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RESEARCH ARTICLE

Geometric sensitivity of statically determinate planar truss families

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Abstract

We define a truss family T_i by a statically determinate truss T_0 and a recursive step $T_{i+1}=f(T_i)$, such that step $f(T_i)$ inserts new joints and bars, while it keeps static determinacy. Such recursive algorithms have been broadly discussed in the literature, e.g. the Henneberg operation 1 is a well-known example. Earlier we introduced the concept of geometric sensitivity index r^g of trusses, here we investigate the sensitivity of truss families, in particular, the limit sensitivity $\lim_{i\to\infty} r^g(T_i)$.

Keywords

geometric sensitivity · Henneberg operations · method of substitute members · topology of trusses · minimal rigidity · truss types

Acknowledgments

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1 Introduction

Generating algorithms for statically determinate trusses was first discussed by Henneberg [5] who proved that each of such trusses can be generated by the repetitions of the so-called Henneberg operations H1 and H2. We discuss this algorithm, another universal generating algorithm, and some other generating operations in section 2.1. The Henneberg algorithm also divides statically determinate trusses into two, disjoint and complementary mathematical classes: *simple trusses* can be constructed solely by applying H1; all other trusses are called *compound trusses*. In this paper we define families of trusses characterized by a single (discrete) parameter *i* and a recursive scheme $T_{i+1}=f(T_i)$. Only such families are investigated, in which each member of a family belongs to the same class, i.e. we can speak of *simple families* and *compound families*. In our earlier works ([7], [15], [16]) we defined the (scalar) *geometric sensitivity index* $0\le r^g\le 1$ associated with a truss; here (in section 3) we extend this definition for families. In particular we investigate the limiting value $\lim_{i\to\infty} r^g(T_i)$ and we show that for simple families $\lim_{i\to\infty} r^g(T_i) \le 0.5$.

Trusses have been classified by engineers based on various properties [4]. While these classifications are not always rigorous in the mathematical sense, they are extremely useful to understand the mechanical behaviour of the truss. In section 2.2 we discuss one of these engineering classifications: the classification based on truss topology, yielding *topology types*. These types do not necessarily belong to the same class (i.e. class of simple or compound trusses), and they are not necessarily disjoint sets, however, their overlap is at most one truss. Families may or may not belong to any type, however, if more than one member of a family belongs to a given type, then so do all the other member.

We use the following assumptions: trusses are supported by links which we call *external bars*. Joints on the fix ground are joined with exactly one external bar and these joints are called *external joints*. The rest bars and joints are called *internal bars* and *internal joints*, respectively. In order to be able to handle uniformly the supported and unsupported trusses, we use the concept of *minimal rigidity* instead of static determinacy. We call a truss minimally rigid, if it is either statically determinate, or it has no external bars, but one can make it statically determinate by adding 3 external bars. We call a problem a *typical truss problem* [15], if it fulfils three criteria:

1. the axes of the forces running to the single joints (i.e. bar forces and loads) are not collinear
2. all bar forces (calculated by linear theory) differ from zero
3. the above criteria are fulfilled also in the case that the geometry or the load is perturbed slightly.

In our investigations - according to the most widely used definitions of a truss (e.g. see [10]) - all bars can stand both compression and tension. However, we mention, that some engineering literature also range tensegrity structures into the class of trusses (such a structure is studied in [11]), although these structures contain slender cables, which cannot stand compression.

2 How to generate trusses, truss families?

2.1 Universal algorithms, generating operations and classification of trusses.

Henneberg gave the following universal algorithm for generating – unsupported - minimally rigid trusses [5]:

A planar framework is minimally rigid if and only if it can be constructed from one rod by the following two operations:

- H1 operation: add a new joint *z* and connect *z* to two distinct existing joints by rods (figure 1a),
- H2 operation: subdivide an existing rod *u-v* by a joint *z* and connect *z* to an existing joint distinct from *u* and *v* (figure 1b).

By applying these operations one has to avoid constructing an infinitesimal mechanism (in [13] Müller-Breslau gives a wide range of these structures). In case of H1 there is solely one condition to fulfil: the axis of the added bars should not be collinear [2]. However, by applying H2, more complex geometric investigations are needed.

