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WAVE SOLUTIONS IN RHEOLOGICAL MEDIA[1]

Patrizia ROGOLINO

Department of Mathematics
University of Messina
Messina, Italy

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Abstract

The propagation of linear acoustic waves in isotropic media in which mechanical relaxation phenomena occur are investigated. The irreversible mechanical processes in the medium due to viscosity and to changes in a tensorial internal variable are analysed with the aid of non-equilibrium thermodynamics. In particular, in this paper, a wave solution applying the method of Laplace transform is proposed. Moreover, the corresponding solutions in Poynting–Thomson, Maxwell, Kelvin–Voigt, Hooke and Newton media are calculated.

Keywords: rheological media, wave solution, non-equilibrium thermodynamics.

1. Introduction

In some previous papers [1-13, 17-21] a theory has been developed for mechanical phenomena in continuous media based on non-equilibrium thermodynamics. Some authors have introduced the following flow laws for shear phenomena in isotropic media

$$\frac{d}{dt}\tilde{\varepsilon}^{(1)}_{\alpha\beta} = \eta_s^{(1,1)}\,\tilde{\tau}^{(1)}_{\alpha\beta} + \eta_s^{(1,0)}\,\frac{d}{dt}\tilde{\varepsilon}_{\alpha\beta}\,, \tag{1.1}$$

$$\tilde{\tau}^{(vi)}_{\alpha\beta} = \eta_s^{(0,1)}\,\tilde{\tau}^{(1)}_{\alpha\beta} + \eta_s^{(0,0)}\,\frac{d}{dt}\tilde{\varepsilon}_{\alpha\beta}\,. \tag{1.2}$$

The scalars $\eta_s^{(i,j)}$ $(i,j = 0,1)$ which occur in (1.1) and (1.2) are usually called phenomenological coefficient. The quantities $\tilde{\varepsilon}^{(1)}_{\alpha\beta}$, $\tilde{\varepsilon}_{\alpha\beta}$, $\tilde{\tau}^{(vi)}_{\alpha\beta}$ and $\tilde{\tau}^{(1)}_{\alpha\beta}$ are deviators of the partial inelastic strain tensor $\varepsilon^{(1)}_{\alpha\beta}$, of the strain tensor $\varepsilon_{\alpha\beta}$, of the viscous stress tensor $\tau^{(vi)}_{\alpha\beta}$, and of the affinity-stress tensor $\tau^{(1)}_{\alpha\beta}$. The flow laws (also called phenomenological equations) may be derived with the aid of non-equilibrium thermodynamics. See, for instance, ref.

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[14] or the first 6 sections of ref. [7]). The phenomenological coefficients $\eta_s^{(0,1)}$ and $\eta_s^{(1,0)}$ satisfy the Onsager–Casimir reciprocal relation

$$\eta_s^{(0,1)} = -\eta_s^{(1,0)} .$$

The deviator $\tilde{A}_{\alpha\beta}$ and the scalar part A of an arbitrary tensor field $A_{\alpha\beta}$ are defined by:

$$\tilde{A}_{\alpha\beta} = A_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta} \sum_{\gamma=1}^{3} A_{\gamma\gamma} , \tag{1.3}$$

$$A = \frac{1}{3} \sum_{\gamma=1}^{3} A_{\gamma\gamma} .$$

Hence,

$$A_{\alpha\beta} = \tilde{A}_{\alpha\beta} + A\,\delta_{\alpha\beta} \tag{1.4}$$

and

$$\sum_{\gamma=1}^{3} \tilde{A}_{\gamma\gamma} = 0.$$

In (1.3) $\delta_{\alpha\beta}$ is the unit tensor. If $A_{\alpha\beta}$ is symmetric $\tilde{A}_{\alpha\beta}$ is also symmetric and reversely. Within the usual procedures of non-equilibrium thermodynamics the following flow laws were obtained for mechanical relaxation phenomena in isotropic media [7-11]:

$$R^{(\tau)}_{(d)0}\,\tilde{\tau}_{\alpha\beta} + \frac{d}{dt}\tilde{\tau}_{\alpha\beta} = R^{(\varepsilon)}_{(d)0}\,\tilde{\varepsilon}_{\alpha\beta} + R^{(\varepsilon)}_{(d)1}\,\frac{d}{dt}\tilde{\varepsilon}_{\alpha\beta} + R^{(\varepsilon)}_{(d)2}\,\frac{d^2}{dt^2}\tilde{\varepsilon}_{\alpha\beta} , \tag{1.5}$$

