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A METHOD FOR OPTIMAL CONTROL UNDER UNCERTAINTY

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Abstract

A new stochastic method and algorithm are presented to solve optimal control problems under uncertainty which are illustrated with two examples of minimum-time control problems.

Keywords: time optimal control, non-linear systems, uncertainty.

1. Introduction

Optimal control problem formulation requires the uncertainty in process dynamics to be taken into account. A number of stochastic algorithms have been developed to deal with the uncertainty, but they either concern the linear systems only or require big computational efforts when being more general. In this paper a new stochastic method and algorithm are presented. The assumption is that the uncertain parameters axe slowly varying, so they are constant in the time interval of interest. It makes computation of the optimal control of non-linear systems much easier compared to the other methods.

2. A New Stochastic Method for Optimal Control under Uncertainty

2.1. Problem Formulation

The following optimal control problem is considered here: Given the process dynamics:

$$\frac{dx}{dt} = f(x, u, p), \quad x(t_0) = x^0, \tag{1}$$

with required expected values x_i^f at the final time t_f of some of the state variables x and probability distributions $d(p)$ of the uncertain parameters p, find values of the control variables u such that the expected value ($E[.]$) of the performance measure:

$$L[u(t)] = G[x_m(t_f)] + \underset{p}{E}\left[\int_{t_0}^{t_f} f_0(x,u,p)dt\right] \tag{2}$$

is maximal taking into account constraint imposed on the control: $u_l \leq u(t) \leq u_u$.

2.2. Algorithm

Here an algorithm is proposed to solve optimal control problems under uncertainty which is based on the necessary conditions for optimality given by STOYANOV and GRANCHAROVA (1993). It can be regarded as an extended version of the gradient methods (LASDON et al, 1967) which in addition takes into account the uncertainty in process dynamics and can solve not only fixed-time problems but time optimal control problems as well. The proposed algorithm can be described by the following steps:

1. Guess the final time t_f if minimum-time problem is to be solved. The first guess for t_f can be generated in the following way: $t_f = T/10$, where T is the time for the process to reach the desired steady state if control variables are set to their steady state values.

2. Guess the control $u(t), t \in [0; t_f]$. It is proposed that initially control variables are set either to their minimal or maximal values allowed.

3. With these values of $u(t)$, integrate the state equations (1) forward in time for different values p^j, $j = 1, 2, \ldots, N$ of the uncertain parameters p and obtain process trajectory $x(t,,p^j)$, $t \in [0; t_f]$, $j = 1, 2, \ldots, N$.

4. With these values ot $u(t)$ and $x(t, p^j)$, integrate the following equations for the adjoint variables λ backward in time for different values of the uncertain parameters:

$$\frac{d\lambda(t,p^j)}{dt} = -\frac{\partial H}{\partial x}, \quad \lambda(t_f, p^j) = \frac{\partial G}{\partial x_m}, \quad j = 1, 2, \ldots, N, \tag{3}$$

where H is the Hamiltonian function.

5. Correct $u(t)$, $t \in [0; t_f]$ by using the necessary condition for optimality as follows:

$$u^{\text{New}}(t) = u^{\text{Old}}(t) + \varepsilon \cdot \frac{\partial}{\partial u}\left[\underset{p}{E}[H(x, u^{\text{Old}}, p, \lambda)]\right], \quad t \in [0; t_f]. \tag{4}$$

Optimise the value of ε so as to maximise the criterion (2) and repeat the algorithm from step 3.

6. Iterate until convergence on the optimality criterion (2) is attained. In case of solving minimum-time problem, it is proposed for the performance measure (2) to have the following form:

$$L[u(t)] = \sum_{i=1}^{n_x} \left(\frac{\underset{p}{E}[x_i(t_f, p)] - x_i^f}{x_i^0 - x_i^f} \right)^2 \rightarrow \min, \tag{5}$$

which expresses the desire to have at the final time the state variables x_i equal to their required values x_i^f.

7. Give a new guess for the final time t_f when a minimum-time problem is to be solved and repeat the algorithm from step 2. Iterate until the optimal value of the criterion (5) becomes small enough. The time end t_f corresponding to this value will be the optimal final time.

2.3. Examples

EXAMPLE 1: An example given by HSU et al (1972) of minimum-time control of the following second-order system:

$$400 \cdot \frac{dx}{dt} = -x + u, \qquad 300 \cdot \frac{dy}{dt} = -y + x \tag{6}$$

was solved by using the proposed algorithm. The optimal control problem was to move the process from the steady state $x = y = u = 40$ to a new steady state $x = y = u = 50$ in minimum time. The control was computed for different values of the final time t_f as it can be seen from *Table 1.* The optimal control is shown in *Fig. 1.*

Table 1

Final time t_f	1000	1200	1400
Optimality criterion $L[u(t)]$	0.1814	0.0410	0.0011
Accuracy on $L[u(t)]$	0.01	0.001	0.0001
Computing time in sec	186	237	375