Trusses, which can be generated by applying solely H1 operations, are called *simple trusses* [3], [6]. Observe, that in these trusses, bar forces can be simply calculated by the joint method. Simple trusses can be constructed in two ways by applying H1 operations:

- setting out from the fix ground we generate a supported truss (figure 2a) [3],
- setting out from a bar we generate an unsupported truss, and we either leave it unsupported or we support it with 3 external bars (figure 2b) [3].

The rest of the trusses are called *compound trusses* after Csonka [3]. Observe, that in the case of supported trusses, the following operation may also be considered as an H2 operation:

- subdivide an existing bar *u-v* of a supported truss by a joint *z* and connect *z* to a new external joint (figure 1c).

In this paper we use the above truss classification (i.e. class of simple and compound trusses), however we mention, that other

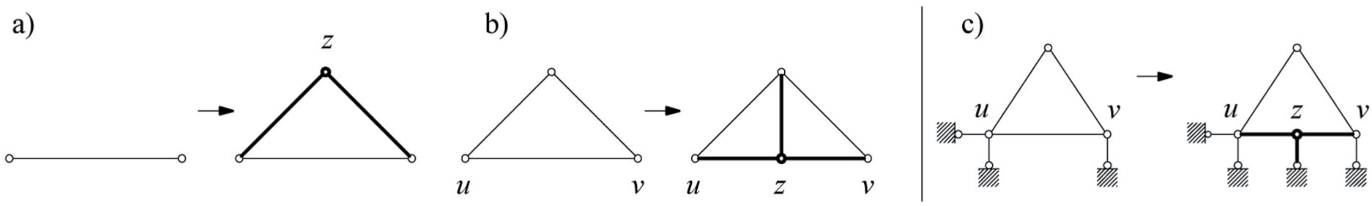


Fig. 1. H1 operation (a) and H2 (b) operation. Another operation (c), which is equivalent to H2 in the case of supported trusses.

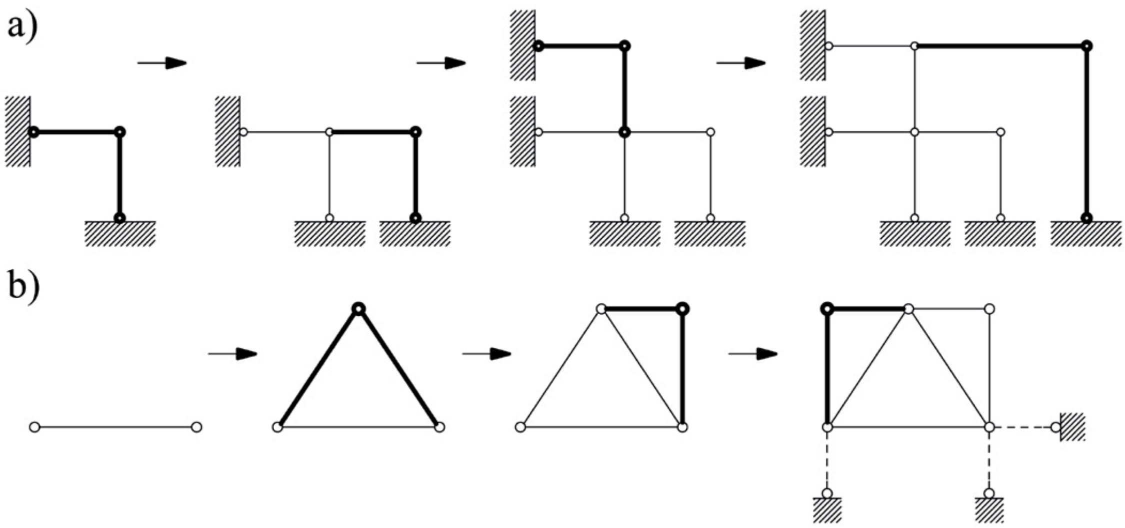


Fig. 2. Two ways of generating simple trusses.

classifications also exist. For example Hibbeler distinguishes three classes [6] (see figure 3):

a) a *simple truss* can be built up solely by H1 operations

b) a *compound truss* cannot be built up solely by H1 operations, and it is formed by connecting two or more simple trusses together. Hibbeler describes three ways to form a compound truss:

- joining two simple trusses by a joint and a bar (see figure 3b*);
- joining two simple trusses by three bars (see figure 3b**);
- substituting some bars of a large simple truss (called *main truss*) by simple trusses (called *secondary trusses*) (see figure 3b***)

c) a *complex truss* cannot be classified either simple or compound.