where

$$R^{(\tau)}_{(d)0} = a^{(1,1)}\,\eta_s^{(1,1)} , \tag{1.6}$$

$$R^{(\varepsilon)}_{(d)0} = a^{(0,0)}(a^{(1,1)} - a^{(0,0)})\,\eta_s^{(1,1)} , \tag{1.7}$$

$$R^{(\varepsilon)}_{(d)1} = a^{(0,0)}(1 + 2\eta_s^{(0,1)}) + a^{(1,1)}\left[\eta_s^{(0,0)}\,\eta_s^{(1,1)} + (\eta_s^{(0,1)})^2\right] , \tag{1.8}$$

$$R^{(\varepsilon)}_{(d)2} = \eta_s^{(0,0)} . \tag{1.9}$$

In these expression $a^{(0,0)}$ and $a^{(1,1)}$ are scalar constants (thermodynamic parameters) which occur in the equations of state. A detailed derivation of

(1.5)–(1.9) is given in section 3 of ref. [6]. The following inequalities were proposed from stability considerations

$$a^{(1,1)} \geq a^{(0,0)} \geq 0;$$

and from the positive definite character of the entropy production

$$\eta_s^{(0,0)} \geq 0\,, \qquad \eta_s^{(1,1)} \geq 0;$$

hence, from the preceding inequalities, one may derive the following correlations:

$$R^{(\tau)}_{(d)0} \geq 0, \qquad R^{(\varepsilon)}_{(d)0} \geq 0, \qquad R^{(\varepsilon)}_{(d)1} \geq 0,$$

$$R^{(\varepsilon)}_{(d)1} - R^{(\tau)}_{(d)0}\, R^{(\varepsilon)}_{(d)2} \geq 0,$$

$$R^{(\varepsilon)}_{(d)1}\, R^{(\tau)}_{(d)0} - R^{(\varepsilon)}_{(d)0} \geq 0.$$

For a detalied discussion see section 5 of ref. [6], section 8 of ref. [7] or section 13 of ref. [14].

Let us denote the tensor of the total strain by $\varepsilon_{\alpha\beta}$. In this paper, we shall assume that the strain is small from a geometrical point of view, i.e.,

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\Big(\frac{\partial}{\partial x_\beta} u_\alpha + \frac{\partial}{\partial x_\alpha} u_\beta\Big)\,, \tag{1.10}$$

where u_1, u_2, and u_3 are the components of the displacement field.

Moreover, we shall assume that the deformations are small from a physical point of view, i.e., the stress tensors $\tau^{(eq)}_{\alpha\beta}$ and $\tau^{(1)}_{\alpha\beta}$ are linear functions of the strain tensors and of the temperature. We shall denote the mechanical stress tensor by $\tau_{\alpha\beta}$. This tensor occurs in the equations of motion and in the first law of thermodynamics. The viscous stress tensor $\tau^{(vi)}_{\alpha\beta}$ is defined by:

$$\tau^{(vi)}_{\alpha\beta} = \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta}\,. \tag{1.11}$$

In this paper we shall assume that

$$\tau = -P_0, \tag{1.12}$$

where τ is the scalar part of the stress tensor and P_0 is the hydrostatic pressure. We assume that it is constant and uniform. Finally, we consider the equation for the conservation of mass

$$\frac{d}{dt}\rho + \rho \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \frac{d}{dt} u_\alpha = 0 \tag{1.13}$$

and the equation of motion

$$\rho \frac{d^2}{dt^2} u_\alpha = \sum_{\beta=1}^{3} \frac{\partial}{\partial x_\beta} \tau_{\alpha\beta} + \rho F_\alpha \,, \tag{1.14}$$

where F_α is the α-component of the force for the unit of mass.

Eqs. (1.13), (1.14), (1.5), (1.12), and (1.10) form a set of 16 equations for the mass density ρ, the three components of the displacement field u, the 6 independent components of the symmetric strain tensor and the 6 independent components of the symmetric stress tensor. We find a solution of the system of equations by applying the method of Laplace Transform. Moreover, we shall consider some special cases of (1.5) in order to find the solution of transverse waves in Poynting–Thomson, Jeffreys, Maxwell, Kelvin–Voigt, Hooke and Newton media.

