

A Visual Path into Differential Geometry for Architects: From Curves to Freeform Surfaces

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Abstract

Today, architectural creativity is increasingly supported by algorithmic design and artificial intelligence tools, yet without a deep understanding of differential geometry, designers often remain passive software users. This paper presents a visual pedagogical approach to teaching differential geometry specifically tailored for architectural engineering students. The methodology balances mathematical precision with architectural applicability through visualization, avoiding self-serving abstraction. Built upon the Van Hiele model of geometric thinking and Cognitive Load Theory, the course utilizes GeoGebra 3D as its primary didactic tool. By focusing on parametric form-finding and form recognition instead of abstract proofs, students are guided toward higher levels of geometric rigor. The article details a specialized syllabus – ranging from surfaces of revolution to freeform geometries. Student projects demonstrate that mastering mathematical threshold concepts significantly increases creative freedom and enables the conscious manipulation of complex forms. These findings suggest that bridging the gap between mathematical theory and architectural practice is essential for educating critical-thinking designers in the digital age.

Keywords

differential geometry, architectural education, parametric design, form-finding, visualization, GeoGebra, Van Hiele model

1 Introduction

Nowadays – primarily due to the explosive use of AI – one of the greatest challenges in architectural engineering education is to train young architects who think critically, do not automatically accept the results provided by computer programs but instead analyze and override them.

The description of geometrical thinking and the possibilities for its development have been addressed in numerous classical and contemporary studies, spanning both secondary and higher education (Alalouch, 2018; Cocchiarella, 2006; Cumino et al., 2021; Kaufmann, 2009; Pék, 2024a; 2024b; Pinho, 2013; Uçar et al., 2023; van Merriënboer et al., 2024).

Before the widespread use of computers, it was natural for architects to receive rigorous training in mathematics, including descriptive geometry, because without it even the flawless design of a simpler building was impossible. With a few iconic exceptions, this constrained architectural creativity. Later, with the emergence of freeform surfaces in architecture and the spread of algorithmic design, differential-geometric data (e.g., Gaussian curvature)

became not only criteria for ex post analysis but also active design parameters.

For these reasons, it has become essential for future architects to acquire basic knowledge of differential geometry that enables them to use today's available programs (e.g., Rhino and Grasshopper) not in a passive way but to "master" them – since using software is not the same as possessing geometric knowledge.

2 Fundamental objectives

The main problem that typically stands in the way of effective teaching of differential geometry is the difference between the ways mathematicians and architects think. Mathematicians often teach differential geometry in abstract spaces and through proofs, and they cannot or do not want to give up mathematical rigor for the sake of comprehensibility. Architects, on the other hand, while using computer programs excellently, know, for example, how to smooth a surface, but they understand less what is happening in the background, because they have less

insight into the deeper differential-geometric relationships of curves and surfaces.

The aim, therefore, was to find the "differential-geometric Goldilocks zone": a course that provides sufficiently deep knowledge for the relationships of curves and surfaces to be understandable and applicable, yet is visual enough for the results to quickly become spectacular and, above all, shapeable.

During the development of the course, the most important considerations were the following:

- To make differential geometry "digestible" to an extent that does not come at the expense of the expected mathematical accuracy. It means that the course required considerable Pedagogical Content Knowledge to transform advanced disciplinary concepts into accessible, learnable representations.
- The students' visual mode of thinking (visual thinking), form recognition, and form-finding need to be strongly developed.
- The creative freedom of the students, as future architects, should increase significantly.

3 Conceptual principles

3.1 Levels of geometric thinking

According to the well-known Van Hiele model (Herbst et al., 2018; introduced by van Hiele-Geldof and van Hiele in 1957), geometric thinking has five levels:

1. Visualization (Level 0);
2. Analysis (Level 1);
3. Abstraction (Level 2);
4. Deduction (Level 3);
5. Rigor (Level 4).

When applying this model at a *micro level*, specifically to differential geometry, students enrolling in the course typically start from approximately the second level. The reason is that students usually acquire the necessary fundamentals of constructive geometry (descriptive geometry) as well as single- and multivariable calculus in the first two years. It follows directly that teaching the elective course presented in this paper is recommended at the end of BSc studies but preferably within an MSc program. In an optimal case, a student who successfully completes the course can reach the fourth level.