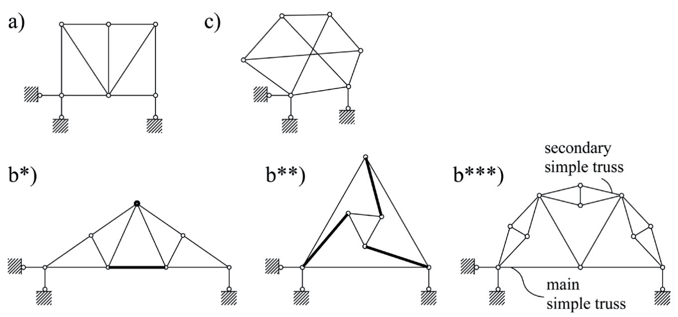


Fig. 3. Examples for the truss classes after Hibbeler: simple truss (a), compound truss (b figures), complex truss (c).

Müller-Breslau gave a method in [13] equivalent to Henneberg's algorithm. Based on his method one can create a universal algorithm, which is suitable for both supported and unsupported trusses. He showed that an arbitrary, minimally rigid, compound truss can be converted to a simple truss by applying the *method of substitute bars* in a suitable number. The following operation is called *bar-substitution*: remove a bar of the minimally rigid truss, and replace it by another bar elsewhere in the truss in such a way, that the truss remains minimally rigid. Since bar-substitution can be reversed [3], thus, H1 and the bar-substitution also create a universal algorithm. Two bars substitute each other in terms of rigidity if and only if [10]:

a) the relative motion of the truss, which would be caused by a zero couple applied in the position of one of the bars, can be stopped by the other bar

b) a zero couple applied in the position of one of the bars induces a bar force in the other bar.

Note that bar-substitution (contrary to the above operations) does not increase the number of the joints and bars of the truss, but it changes the adjacency relationships between the joints, i.e. it varies the *topology* of the truss. The topology of a truss can be described by the *adjacency* (or *topology*) matrix $\mathbf{A}$ defined as follows:

$a_{ij}=1$, if joint i and joint j are adjacent,

$a_{ij}=0$, if not,

$a_{ii}=1$.

We mention that in case the first criterion of the typical truss problem explained in Section 1 is fulfilled, one can replace the above condition (b) with a condition, which does not require any bar force calculation. This condition contains the concept of *rigid core* [15], which can be obtained solely from the topology of the truss. We define the rigid core of a subset $R \subset T$ of a minimally rigid truss T (R is a set of joints) as the smallest, minimally rigid subset M(R) such that $R \subset M(R) \subset T$. It was proven in [15] and in [7] that an equilibrium load applied on an H set of hinges (such that $H \subset T$) induces bar forces exactly in the rigid core of H (in case the first criterion of the typical truss problem is fulfilled). Thus, condition (b) can be replaced by condition (b'):

b') two bars substitute each other in terms of rigidity if and only if the rigid core of the endpoints of the removed bar contains the substituting bar.

Besides those operations, which are suitable for creating universal algorithms, some further operations also exist, e.g. the operations introduced earlier at the Hibbeler classification (see figures 2b). In figure 4a-d we show some more of them:

a) joining three trusses by three joints [3]

b) X-replacement: we replace a bar-crossing to a joint [3], [17]

c) gluing two trusses along a bar [17]

d) vertex-splitting [17]:

d1) double a joint *z* and two bars connected to *z*, and distribute the rest of the bars connected to *z* among *z* and the new joint *z*'

d2) double a joint *z* and one bar connected to *z*, distribute the rest of the bars connected to *z* among *z* and the new joint *z*', and add the bar *z*-*z*'.