A detalied discussion of the propagation of sound waves in fluids with volume viscosity and a scalar internal variable is given by De Groot and Mazur. (See sections 3, 4, and 5 of chapter 12 of ref. [16].)

2. Transverse Waves

It is the purpose of this paper to find solutions of the equations discussed in the last paragraph. We consider plane waves which propagate in the direction of the x_1-axis, while u has the direction of the x_3-axis. Hence, we deal with transverse acoustic waves. In particular we suppose that

$$u = u(x_1, t) = u_3(x_1, t) e_3, \tag{2.1}$$

where e_3 is the unit vector in the direction of the positive x_3-axis.

It follows from the last assertions that

$$u_1(x_1, t) = u_2(x_1, t) = 0. \tag{2.2}$$

From the definition (1.10) for $\varepsilon_{\alpha\beta}$ and the assumption (2.2) for the displacement field u, we obtain

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\partial u_3}{\partial x_1} & 0 & 0 \end{pmatrix} \tag{2.3}$$

since $\frac{\partial u_\alpha}{\partial x_\beta}$ vanishes unless $\alpha = 3$ and $\beta = 1$. Since, in our case the trace of $\varepsilon_{\alpha\beta}$ vanishes, we have

$$\tilde{\varepsilon}_{\alpha\beta} = \varepsilon_{\alpha\beta} \,. \tag{2.4}$$

We shall limit ourselves to linear waves. Hence, we assume that

$$\frac{d}{dt} = \frac{\partial}{\partial t}; \tag{2.5}$$

we shall look for solutions for the stress field $\tau_{\alpha\beta}$ which are of the form

$$\tau_{\alpha\beta} = \begin{pmatrix} -P_0 & 0 & \tau_{13} \\ 0 & -P_0 & 0 \\ \tau_{31} & 0 & -P_0 \end{pmatrix}, \tag{2.6}$$

where P_0 is the constant hydrostatic pressure. Hence, from (2.6) and (1.3), one has

$$\tilde{\tau}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \tilde{\tau}_{13} \\ 0 & 0 & 0 \\ \tilde{\tau}_{31} & 0 & 0 \end{pmatrix}. \tag{2.7}$$

In (2.6) and (2.7), we have

$$\tilde{\tau}_{13} = \tilde{\tau}_{31} = \tau_{31} \tag{2.8}$$

by virtue of the symmetry of the stress tensor and its deviator.

Furthermore, we shall assume that F vanishes (no volume forces). Hence, the equation of motion (1.14) becomes

$$\rho \frac{\partial^2}{\partial t^2} u_\alpha = \sum_{\beta=1}^{3} \frac{\partial}{\partial x_\beta} \tau_{\alpha\beta}, \tag{2.9}$$

where we also used (2.5). It follows from (2.2) and (2.6) that both sides of (2.9) vanish for $\alpha = 1$ and $\alpha = 2$. For $\alpha = 3$, (2.9) becomes

$$\rho \frac{\partial^2}{\partial t^2} u_3 = \frac{\partial}{\partial x_1} \tau_{13}, \tag{2.10}$$

since P_0 is constant. Using (2.2) and (2.5), it follows that the mass conservation law (1.13) becomes

$$\frac{\partial \rho}{\partial t} = 0 \tag{2.11}$$

and this equation is satisfied if we assume that

$$\rho = \rho_0, \tag{2.12}$$

where ρ_0 is constant. Again, using (2.5), the stress–strain relation (1.5) becomes

$$R^{(\tau)}_{(d)0}\,\tilde{\tau}_{\alpha\beta} + \frac{\partial}{\partial t}\tilde{\tau}_{\alpha\beta} = R^{(\varepsilon)}_{(d)0}\,\tilde{\varepsilon}_{\alpha\beta} + R^{(\varepsilon)}_{(d)1}\,\frac{\partial}{\partial t}\tilde{\varepsilon}_{\alpha\beta} + R^{(\varepsilon)}_{(d)2}\,\frac{\partial^2}{\partial t^2}\tilde{\varepsilon}_{\alpha\beta} \tag{2.13}$$