According to Van Hiele, to reach each level, students need to progress through certain learning phases:

1. Information/Orientation (Phase 1);
2. Guided orientation (Phase 2);

3. Explication (Phase 3);
4. Free orientation (Phase 4);
5. Integration (Phase 5).

As for the first phase, students know the necessary basics from their previous studies to about 80–90%. The second and third phases are key tasks of the lecture and the practical classes. The fourth and fifth phases can be realized through extra exercises available in the practical classes, as well as through the take-home project assignments. Some possible methods are presented in the following.

3.2 Briefly on methodological principles

Competencies can never be developed in the absence of adequate professional foundations; therefore, one of the hidden objectives of the course is to develop purely mathematical skills. The explicit objective is to increase students' ability in form recognition and form-finding, so that the future architect's creativity is not curtailed by a lack of this knowledge.

One of the main pedagogical methods of the practical classes is Cognitive Scaffolding: to provide support to students that enables them to solve complex tasks that they would not yet be able to solve on their own. The other didactic method applied is Cognitive Load Theory (Sweller et al., 2011): the interrelated elements of the theoretical part are taught in a consolidated block, in an easily accessible form. The topics are built on each other, and within each topic the principle of gradual progression is strongly present. This can be implemented through easier, decomposed tasks within each topic, followed by increasingly difficult tasks (partly building on previous ones). At the end of the semester, students must complete a project assignment in which they can work fully independently, while applying effectively the methods learned during the course.

3.3 Advantages of using dynamic software

To solve the tasks, we use GeoGebra (International GeoGebra Institute, online), a freely available program that is known in many parts of the world. Before the 3D version of the software, it could mainly be used effectively in primary and secondary school classes. GeoGebra's 3D commands and Computer Algebra System (CAS) integration have reached a level in recent years that makes them truly suitable for supporting a university course: it can handle surfaces given in parametric (possibly implicit) form.

GeoGebra is extremely effective in this course because it intentionally deprives students of the convenience provided by the classic design software used in architecture

– it displays only what has been mathematically defined. In short: GeoGebra is an excellent didactic development tool in higher education. It can develop students' competence to use "convenient" design software later with deep understanding. The real strength of the software, however, lies in its fully dynamic nature: all initial input data (coordinates of objects, parameters of equations, etc.) can be modified – and this is nothing other than the basis of parametric design.

4 Methodological implementation

4.1 A possible syllabus

The basic principle of the syllabus is to discuss each important surface type used in architecture in much greater depth than basic instruction allows. An essential consideration was that the topics should support conceptual understanding and form recognition in design.

Alongside the classic monograph by Pottmann et al. (2007), *Architectural Geometry*, we aimed to create a kind of didactic bridge to Lastra's book *Parametric Geometry of Curves and Surfaces* (Lastra, 2021; Pottmann et al., 2007). In the former, the parametric or implicit descriptions of curves and surfaces appear only at the level of mention, without practice exercises and skill development – this is obvious, since that is not the aim of the book. The latter may be an optimal textbook in the foundation years of a PhD program, due to its more substantial mathematical content. (Interestingly, Lastra's book (Lastra, 2021) also contains figures created with GeoGebra.)

Considering the earlier methodological principles and geometric knowledge that can be applied effectively in architecture, the following syllabus – roughly divisible into weeks – can be built, assuming one one-hour lecture and one two-hour practical class per week:

- Fundamentals of curve theory (second-order curves);
- Fundamentals of surface theory (second-order surfaces);
- Surfaces of revolution;
- Translational surfaces;
- Ruled surfaces;
- Developable surfaces;
- Helical surface;
- Pipe surfaces (briefly);
- Freeform curves (Bézier, B-spline, NURBS);
- Freeform surfaces (Bézier, B-spline, NURBS).