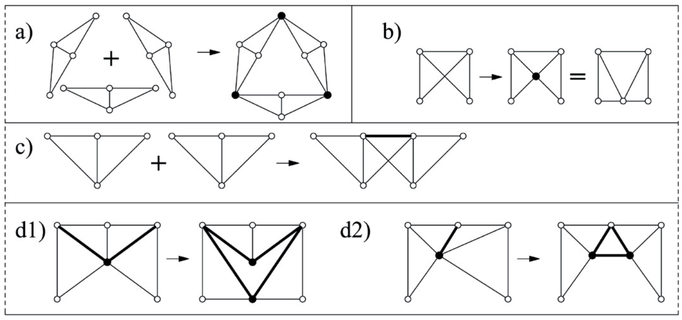


Fig. 4. Some further generator operations.

2.2 Statically determinate topology types of trusses

Trusses have been also classified by engineers based on truss topology, yielding topology types [4]. These types do not necessarily belong to the same mathematical class (i.e. simple or compound class). For example types shown in figure 6 belong to the simple class, on the other hand, types shown in figure 5 have simple and compound elements as well. What is more, types are not necessarily disjoint sets, however, their overlap is at most one truss (for example the truss shown in figure 8 in the second column and in the second row is a Pratt truss and a Warren truss with verticals at the same time). We discuss the main statically determinate topology types [4],

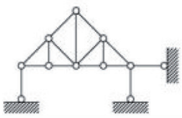
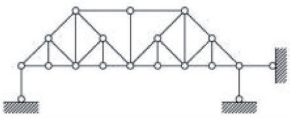
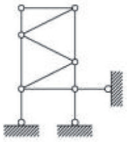
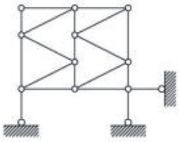
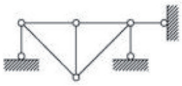
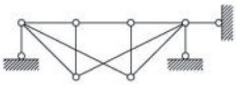
truss types	truss samples for the truss type	
1. subdivided		
2. double K		
3. Bollman		

Fig. 5. Topology types which also have simple (see the first samples) and compound (see the second samples) truss representatives.

[8], [9] for the reason that some truss families investigated in the present paper belongs to these types.

In figures 6 and 7 we introduce and illustrate by realised structures the most common statically determinate topology types. Figure 6 shows types belonging to the class of simple trusses (in case the type has a different expression in English and German engineering literature [14], we give the German expression in brackets):

a) Pratt and Howe truss (N-Fachwerk / Pfostenfachwerk / Ständerfachwerk)
b) Warren truss (V-Fachwerk / Strebenfachwerk)
c) Warren truss with verticals (WM-Fachwerk / Strebenfachwerk mit Hilfspfosten)
d) K-truss
e) statically determinate double Warren truss (statisch bestimmtes Rautenfachwerk ohne Hilfspfosten)
f) Fink truss

