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TIME OPTIMAL CONTROL OF HIGH-DIMENSIONAL SYSTEMS

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Abstract

For time optimal control of high-dimensional systems, we use time stages of varying length, and iterative dynamic programming (IDP) to search simultaneously for the switching times and the stage lengths. The procedure is evaluated with a twenty plate gas absorber having two control variables. To obtain convergence in reliable manner, the use of continuation as shown in this paper is very effective. The accurate switching times yield better results than have been reported in the literature. The computational procedure is straightforward and the computations can be readily carried out on a personal computer.

Keywords: time optimal control, iterative dynamic programming.

Introduction

In optimal control, the minimum time control problem, which is very important for practical applications, has received considerable attention. Although for linear systems, significant progress has been made recently, there still remains the problem of how to handle computationally systems of very high dimension. In this paper, we examine the use of iterative dynamic programming (IDP), as outlined by BOJKOV and LUUS (1994a), to a high-dimensional problem.

Twenty Plate Gas Absorber

Let us consider a twenty plate gas absorber with two control variables that was used for time suboptimal control by WONG and LUUS (1980). The model is based on the six plate gas absorber described in detail by LAPIDUS and LUUS (1967), and used for time-optimal control studies by BASHEIN (1971), LUUS (1974) and ROSEN et al. (1987). The system is described by

$$\frac{\mathrm{dx}}{\mathrm{d}t} = \mathbf{Ax} + \mathbf{Bu}\,, \qquad (1)$$

where

$$\mathbf{A} = \text{tridiag}[\,0.5388 \quad -1.1730 \quad 0.6342\,]\,, \tag{2}$$

and the (20×2) control coefficient matrix is given by

$$\mathbf{B}^T = \begin{bmatrix} 0.5388 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0.6342 \end{bmatrix} \tag{3}$$

The control variables are constrained by

$$0.0 \le u_1 \le 1.0\,, \tag{4}$$

$$-0.4167 \le u_2 \le 0.972\,, \tag{5}$$

and the initial state is given by

$$\mathbf{x}^T(0) = \begin{bmatrix} -21.5 & -39.9 & -55.4 & -68.7 & -79.9 \\ -89.5 & -97.6 & -104.5 & -110.4 & -115.4 \\ -119.6 & -123.2 & -126.3 & -128.9 & -131.1 \\ -133.0 & -134.6 & -135.9 & -137.1 & -138.1 \end{bmatrix} \times 10^{-3}. \tag{6}$$

We want to take the system from the initial state to the desired final state

$$|x_i(t_f)| \le \epsilon\,, \qquad i = 1, 2, \ldots, 20 \tag{7}$$

in minimum time t_f, where $\epsilon = 1.0 \times 10^{-3}$.

Instead of using the sum of squares of the final state as the performance index, here we use the augmented performance index introduced by BOJKOV and LUUS (1994b) for a six plate gas absorber. The performance index includes the final time, and also penalizes any final state variable which is larger than ϵ at the final time t_f :

$$J = \sigma \sum_{i=1}^{20} x_i(t_f)^2 + \mu \sum_{i=1}^{20} \left(|x_i(t_f)| - \epsilon\right)^2 + \delta t_f\,, \tag{8}$$

where

$$\sigma = \begin{cases} 0 & \text{if all } \ |x_i| \le \epsilon\,, \\ 1 & \text{if any } \ |x_i| > \epsilon\,, \end{cases} \tag{9}$$

$$\mu = \begin{cases} 0 & \text{if all } \ |x_i| \le \epsilon\,, \\ 1 & \text{if any } \ |x_i| > \epsilon\,, \end{cases} \tag{10}$$

and

$$\delta = \begin{cases} \dfrac{\epsilon^2}{1200} & \text{if all } \ |x_i| \le \epsilon\,, \\ 0 & \text{if any } \ |x_i| > \epsilon\,. \end{cases} \tag{11}$$

Having t_f included in the performance index, in the above form, allows the final time to be reduced when the final state constraints are satisfied.

