b) Different moment-diagrams considered at design;
c) Moment diagram corresponding to a unit force, applied in work method.

Values of deflection obtained from the proposed method and those determined according to the traditional method were compared in the same way as moments. According to the traditional method, the deflection was computed by the formula

$$a_{\mathrm{midspan}} = \frac{1}{384} \cdot \frac{p_{\mathrm{qp}} \cdot L^4}{E_{\mathrm{c,eff}} \cdot I} \tag{9}$$

where a_{midspan} is the deflection at midspan, p_{qp} is total load at the quasi-permanent combination of actions, L is the span of the element.

If the work method is applied for the computation of deflection, a moment diagram corresponding to a unit force is employed (Fig. 6c), and the two moment diagrams are graphically integrated. Since the region of negative moments of the first diagram is integrated with small values of the second diagram, it is the midspan regions of the diagrams that account for a considerable part of the result, i.e. the deflection. Therefore, in Formula 9 above, $I_{2,\mathrm{midspan}}(E_{\mathrm{c,eff}})$ and I_c were substituted for I, and the deflection was obtained by interpolating between the results by ζ. When the moment-diagram was shifted upwards at design, the proposed moment-diagram also moved upwards, what led to a decrease in the midspan moment in service state. This explains why the deflection obtained from the proposed method is smaller than the deflection computed traditionally. Accordingly, shifting the moment-diagram downwards at design led to a downward move of the diagram in service state, i.e. to an increase of the midspan moment as compared to the moments assuming constant stiffness. This resulted in greater deflection than the one calculated by the other method. These results are illustrated by Fig. 8 below.

While shifting the moment diagram downwards by 15 percent at design led to a deflection (a_{prop}) that is up to 20 percent greater than a_{trad}, shifting the diagram downwards by 30 percent resulted in an increase of the deflection (a_{prop}) of up to 40 percent as compared to a_{trad}.

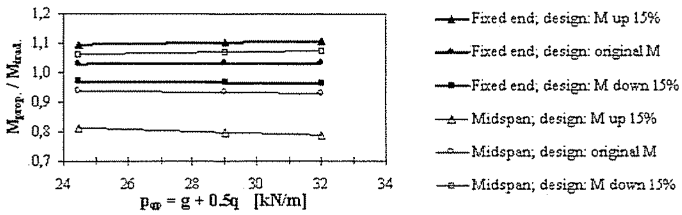


Fig. 7. Ratio of moments computed according to the proposed, M_{prop}, and the traditional method, $M_{trad.}$, in service state, considering three different cases at design

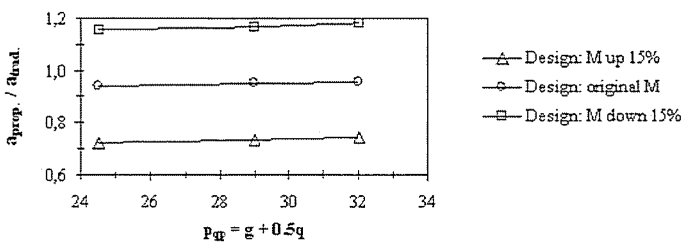


Fig. 8. Ratio of values of midspan deflection computed according to the proposed method, $a_{prop.}$, and the traditional method, $a_{trad.}$, considering three different cases at design

The different values of the live load and the different cases at design resulted in several different values of reinforcement area for the same concrete cross section. It provided data for an analysis on what values of the moment of inertia belong to different values of the reinforcement ratio. This may be useful at the analysis of flat slabs, where there are greater differences in the reinforcement ratio than in case of beams. Increasing the reinforcement ratio by 100 percent leads to an increase of the moment of inertia of approx. 60 percent. However, as Fig. 9 indicates, the function is not perfectly linear.

8 Conclusions

The paper discussed how the effects of cracked, elastic state of cross-sections of flexural elements and creep influence the effects of actions and deformations. A method was proposed to take these effects into consideration. Results of a numerical analysis demonstrated that plastic redistribution of moments at design led to a different moment distribution from the one assuming constant stiffness. In certain cases, this resulted in up to 20 percent greater deformations than those obtained by a traditional method (see Fig. 8).

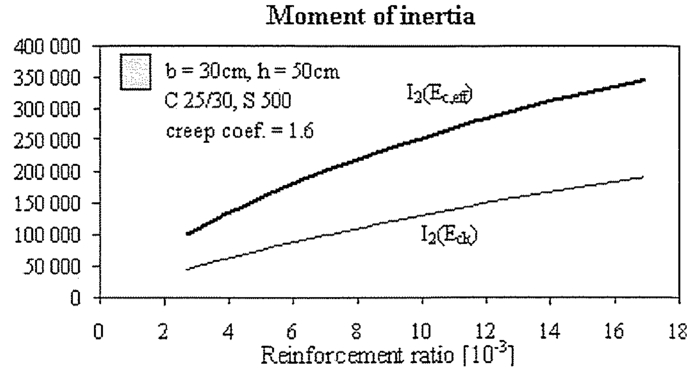


Fig. 9. Moment of inertia – reinforcement ratio diagram

9 Future Work

A future task of the research is studying continuous beams assuming different loading schemes (i.e. multi-parameter loading). Concerning limitation of crack width, the influence of service moments (corresponding to the flexural stiffness of the element in state 2) on crack width will also be examined. Further tasks include studying flat slabs by numerical analysis, taking into account the considerations introduced in the paper. Performing experiments in the near future is also under consideration. The ultimate objective of the research is to develop a method for the accurate computation of deformations of reinforced concrete continuous beams and flat slabs, which can be applied in engineering practice.

References

1 **Deák Gy**, *Használhatósági követelmények, használati biztonság: (Serviceability Requirements, Service Reliability)*, Proceedings of the TARTÓK 2000 – VI. Magyar Tartószerkezeti Konferencia, Budapest, 2000.

2 *CEB-FIP Model Code (1990). Comité Euro-International du Béton*, 1991.

3 **Deák Gy, Rákóczy K**, *Síklemez födémek vizsgálata a használati állapotban: (Analysis of Flat Slabs in Serviceability Limit State)*, Tudományos közlemények (1998). (BME Építőmérnöki Kar, Vasbetonszerkezetek Tanszék.

4 *Eurocode 1: ENV 1991-1: Basis of design and actions on structures.*

5 *Eurocode 1: ENV 1991-2-1: Actions on structures; Densities, self-weight and imposed loads.*

6 *Eurocode 2: prEN 1992-1-1:2003: Design of Concrete Structures – Part 1-1: General Rules and Rules for Buildings.*

7 **Ghali A, Favre R**, *Concrete Structures; Stresses and Deformations*, London, 1986.

8 **Jankó L**, *A beton kúszásának hatása az igénybevételekre: (The Effect of Concrete Creep on the Effects of Actions)*. 1, Vasbetonépítés (2000), no. 2, 38–49; 2, Vasbetonépítés (2000), no. 3, 67–75.

9 **Massányi T, Dulácska E**, *Statikusok könyve; Magasépítés: (Book of Structural Engineering; Architectural Engineering)*, Műszaki Könyvkiadó, Budapest, 2000.

10 *MSZ 15022/1: Építmények teherhordó szerkezeteinek erőtani tervezése: (Design of Loadbearing Structures of Buildings. Reinforced Concrete Structures)*. Hungarian Standard.

11 **Rákóczy K, Deák Gy**, *The Effect of Flexural Rigidity on the Moment and Deflection of Statically Indeterminate Reinforced Concrete Elements*, Proceedings of the 2^{nd} Int. Ph.D. Symposium in Civil Engineering, Budapest, 1998.