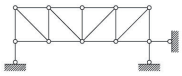
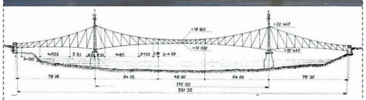
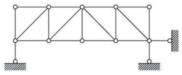
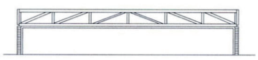
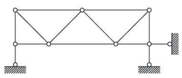
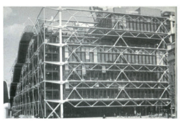
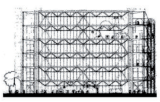
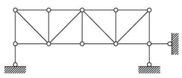
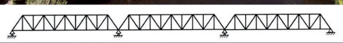
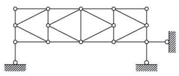
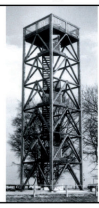
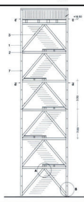
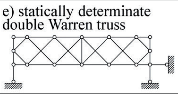
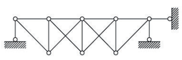
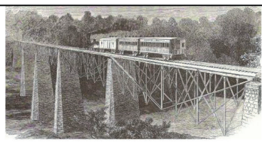
truss type	realized truss sample photo and structural form	main data
a1) Pratt truss 	 	name of the building/structure: Szabadság híd place: Budapest, Hungary architect/engineer: János Feketeházy year: 1896 material: cast iron span: 171 m number of trussbars: 193 sources of pictures: http://commons.wikimedia.org/wiki http://forum.index.hu/Article/showArticle?go=48730274&t=9023878
a2) Howe truss 	 	name of the building/structure: Airplane hangar place: Locarno, Switzerland architect/engineer: Giacomazzi@Associati Architetti year: 1996 material: timber and steel span: 40 m number of trussbars: 25 source of pictures: Herzog, Natterer, Schweitzer, Winter, Volz: Holzbau Atlas (Edition Detail), 2003, Kösel Gmbh & Co. KG, Kempten
b) Warren truss 	 	name of the building/structure: Pompidou Center place: Paris, France architect/engineer: R. Piano, R. Rogers year: 1977 material: steel span: 48 m number of trussbars: 27 sources of pictures: Vámossy Ferenc: Az építészet története - A Modern Mozgalom és a későmodern, Nemzeti Tankönyvkiadó Budapest, 2002 http://ferb.buzznet.com/user/photos/pompidou-center-section/?id=1983370
c) Warren truss with verticals 	 	name of the structure: Railroad bridge in Tokaj place: Tokaj, Hungary architect/engineer: no data year: no data material: steel span: 3*68 m number of truss bars: 3*37 source of picture: http://vasutihid.hu/legifoto.html
d) K-truss 	 	name of the building/structure: Look-out tower place: Venne, Germany architect/engineer: Hochbauamt Osnabrück year: 1976 material: timber hieght of the tower: 18 m number of truss bars: 36 source of pictures: Herzog, Natterer, Schweitzer, Winter, Volz: Holzbau Atlas (Edition Detail), 2003, Kösel Gmbh & Co. KG, Kempten
e) statically determinate double Warren truss 	sample not found	
f) Fink truss 	 	name of the structure: Green River Railroad bridge place: Kentucky, USA architect/engineer: A. Fink year: ~1860 material: cast iron span: no data number of truss bars: 59 sources of pictures: http://bonnievilleky.blogspot.com/2010/04/old-pillar-over-green-river-in.html http://en.wikipedia.org/wiki/File:Bridges_20.png

Fig. 6. Topology types belonging to the class of simple trusses.

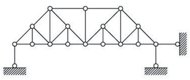
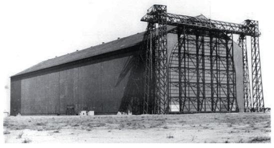
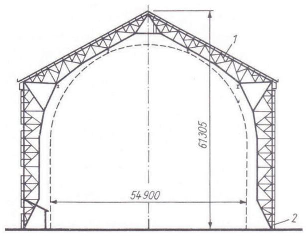
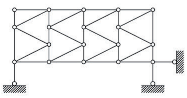
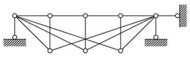
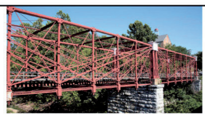
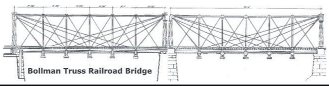
truss type	realized truss sample	
	photo and structural form	main data
a) subdivided truss 	 	name of the building/structure: Airship Hangar place: Karachi. Pakistan architect/engineer: no data year: 1925 material: steel span: 70 m number of trussbars: 303 sources of pictures: http://www.airshipsonline.com/sheds/images/krchshed.jpg O. Büttner, H. Stenker: Stahlhallen - Entwurf und Konstruktion, VEb Verlag für Bauwesen, Berlin, 1986
b) double K-truss 	 	name of the building/structure: "K-híd" place: Budapest, Hungary architect/engineer: FŐMTERV year: 1955 material: steel span: 98 m number of trussbars: 272 source of picture: http://hu.wikipedia.org/wiki/K-h%C3%ADd
c) Bollman truss 	 	name of the structure: Bollman bridge place: Savage, Maryland, USA architect/engineer: W. Bollman year: 1852 material: iron span: 2*49 m number of truss bars: 81 sources of pictures: http://fineartamerica.com/featured/bollman-truss-bridge-at-savage-in-maryland-william-kuta.html http://www.historicbridges.org/maryland/bollman/

Fig. 7. Topology types, in which almost all elements of the type are compound trusses.

Figure 7 shows such types, in which almost all elements belong to the class of compound trusses:

a) subdivided truss
b) double K-truss
c) Bollman truss

3 Geometric sensitivity of truss families

In previous sections we showed the generating operations, from which the recursive algorithms of the families are built up; we introduced the truss classes, to which the families belong to, and we illustrated the topology types, among which some of the families can be ranged. Our investigations about *geometrical sensitivity* of the families concern typical truss problems. We begin the discussion with the explanation of the concept *geometric sensitivity*.