All computations were performed on a 486DX2/66 personal computer using IBM's OS/2 version 2.1 operating system and WATCOM's FORTRAN 77^{32} version 9.5 compiler. For the integration of the differential equations, the FORTRAN subroutine DVERK of HULL et al (1976) was used with an integration tolerance of 10^{-8}. The random allowable values for control were generated using the compiler's random number generator URAND( ).

Instead of attempting to solve this problem by using a very large number of stages P for the optimization by IDP, we suggest to use continuation on the control and stage lengths obtained with only $P = 10$ time stages. For IDP, we choose as parameters 3 state grid points ($N = 3$), 15 allowable values for control ($R = 15$), and a region contraction factor $y = 0.80$. To ensure reliable convergence, a multipass scheme consisting of 30 IDP passes of 30 iterations each is used, where the policy at the end of a pass is used as the initial policy for the subsequent pass.

For the first step in the continuation approach, we use as initial control the lower bound values α of the control variables and as region size $r_j = 1.5\,(\beta_j - \alpha_j)$. For the stage parameters, we choose as initial stage length $v(k)^{(0)} = 0.5$ for $k = 1, 2, \ldots, 10$; and as initial region size $s(k)^{(0)}$, 0.02, 0.03, ..., 0.06 for $k = 1, 2, \ldots, 10$. As can be seen in the first row of *Table 1,* the algorithm yielded without difficulty a final time of 26.694 min in approximately 20 IDP passes of 30 iterations each. The five time stage control policy corresponding to this final time yields a final state

$$\mathbf{x}^T(29.694) = \begin{bmatrix} 8.2 & -12.3 & 11.5 & 6.2 & -1.9 \\ -10.3 & 16.4 & -18.4 & -15.1 & -6.9 \\ 4.6 & 16.1 & 23.1 & 21.0 & 6.8 \\ -16.7 & 33.6 & -14.4 & 39.8 & -20.2 \end{bmatrix} \times 10^{-3} \qquad (12)$$

with a performance index value of $1.196 \cdot 10^{-2}$.

For the second step of the continuation, we use the five stage control policy obtained in step 1. For the remaining time stages we again assign as initial control $u_j(k)^{(0)} = a_j$, for $k = 6, 7, \ldots, 10$; and as initial stage length $v(k)^{(0)} = 0.5$ for $k = 6, 7, \ldots, 10$. For the control region size, we use $r_j(k) = 1.5\,(\beta_j - \alpha_j)$, for $k = 1, 2, \ldots, 10$, and for the stage length we use as initial region size $s(k)^{(0)}$, 0.02, 0.03, ..., 0.06 for $k = 1, 2, \ldots, 10$. Here the algorithm yielded without difficulty a performance index value of $1.652 \cdot 10^{-3}$ at a final time of 30.759 min. These results, shown in the second row of *Table 1,* were also obtained in fewer than 20 IDP passes. The

Table 1

Value of the final time using a continuation approach with $P = 10$, for the twenty stage gas absorber, as a function of the continuation step and the initial stage region size parameter $s(k)^{(0)}$, using for IDP $N = 3$, $R = 15$ and $y = 0.80$. The numbers in parentheses indicate the number of passes required to reach the given final time to a maximum of 30 passes

Continuation step	Final time t_f (min)				
	Initial stage length $s(k)^{(0)}$				
	0.2	0.3	0.4	0.5	0.6
1	21.708*	22.350*	26.694 (30)	26.694 (24)	26.694 (21)
2	30.759 (30)	30.759 (21)	30.759 (18)	30.759 (15)	30.759 (15)
3**	37.009 (52)	37.053 (43)	37.049 (41)	37.061 (37)	37.062 (36)