Students' deepening of knowledge can be supported by lecture notes expanded with exercises (Pék, 2023a), as well as by a collection of dynamic GeoGebra files (Pék, 2023b).

4.2 Theoretical foundations

As mentioned earlier, we assume the presence of so-called threshold concepts. In differential geometry, for example, the fundamental data of a surface are precisely such threshold concepts; therefore – in our view – their summarization, review, and especially their deepening, following the principle of spiral teaching, is indispensable.

It is worth addressing, at the beginning of the course, a more thorough presentation of curve- and surface-theoretical knowledge compared to previous studies. Here, basic concepts of curve theory (Frenet frame, curvature, torsion) can be discussed in detail, as well as the fundamentals of surface theory (tangent plane, normal, surface curves, principal directions, principal curvatures, Gaussian and mean curvature, classification of surfaces). This knowledge can also be presented with mathematically precise formulas; however, the related theoretical theorems and proofs can be omitted. The main reason is that, although due to the practice-oriented approach it is important that the classic formulas be available in a few sections, it is desirable to stop at this point in terms of mathematical depth. A thematic collection of formulas is also useful because students have at their disposal a reliable resource that they can later apply, for example, with the help of Wolfram Alpha. Instead of a precise derivation of proofs, the main objective is to develop an appropriate way of thinking. This goal is served extremely well by GeoGebra, where dynamic 3D figures provide the "intuitive proof".

In the later sections of the course, the main (primarily parametric, more rarely implicit) descriptions of the individual surface types can be introduced. The parametric form is more advantageous because most computer-aided design programs support this input mode. Moreover, the correct specification of parameter domains provides one of the pillars of later independent and creative surface shaping ability.

The general form of surface types common in architecture often requires only a geometric concept on the part of the designer. Therefore, it is sufficient if the theoretical block is limited to writing down these formulas, while discussing – at the necessary level of detail – the ideas that lead to them. Despite the apparent simplification, this is a very important phase from a pedagogical point of view, because at this stage students can learn, in a latent way, pure (but not self-serving) mathematics, while the motivation arises from natural interest.

For example, in the topic of Surfaces of revolution, it is not problematic to make students understand a parametric description of the surface by examining the com-

ponent functions. In the case of rotating a plane curve $c(u) = (c_1(u), 0, c_2(u))$, $u \in I \subset \mathbb{R}$ in the xz -plane about the z -axis, the parametric form of the surface is:

$$F(u, v) = (c_1(u) \cdot \cos v, c_1(u) \cdot \sin v, c_2(u)), \quad (1)$$

where $u \in I, v \in [0, 2\pi]$ (see Fig. 1).

In the theoretical lecture it is worthwhile to devote time to analyzing the above formula: how we obtain a surface of revolution from a meridian curve, how we can determine a parallel circle at a given height, etc.

This is also where the classification of surfaces known from previous studies can be revisited. Staying with the example above, with students' active involvement, all surfaces of revolution already known can be listed. From a didactic point of view, this is once again an application of the principle of spiral curriculum.

4.3 Types of exercises

The practical exercises can be divided into two main types: a set of examples aimed at form-finding and a set aimed at form recognition. Accordingly, in both types we can proceed from tasks that initially require merely the algorithmic application of the formula learned in the theoretical part, towards tasks that cultivate independence and require complex and creative solutions (see Van Hiele's Phase 4), with a gradual increase in difficulty.

Form recognition and form-finding are key competencies for a future architect. This is the ability that is currently a human privilege; artificial intelligence operates in these tasks in a limited way and very often incorrectly. In the practical classes, it is worth devoting time to analyzing a building or form first independently by the instructor, later involving the students, so that eventually the students become able to reconstruct a building (or building detail) individually from photographs (possibly videos).