Due to minor manufacturing or constructional inaccuracies the location of some joints of a truss may differ slightly from the location originally designed. In case the location of an unloaded, not V-type, internal joint is perturbed in a loaded truss (such that all bar forces calculated by the linear theory differ from zero), the internal forces will change in a certain set of the bars. We call the joint perturbed an *imperfect joint* and the degree of the perturbation *geometric imperfection*. The set of those bars (and the joints connected to these bars), in which bar forces change due to almost every small dislocation of a denoted imperfect joint j, is called the *influenced zone* of joint j. The *geometric sensitivity matrix* $\mathbf{R}^g$ of a truss is defined as a matrix, which makes connections between the internal joints and their influenced zones in the following way:

$r^g_{ij}=1$, if the influenced zone of the internal joint j contains bar i,
$r^g_{ij}=0$, if not.

To measure the geometric sensitivity of a truss, we introduced [16] the scalar concept of *geometric sensitivity index* $0 \le r^g \le 1$, which can be obtained from $\mathbf{R}^g$ as follows:

$$r^g = \frac{1}{bn}\sum_{i=1}^{r}\sum_{j=1}^{n} r^g_{ij},$$

where b and n denotes the total number of bars and the total number of internal joints, respectively.

In earlier papers [16], [7] we showed that influenced zones can be calculated solely from the topology of the truss, thus, no bar force calculations are needed. We proved that in minimally rigid trusses - in case of typical truss problems - the influenced zone of a denoted joint corresponds to the rigid core of the *star* of the joint. The star of a denoted joint i consists of i and the bars and joints joined to i. We restrict that in case i is a V-joint, then the rigid core of the star of i equals zero. The above equality between the influenced zone of a denoted joint and the rigid core of the star of the denoted joint means, that $\mathbf{R}^g$ can be obtained in the following way:

$r^g_{ij}=1$, if the rigid core of the star of the internal joint j contains bar i,
$r^g_{ij}=0$, if not.

Since any rigid core can be determined solely by the topology of the truss, the matrix $\mathbf{R}^g$ and the index r^g can be too.

The geometric sensitivity of truss families can be characterized by the function $r^g(T_i)$ and its infinite limit $\lim_{i\to\infty} r^g(T_i)$. To obtain function $r^g(T_i)$, the r^g of the first k members of the family (where $5 \leq k \leq 10$) were calculated numerically on the basis of the topology: in each member of the family we determine the rigid cores of the stars of the internal joints, we fill in the matrix $\mathbf{R}^g$ according to (1), and we calculate the index r^g. After, the general formula for $r^g(T_i)$ was deduced from these results.

Figures 8 and 9 show simple and compound truss families, and their geometrical sensitivity, respectively. The following data are given in the columns of the figures:

- serial number of the family, and the topology type of it (if the family is a subset of a type)
- figure illustrating the topology of the initial truss T_0
- figure illustrating the topology of truss T_1
- recursive algorithm $f(T_i)$
- geometric sensitivity $r^g(T_i)$ of the members T_i
- limit sensitivity of the family $\lim_{i\to\infty} r^g(T_i)$.

We mention, that based on the function type of $n(i)$ (where n and i denote the number of the joints and the serial number of the family member, respectively) three different recursive algorithms took place in our examples: linear (e.g. see the first seven families in figure 8), power (e.g. see family No. 8 in figure 8), and exponential algorithms. In our examples all exponential algorithms generate fractal trusses [1], [12] (e.g. see family No. 7 and 8 in figure 9), however, this is not necessary.

Comparing the limit sensitivities of the truss families, we can observe, that the limit sensitivities of compound families adopt each possible value $0 \leq \lim_{i\to\infty} r^g(T_i) \leq 1$, while the limit sensitivities of simple families are at most 0.5. In our opinion it is associated with the fact, that the simple truss families can be constructed solely by H1 operations, while the rest of the families cannot.