* Five stage policy

** A maximum of 60 IDP passes for the third continuation step

control policy consists of six stages and yields the improved final state

$$\mathbf{x}^t(30.759) = \begin{bmatrix} 3.9 & 5.7 & 5.0 & 2.2 & -1.7 \\ -5.3 & -7.1 & -6.4 & -3.1 & 1.9 \\ 6.8 & 9.0 & 6.4 & -1.0 & -9.4 \\ -10.5 & 2.0 & 15.3 & -10.7 & 1.8 \end{bmatrix} \times 10^{-3} . \tag{13}$$

For the third continuation, we again submit the stage lengths and the control policies of the previous run. As for the last four time stages, we use as initial control $u_j(k)^{(0)} = \alpha_j$ with initial stage length $v(k)^{(0)} = 0.5$. By using the same region sizes for the control and stage lengths as above, we obtain the eight stage control policy shown in *Fig. 1* with a final time $t_f = 37$ min. The results for the five runs performed are given in the last row of *Table 1,* and have the final state

$$\mathbf{x}^T(37.053) = \begin{bmatrix} 0.7 & 0.9 & 0.6 & -0.0 & -0.7 & \\ -1.0 & -0.8 & -0.1 & 0.7 & 1.0 & \\ & 0.6 & -0.4 & -1.0 & -0.6 & 0.7 \\ 0.9 & -1.0 & 0.4 & -0.1 & 0.6 & . \end{bmatrix} \times 10^{-3} \tag{14}$$

with the variables x_6, x_{10}, x_{13} and x_{16} at the boundary ϵ.

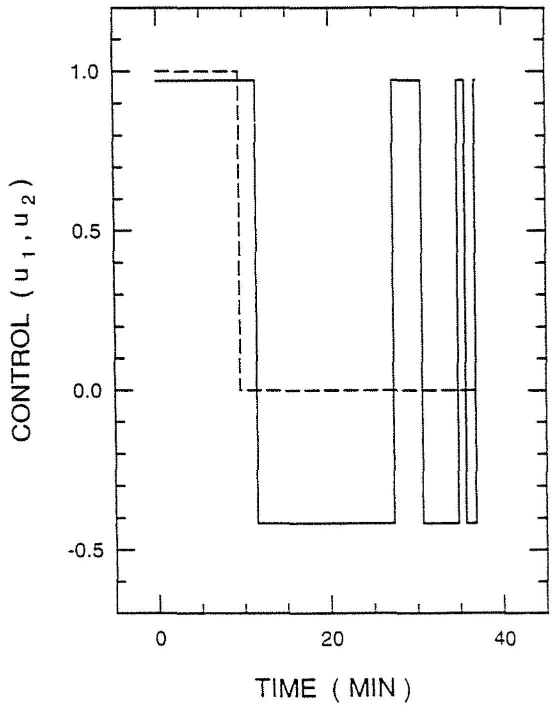

Fig. 1. Eight stage time optimal control policy for the twenty plate gas absorber with a final time $t_f = 37$ min

As is seen in *Table 1,* when using this approach to obtain the minimum time for this system, we obtain for different starting conditions results for the final times in the vicinity of 37 min. The best result obtained here is 37.009, which is an improvement of 34 per cent over the final time $t_f = 56$ min obtained by WONG and LUUS (1980) by using model reduction and feedback control. The average run for 30 passes of 30 iterations each required approximately 4.5 hours of CPU time. The results obtained here show that IDP with the continuation scheme outlined in this paper provides an effective means of reaching the close vicinity of the origin for high dimensional minimum time problems.

Conclusions

Iterative dynamic programming using variable stage lengths and continuation on the control provides an attractive means for determining the time optimal control policy of high dimensional systems by simultaneously searching for the stage lengths and the values for control. For the twenty stage gas absorber presented here, the algorithm yielded the global optimal control policy, resulting in a 34 % improvement in the final time obtained by other means. The method provides a convenient way of determining the exact switching times, which are extremely important for bang-bang control problems. This procedure can be easily used on a personal computer.

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