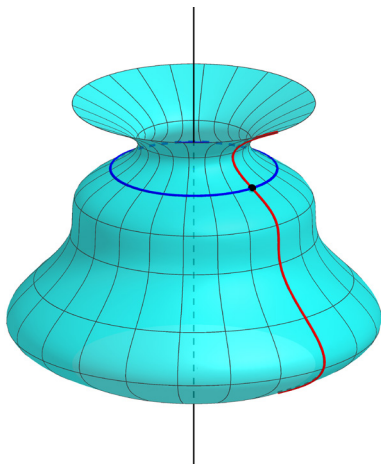


Fig. 1 A surface of revolution

The gradual development of students in small steps can be achieved within each topic by increasing the difficulty of the tasks; while the complex deepening of knowledge is realized by building the topics on each other and increasing their complexity. A suitable example may be the following task (Figs. 2 to 4). Approximate the roof structure of the building shown in Fig. 2 with a ruled surface.

In the spirit of differentiated learning, after some independent thinking, the following addition may be provided: a circle of radius r , centered at the origin, in the xy -plane rotated about the x -axis by an angle φ :

$$c(u) = (r \cdot \cos u, r \cdot \sin u \cdot \cos \varphi, r \cdot \sin u \cdot \sin \varphi),$$

where $u \in [0, 2\pi]$.

It is not a problem – indeed, in some cases it can even be an explicit expectation – that students approximate the shape of a building detail with simpler forms; this can facilitate the phases of design and, especially, construction. Deepening students' knowledge can be greatly supported if, in these tasks, they individually vary the parameter domains and other running parameters, because this leads to the desired pedagogical goal: to teach students the mathematical background of parametric design software.



Fig. 2 An example of form-recognition exercise 1 (detail, source: BIG, 2023; edited by author)