For $\alpha = 1$ and $\beta = 3$, it becomes

$$R^{(\tau)}_{(d)0}\,\tilde{\tau}_{13} + \frac{\partial}{\partial t}\tilde{\tau}_{13} = R^{(\varepsilon)}_{(d)0}\,\tilde{\varepsilon}_{13} + R^{(\varepsilon)}_{(d)1}\,\frac{\partial}{\partial t}\tilde{\varepsilon}_{13} + R^{(\varepsilon)}_{(d)2}\,\frac{\partial^2}{\partial t^2}\tilde{\varepsilon}_{13}. \tag{2.14}$$

3. Wave Solution

We obtain solutions for the problem given by relations (2.10), (2.12), and (2.14) in the case of specific boundary conditions, using the method of Laplace Transform. The displacement field and the deviator of the stress tensor are taken zero at each point at the initial time and this condition is given by

$$u_3(0, x_1) = 0, \qquad \left(\frac{\partial}{\partial t} u_3(t, x_1)\right)_{t=0} \tag{3.1}$$

and

$$\tilde{\tau}_{13}(0, x_1) = 0, \qquad \left(\frac{\partial}{\partial t} \tilde{\tau}_{13}(t, x_1)\right)_{t=0}. \tag{3.2}$$

We use the following notations to denote the Laplace transform of the functions considered with respect to the variable t; the transformation variable is s,

$$\overline{u}_3(s, x_1) = \mathcal{L}[u_3(t, x_1)], \tag{3.3}$$

$$\overline{\tilde{\tau}}_{13}(s, x_1) = \mathcal{L}[\tilde{\tau}_{13}(t, x_1)]. \tag{3.4}$$

Using the initial condition (3.1), (3.2), relation (1.10), and *Eq.* (2.10), and doing the Laplace transform, *Eqs.* (2.12) and (2.14) take the following forms:

$$\rho_0 (s^2 \overline{u}_3) = \frac{d}{dx_1} \overline{\tau}_{13}, \tag{3.5}$$

$$R^{(\tau)}_{(d)0}\, \overline{\tilde{\tau}}_{13} + s\overline{\tilde{\tau}}_{13} = \frac{1}{2}\left[R^{(\varepsilon)}_{(d)0} \frac{d}{dx_1}\overline{u}_3 + R^{(\varepsilon)}_{(d)1} (s \frac{d}{dx_1}\overline{u}_3) + R^{(\varepsilon)}_{(d)2} (s^2 \frac{d}{dx_1}\overline{u}_3)\right]. \tag{3.6}$$

By solving the system of *Eqs* (3.5) and (3.6), we obtain the following ordinary differential equation

$$\frac{d^2}{dx_1^2}\overline{u}_3 - 2\rho_0 s^2 \frac{(R^{(\tau)}_{(d)0} + s)}{(R^{(\varepsilon)}_{(d)0} + sR^{(\varepsilon)}_{(d)1} + s^2 R^{(\varepsilon)}_{(d)2})}\overline{u}_3 = 0 \tag{3.7}$$

We use the following notation

$$h^2(s) = 2\rho_0 s^2 \frac{(R^{(\tau)}_{(d)0} + s)}{(R^{(\varepsilon)}_{(d)0} + sR^{(\varepsilon)}_{(d)1} + s^2 R^{(\varepsilon)}_{(d)2})}. \tag{3.8}$$

Hence, we obtain the following general solution of the differential equation (3.7)

$$\overline{u}_3 = c_1 e^{-h(s)x_1} + c_2 e^{h(s)x_1}. \tag{3.9}$$

If we restrict our examination to bounded values of $u_3(t,x_1)$ for $x_1 \to \infty$ then, in consequence, $\overline{u}_3(s,x_1)$ is bounded and the constant c_2 should be zero. The other constant is specified by the following boundary condition:

$$u_3(t,0) = A^*, \tag{3.10}$$

hence

$$\overline{u}_3(s,x_1) = \frac{A^*}{s}. \tag{3.11}$$

For the last condition and (3.11), the solution becomes:

$$\overline{u}_3(s,x_1) = \frac{A^*}{s} e^{-h(s)x_1}. \tag{3.12}$$

We perform inverse transformation by expansion of the function $e^{-h(s)x_1}$ into series in the following form:

$$\sum_{n=0}^{\infty}\Big((-1)^{2n+1}\frac{1}{(2n+1)!}\Big(\frac{2\rho_0}{R^{(\varepsilon)}_{(d)2}}\Big)^{n+\frac{1}{2}} x_1^{2n+1}\frac{s^{2n+1}(R^{(\tau)}_{(d)0}+s)^{n+\frac{1}{2}}}{(s+a)^{n+\frac{1}{2}}(s+b)^{n+\frac{1}{2}}}+$$
$$(-1)^{2n}\frac{1}{(2n)!}\Big(\frac{2\rho_0}{R^{(\varepsilon)}_{(d)2}}\Big)^{n} x_1^{2n}\frac{s^{2n}(R^{(\varepsilon)}_{(d)0}+s)^{n}}{(s+a)^{n}(s+b)^{n}}\Big),$$

where $-a$ and $-b$ are the roots of the equation $s^2 + s\frac{R^{(\varepsilon)}_{(d)1}}{R^{(\varepsilon)}_{(d)2}} + \frac{R^{(\varepsilon)}_{(d)0}}{R^{(\varepsilon)}_{(d)2}} = 0.$

Since the series is convergent, the Laplace transformation can be applied term by term. By applying the convolution theorem [15] and making use of the fact that for positive integer values of n we have $\frac{1}{\Gamma(-n)} = 0$, we obtain the following final result:

$$u_3(t,x_1) = \sum_{n=0}^{\infty} A_n \int_0^t \Big[\frac{d^{2n}}{d\tau^{2n}}\Big(\tau^{-n-\frac{3}{2}}e^{-R^{(\tau)}_{(d)0}\tau}\Big)\Big]\Big(\frac{t-\tau}{a-b}\Big)^n$$
$$e^{-\frac{(a+b)}{2}(t-\tau)} I_n\Big(\frac{(a-b)}{2}(t-\tau)\Big)d\tau, \tag{3.13}$$

where

$$A_n = A^*(-1)^{2n+1}\frac{1}{(2n+1)!}\frac{\sqrt{\pi}}{\Gamma\big(n+\frac{1}{2}\big)\Gamma\big(-n-\frac{1}{2}\big)}\Big(\frac{2\rho_0}{R^{(\varepsilon)}_{(d)2}}\Big)^{n+\frac{1}{2}} x_1^{2n+1}$$

and $I_n\Big(\frac{(a-b)}{2}(t-\tau)\Big)$ denote the n-order modified Bessel function.

4. Inelastic Media with Memory (Poynting–Thomson Media or Standard Linear Solids)

The stress–strain relation for shear phenomena in inelastic media with memory and for the media which we shall consider in the remaining sections of the present paper may be regarded as special cases of (1.5). Viscous shear effects do not occur in inelastic media with memory, i.e., $\tilde{\tau}^{(vi)}$ vanishes. Hence, in this case we have

$$\eta_s^{(0,0)} = \eta_s^{(0,1)} = 0 \tag{4.1}$$

and consequently (1.8) and (1.9) become

$$R_{(d)1}^{(\varepsilon)} = a^{(0,0)}, \tag{4.2}$$

$$R_{(d)2}^{(\varepsilon)} = 0, \tag{4.3}$$

while (1.6) and (1.7) remain unaltered. Hence, in the stress–strain relation (1.5) the term with the second derivative of the deviator of the strain tensor vanishes, i.e., (1.5) reduces to the Poynting–Thomson equation. By virtue of (1.6), (1.7), (4.2), and (4.3), equality (3.8) takes the following form

$$h^2(s) = \frac{2\rho_0}{R_{(d)1}^{(\varepsilon)}} \frac{s^2(s+a)}{(s+b)}, \tag{4.4}$$

where

$$a = R_{(d)0}^{(\varepsilon)} \quad \text{and} \quad b = \frac{R_{(d)0}^{(\varepsilon)}}{R_{(d)1}^{(\varepsilon)}}. \tag{4.5}$$

By applying the same method as in the last section, we obtain the following final result

$$u_3(t,x_1) = \sum_{n=0}^{\infty} B_n \int_0^t \left[\frac{d^{2n}}{d\tau^{2n}}\left(e^{-a\tau}\tau^{-n-\frac{3}{2}}\right)\right] e^{-b(t-\tau)}(t-\tau)^{n-\frac{1}{2}}\,d\tau,$$

where

$$B_n = A^*(-1)^{2n+1}\frac{1}{(2n+1)!}\frac{1}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(-n-\frac{1}{2}\right)}\left(\frac{2\rho_0}{R_{(d)1}^{(\varepsilon)}}\right)^{n+\frac{1}{2}} x_1^{2n+1}.$$