4 Summary

In this paper the geometrical sensitivity of statically determinate planar truss families was investigated. A truss family is defined by an initial truss and a recursive step, which keeps static determinacy. These steps are truss generating algorithms, which contain generating operations. A literature overview was presented about the generating operations, and about universal truss generating algorithms. It was shown that the correct bar-substitution can be carried out by using the concept of rigid core. The families were ranged into the truss classes according to whether they can be built up solely by H1 operations or not. Beside this mathematical classification another – engineering – classification was discussed based on the truss topology types.

The formulas for geometric sensitivity $r^g(T_i)$ of truss families were determined in deductive way. Investigations showed, that while compound families may have limit sensitivities with all possible values $0 \leq \lim_{i\to\infty} r^g(T_i) \leq 1$; the simple families have limit sensitivities equal to at most 0.5.

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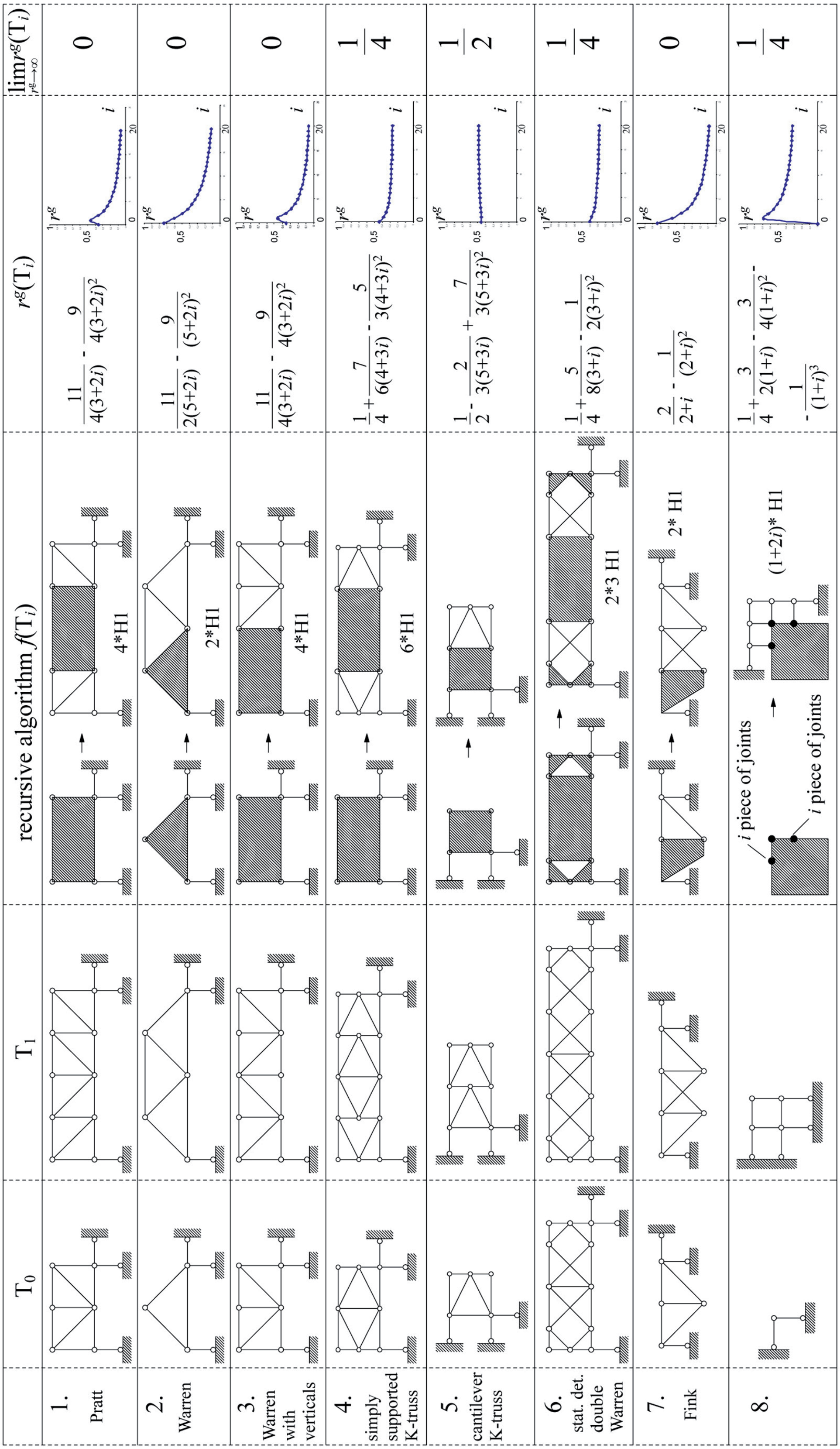