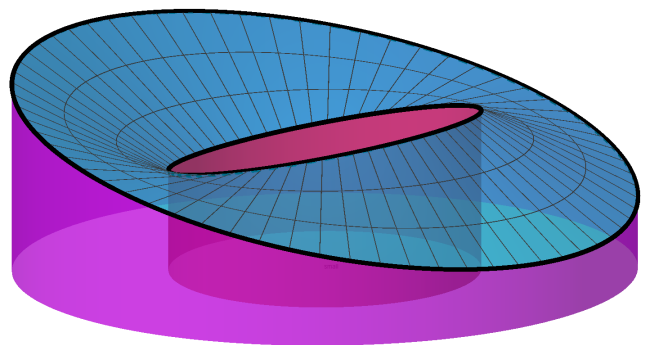


Fig. 3 An example of form-recognition exercise 2

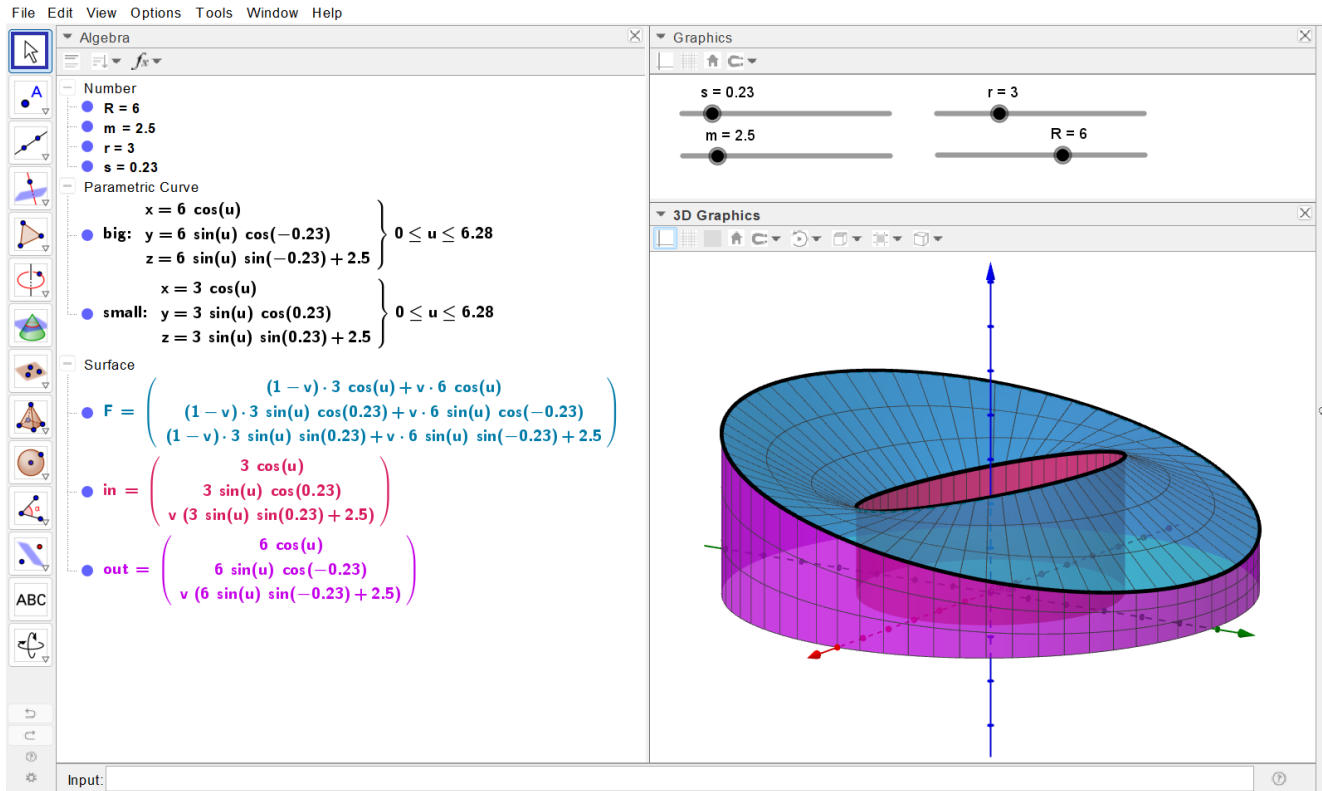


Fig. 4 An example of form-recognition exercise 3

In tasks requiring form-finding ability, the goal is that the future architect can design buildings with special shapes even when working under certain constraints. This is an obviously recurrent problem during design; therefore, these tasks require the application of classic engineering skills. Remaining with the topic of ruled surfaces, we present an example. Four points are given:

$$A = (7, 8, 1), \quad B = (3, -4, 7), \\ C = (-4, 10, 4) \quad \text{and} \quad D = (-8, -2, 0).$$

Determine a parametric form of the saddle surface defined by the skew quadrilateral $ABDC$. What can we say about the axis direction of the surface (see Fig. 5)?

An additional task can be excellently built upon this exercise, where, for example, the intersection curve with a cylinder of vertical axis must be determined (see Van Hiele's Phase 5 and Level 4).

In every task type, it can be required that students compute the parametric forms of the surfaces on paper, and only in the case of "fine-tuning" may intuitive modification of the parameters be allowed – these are the points where the hidden development of mathematical competencies takes place, because this is the key to a deeper understanding of the changeability of curves and surfaces. If this development is successful, students can modify surfaces not only

aesthetically but also in terms of their geometric and physical properties; this, in turn, directly enhances their creativity.

4.4 Tests

For comprehensive assessment, two mid-semester tests are recommended, in which the questions include both task types: form recognition and form-finding. To identify talented students, it is also advisable to ask a question that is not discussed during the practical classes but can be inferred from the course material.

In this course, students may use all course materials and the internet to solve the test tasks, since the goal is not to check memorisation of formulas but to measure their effective and inventive application.

4.5 Project assignment

To reach Rigour, i.e., the highest level, students also receive an independent project assignment. The project is the complete elaboration of an already existing or an invented (but possibly reality-inspired) building/building detail in GeoGebra (International GeoGebra Institute, online), as well as its *mathematical* documentation, i.e., including the necessary detailed calculations. The didactic goal is explicitly to strengthen individual thinking, activate a mathematical mindset, and apply what has been learned in a complex and