5. Viscoinelastic Media without Memory (Jeffrey Media)

If a viscoinelastic medium has no memory, we have (see section 4 of ref. [4] and section 17 of ref. [6])

$$a^{(1,1)} = a^{(0,0)}. \tag{5.1}$$

From (1.7), one has

$$R^{(\varepsilon)}_{(d)0} = 0, \tag{5.2}$$

while $a^{(1,1)}$ may be replaced by $a^{(0,0)}$ in (1.6) and (1.8). Hence, the term with $\tilde{\varepsilon}_{\alpha\beta}$ vanishes in the stress–strain relation (1.5), i.e., (1.5) reduces to Jeffrey's equation.

By virtue of *Eqs.* (1.6), (1.8), (5.2), and (1.9), we obtain

$$h^2(s) = 2\rho_0 s \frac{(s+a)}{R^{(\varepsilon)}_{(d)2}(s+b)} \tag{5.3}$$

from (3.8), where

$$a = R^{(\tau)}_{(d)0} \quad \text{and} \quad b = \frac{R^{(\varepsilon)}_{(d)1}}{R^{(\varepsilon)}_{(d)2}}. \tag{5.4}$$

In this case, we obtain the following solution

$$u_3(t, x_1) = \sum_{n=0}^{\infty} C_n \int_0^t \Big[\frac{d^{n-\frac{1}{2}}}{d\tau^{n-\frac{1}{2}}}\Big(e^{-a\tau}\tau^{-n-\frac{3}{2}}\Big)\Big] e^{-b(t-\tau)}(t-\tau)^{n-\frac{1}{2}}\, d\tau,$$

where

$$C_n = A^*(-1)^{2n+1}\frac{1}{(2n+1)!}\frac{1}{\Gamma\Big(n+\frac{1}{2}\Big)\Gamma\Big(-n-\frac{1}{2}\Big)}\Big(\frac{2\rho_0}{R^{(\varepsilon)}_{(d)2}}\Big)^{n+\frac{1}{2}} x_1^{2n+1}.$$

6. Inelastic Media without Memory (Maxwell Media)

If a Jeffreys medium has no viscosity, we have (see section 4 of ref. [4])

$$\eta_s^{(0,0)} = \eta_s^{(0,1)} = 0 \tag{6.1}$$

and from (1.6), (1.8), and (1.9) — moreover, since $a^{(1,1)} = a^{(0,0)}$ — one has

$$R^{(\tau)}_{(d)0} = a^{(0,0)}\eta_s^{(1,1)}, \tag{6.2}$$

$$R^{(\varepsilon)}_{(d)0} = 0, \tag{6.3}$$

$$R^{(\varepsilon)}_{(d)1} = a^{(0,0)}, \tag{6.4}$$

$$R^{(\varepsilon)}_{(d)2} = 0. \tag{6.5}$$

Hence, the term on the right-hand side of the stress–strain relation (1.5) with the deviator of the strain tensor $\tilde{\varepsilon}_{\alpha\beta}$ and the second time derivative of this tensor vanish; i.e., (1.5) reduces to the equation of Maxwell media. Using (6.2)–(6.5), we obtain

$$h^2(s) = \frac{2\rho_0}{a^{(0,0)}} s^2 + 2\rho_0\, \eta_s^{(0,0)}\, s \tag{6.6}$$

from (3.8), and using the relation

$$f(s) = \frac{A^*}{s},$$

Eq. (3.12) will have the following final solution

$$u(t, x_1) = e^{-\frac{b}{a}x_1} f\left(t - \frac{x_1}{a}\right) - a \int_{\frac{x_1}{a}}^{t} f(t-\tau) e^{-b\tau} \frac{\partial}{\partial x_1} J_0\left(\frac{b}{a}\sqrt{x_1^2 - a^2\tau^2}\right) d\tau, \tag{6.7}$$

where

$$a = \frac{1}{\sqrt{\frac{2\rho_0}{a^{(0,0)}}}} = \frac{\sqrt{a^{(0,0)}}}{\sqrt{2\rho_0}},$$

$$b = \frac{\eta_s^{(0,0)}\, a^{(0,0)}}{2}$$

and $J_0\left(\frac{b}{a}\sqrt{x_1^2 - a^2\tau^2}\right)$ is the 0-order Bessel function.