Fig. 8. Geometric sensitivity of simple truss families

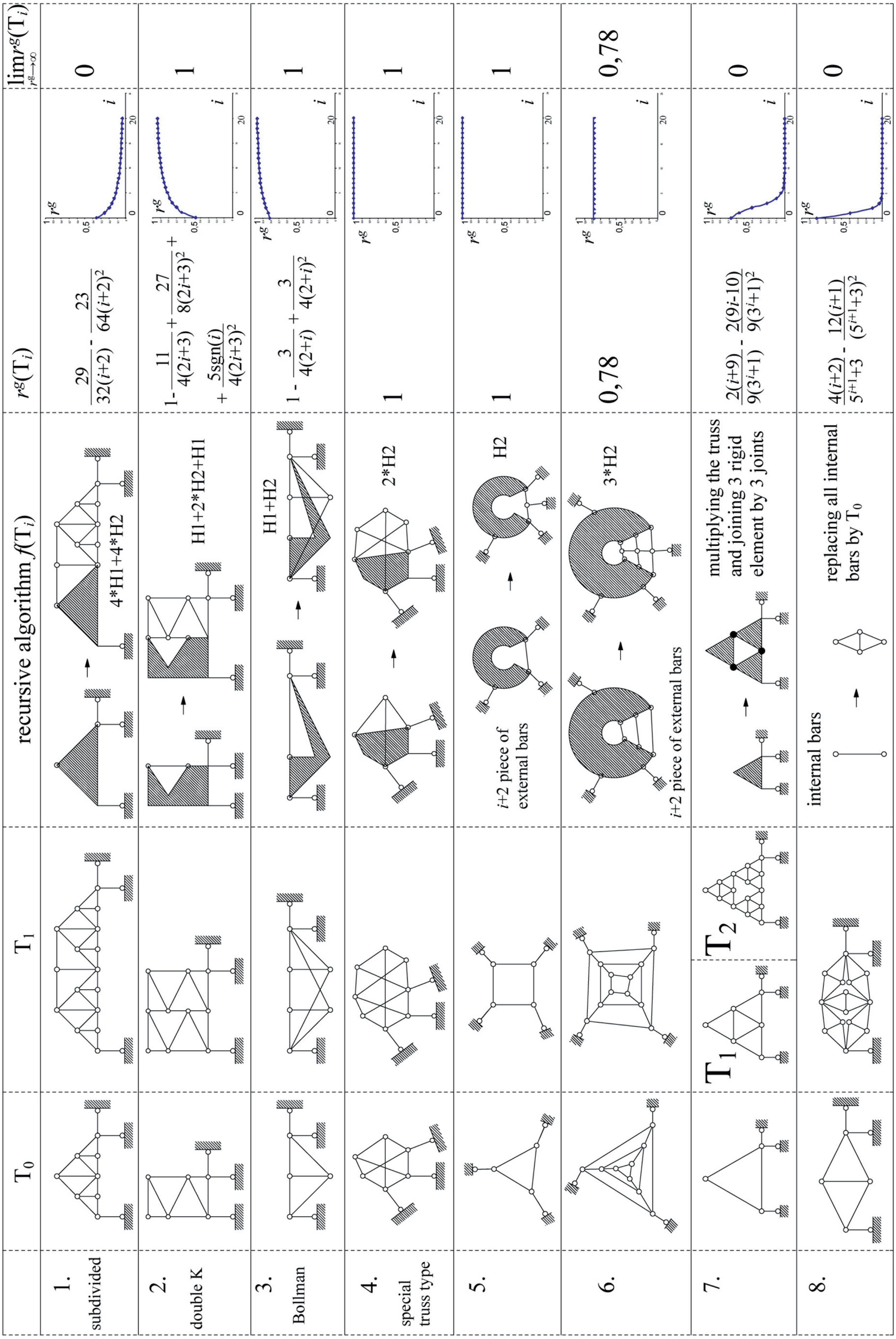

Fig. 9. Geometric sensitivity of compound truss families