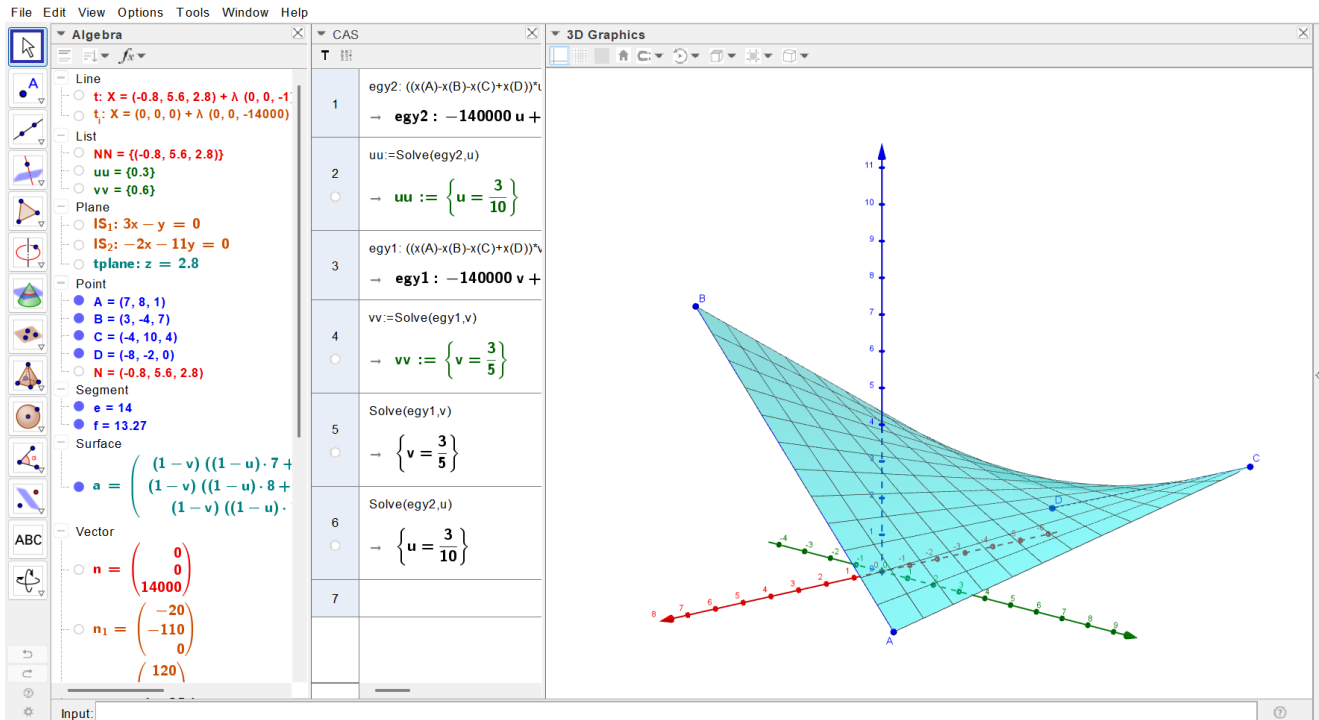


Fig. 5 An example of form-finding exercise

creative manner. Based on experience, students' development appeared most strongly in this task – in most cases, they experimented with significantly more complex forms than what they had seen and learned during the semester.

Interestingly, several pieces of feedback were received from students both during consultations and in the anonymous course evaluation that it was during the solution of the project assignment that they understood what they had learned during the semester in a complex way – that is, they reached the level of conceptual change and consolidation, and their knowledge became transferable.

5 Results

The course presented in this paper has been offered so far on two occasions, in both cases with positive feedback (Table 1). Several students continued their studies in a PhD program; some of them regarded this course explicitly as preparatory studies for it.

In what follows, we would like to present a few student works, preserving the anonymity of their authors. (The students consented to the research use of their project assignments.)

In the first example (Fig. 6), the student employed a variety of surface types, the majority of which are surfaces of revolution or generalized helical surface. The design was inspired by a futuristic building featured on social media. Reconstructing such a model requires advanced

Table 1 Some results

	First offering*	Second offering
Number of students enrolled	18	8
Average student grade (1–5)	4.39	4.25
Number of anonymous evaluations submitted	7	7
Anonymous evaluation of the lecture (Likert scale, 1–6)	6 (100%)	6 (100%)
Anonymous evaluation of the practical class (Likert scale, 1–6)	6 (100%)	6 (100%)

* In the first offering, twice as many students could enroll in the course as in the second offering.

form-recognition competencies to perform a high-level geometric analysis of the visual inspiration.