7. Viscoelastic Media (Kelvin–Voigt Media)

If no inelastic phenomena occur in a Jeffrey medium, one has (see section 4 of ref. [4])

$$\eta_s^{(1,1)} = \eta_s^{(0,1)} = 0, \tag{7.1}$$

and, hence, by virtue of (1.6), (1.8), and (1.9), we have

$$R^{(\tau)}_{(d)0} = 0, \tag{7.2}$$

$$R^{(\varepsilon)}_{(d)0} = 0, \tag{7.3}$$

$$R^{(\varepsilon)}_{(d)1} = a^{(0,0)}, \tag{7.4}$$

$$R^{(\varepsilon)}_{(d)2} = \eta_s^{(0,0)}. \tag{7.5}$$

Using (7.2)–(7.5), expression (3.8) becomes

$$h^2(s) = \frac{2\rho_0 s^2}{\left(a^{(0,0)} + s\eta_s^{(0,0)}\right)}, \tag{7.6}$$

hence, the final solution is

$$u_3(t, x_1) = \sum_{n=0}^{\infty} D_n \frac{d^{2n}}{dt^{2n}}\left(e^{-bt}t^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} F_n \frac{d^{2n-1}}{dt^{2n-1}}\left(e^{-bt}t^{n-1}\right),$$

where

$$D_n = A^*(-1)^{2n+1}\frac{1}{(2n+1)!}\frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{2\rho_0}{a^{(0,0)}}\right)^{n+\frac{1}{2}} x_1^{2n+1}$$

and

$$F_n = A^*(-1)^{2n}\frac{1}{(2n)!}\frac{1}{(n-1)!}\left(\frac{2\rho_0}{a^{(0,0)}}\right)^{n} x_1^{2n}.$$

8. Elastic Media (Hooke Media)

If no viscous effects occur in a Kelvin medium, one has

$$\eta_s^{(0,0)} = \eta_s^{(1,1)} = \eta_s^{(0,1)} = 0 \tag{8.1}$$

and from (1.6)–(1.9), we have

$$R^{(\tau)}_{(d)0} = 0, \tag{8.2}$$

$$R^{(\varepsilon)}_{(d)0} = 0, \tag{8.3}$$

$$R^{(\varepsilon)}_{(d)1} = a^{(0,0)}, \tag{8.4}$$

$$R^{(\varepsilon)}_{(d)2} = 0. \tag{8.5}$$

In this case we obtain

$$h^2(s) = \frac{2\rho_0}{a^{(0,0)}} s^2 \tag{8.6}$$

and

$$u_3(t, x_1) = A^* U(t - B),$$

where

$$B = \sqrt{\frac{2\rho_0}{a^{(0,0)}}} x_1$$

and $U(t - B)$ is the unit step function, i.e., it is 0 if $t < B$ and 1 if $t > B$.

9. Fluids (Newtonian Media)

If, for a Kelvin–Voigt medium, the quantity $a^{(0,0)}$ vanishes, we have

$$a^{(1,1)} = a^{(0,0)} = \eta_s^{(1,1)} = \eta_s^{(0,1)} = 0. \tag{9.1}$$

From (1.6)–(1.9) and using (9.1), one has

$$R^{(\tau)}_{(d)0} = 0, \tag{9.2}$$

$$R^{(\varepsilon)}_{(d)0} = 0, \tag{9.3}$$

$$R^{(\varepsilon)}_{(d)1} = 0, \tag{9.4}$$

$$R^{(\varepsilon)}_{(d)2} = \eta_s^{(0,0)}. \tag{9.5}$$

Consequently, expression (3.8) and the final result of the inverse Laplace transformation become

$$h^2(s) = \frac{2\rho_0}{\eta_s^{(0,0)}} s, \tag{9.6}$$

$$u_3(t, x_1) = A^* erfc \frac{B}{2\sqrt{t}}, \tag{9.7}$$

where

$$B = \sqrt{\frac{2\rho_0}{\eta_s^{(0,0)}}} x_1$$

and $erfc$ is the complementary of the error integral; it is defined as

$$erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du\,.$$

Then one sees that the solution does not depend on the internal variables but only on the shear viscosity.

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