The second case (Fig. 7) comprises an ensemble of various ruled surfaces, specifically cylinders and conoids. The student's proficiency in form-finding is evidenced not only by the structural complexity but also by the deliberate selection of parameter domains. Through the application of non-trivial conoids, the resulting surfaces exhibit a quasi-freeform aesthetic.

The subsequent project (Fig. 8) required rigorous mathematical preparation, as the central and internal components of the surface assembly are joined with tangent continuity (C^1). Furthermore, certain generatrices of the outer, wave-like surface function as tangents to the central surface at specific points.

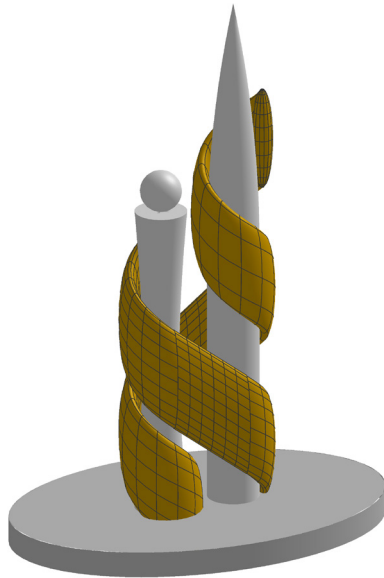


Fig. 6 An example of students' project works No. 1

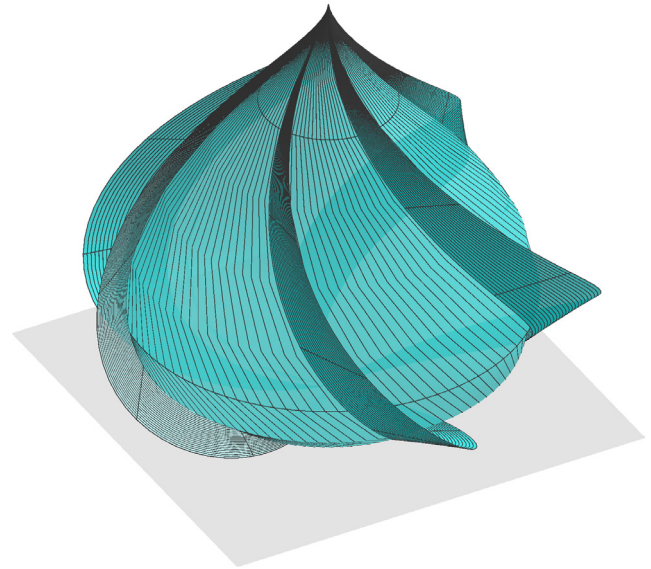


Fig. 9 An example of students' project works No. 4

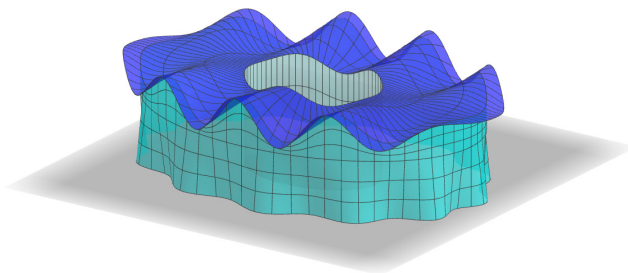


Fig. 7 An example of students' project works No. 2

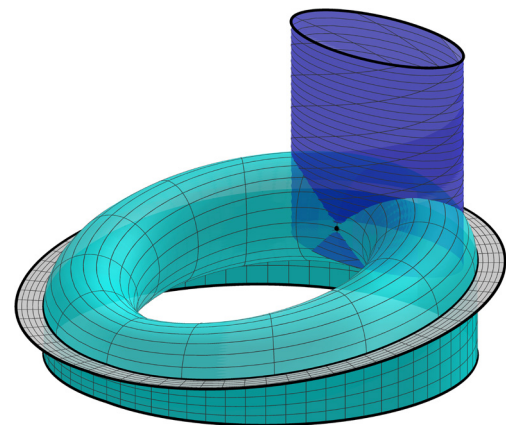


Fig. 10 An example of students' project works No. 5

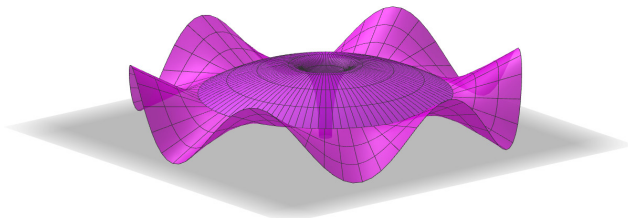


Fig. 8 An example of students' project works No. 3

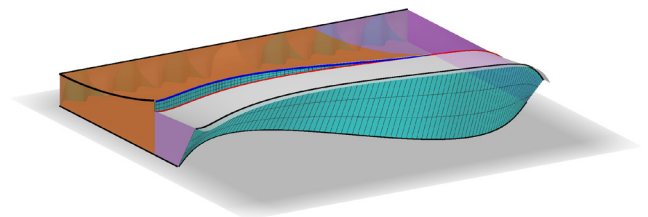


Fig. 11 An example of students' project works No. 6a

The fourth case (Fig. 9) features a seemingly simple surface of revolution; however, the external "wing-like" elements originate from specific parameter lines of the primary surface. The boundary parameter lines of these external components intersect at two points located on the axis of the surface of revolution – a construction requiring both mathematical precision and advanced form-finding capabilities.

Fig. 10 illustrates the intentional application of ruled surfaces and the sophisticated application of variable-radius canal surfaces. The student demonstrated significant technical proficiency by managing the intersection of this complex, tapering toroidal surface and an elliptical cylinder, requiring precise control over the respective parameter domains.

The final example (Figs. 11 and 12) demonstrates the construction of an entire building complex through the

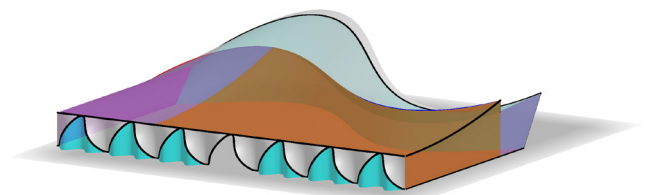


Fig. 12 An example of students' project works No. 6b

multiple application of conoids and ruled surfaces defined by two directrices. By strategically selecting the directrices of these diversely generated surfaces, a freeform effect is successfully achieved.

The originality, richness of detail, and differential-geometric precision (such as the smooth transition between surfaces) of the submitted projects were remarkable, particularly given that none of the students had previously been able to represent surfaces – beyond standard primitives like spheres or cones – parametrically using GeoGebra (International GeoGebra Institute, online).

We would like to highlight the fact that students modified the parameter domains and the parameters determining the shape of the surface with mathematical awareness; they selected the different surface types based on geometric considerations. The construction of the building or form was no longer purely aesthetic but proceeded along differential-geometric principles.

6 Conclusion

Based on the project works seen above and on our experiences during teaching the course – if we consider this as action research – we believe that teaching

differential geometry is not impossible for architectural engineering students. The lack of an abstract mathematical background is a surmountable obstacle, and the use of GeoGebra (International GeoGebra Institute, online) can bridge the gap between abstract differential-geometric concepts and architectural form-making. If we avoid foundations that are too precise – almost self-serving – for engineering thinking, and if we omit proofs, more time remains for visualisation and practice-oriented parameterisation.

A methodological key, therefore, may be to introduce an application-centred perspective into the syllabus as early as possible, and to set tasks that encourage creativity. As a result, we can provide the world with young architects for whom the lack of differential-geometric foundations is no longer a hindering factor and does not limit their creativity; rather, the acquired geometric knowledge is integrated deeply and in a complex way into their competencies.

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