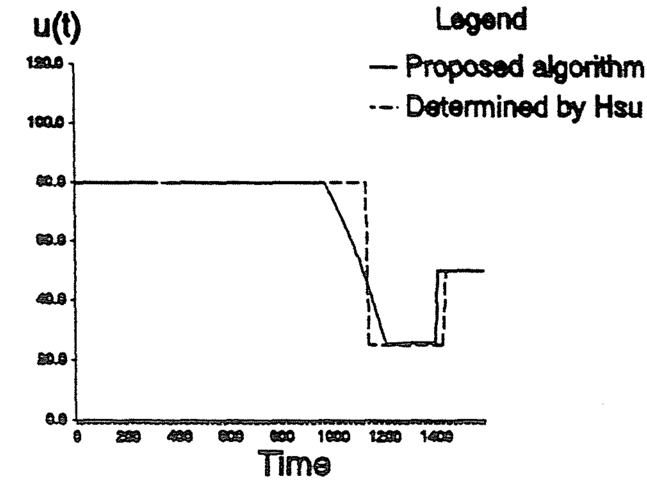

Fig. 1. Optimal control policy

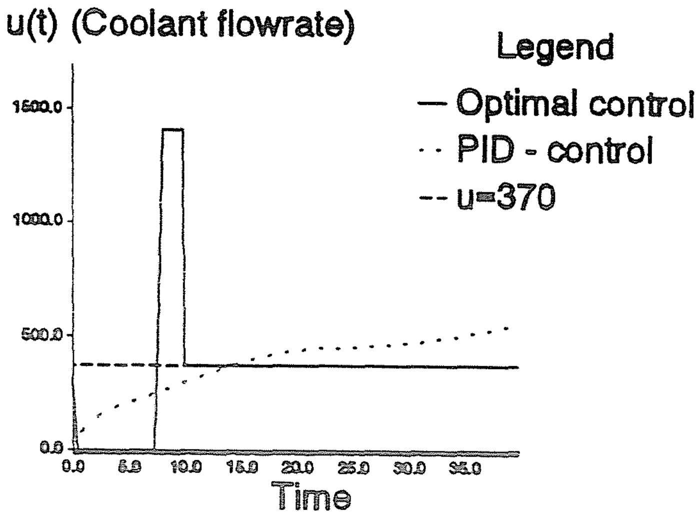

Fig. 2. Optimal control policy

EXAMPLE 2: A problem of optimal start-up under uncertainty of a continuous stirred tank reactor (CSTR) in which a first-order irreversible reaction $A \rightarrow B$ takes place was solved. This example is taken from (HICKS and RAY, 1971) where the control problem is solved in the absence of uncertainty. The mass and heat balance of the CSTR expressed through dimensionless concentration y_1 and temperature y_2 are:

Table 2

N	24.0	24.4	25.2	26.0	26.4
$D(N)$	0.05	0.2	0.5	0.2	0.05

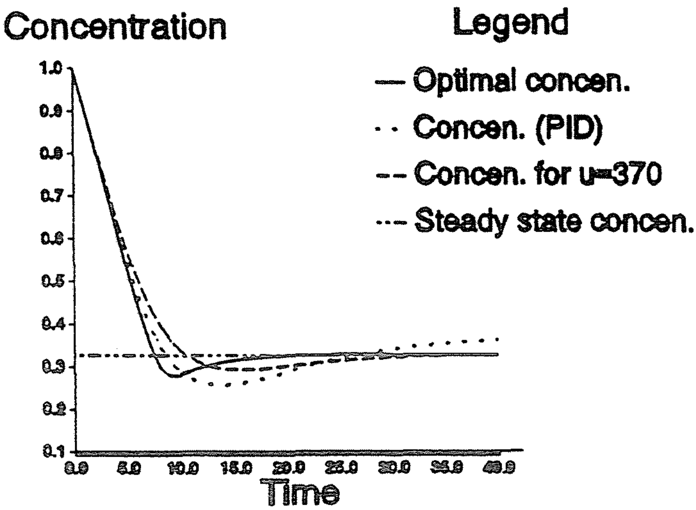

Fig. 3. Concentration trajectory for $N = 24.4$

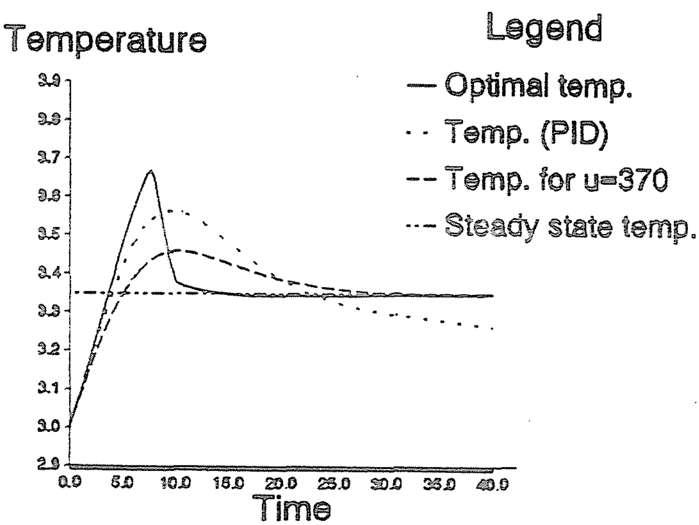

Fig. 4. Temperature trajectory for $N = 24.4$

$$\frac{dy_1}{dt} = \frac{(1-y_1)}{\theta} - r(N), \ \frac{dy_2}{dt} = \frac{(y_f - y_2)}{\theta} + r(N) - \alpha.u.(y_2 - y_c). \quad (7)$$

It is supposed that the parameter N is uncertain with discrete probability distribution given in *Table 2*. The following problem was solved: Move the CSTR from the initial state $y_1(0) = 1$, $y_2(0) = y_f$ to the steady state corresponding to the value $u = 370$ of the control, in minimum time, having uncertainty in parameter N and constraint imposed on the control: $0 \leq u(t) \leq 1500$. The time-optimal response of the CSTR found by applying the proposed algorithm is shown in *Figs. 3* and *4* for a possible value $N = 24.4$ of the uncertain parameter. There it is compared with the response obtained for control set to its steady state value and with the response using a proportional-integral-derivative (PID) control. The control actions are shown in *Fig. 2*. It can be seen that the optimal response obtained by using the developed algorithm reaches the steady state more quickly compared to the two other responses.

References

1. STOYANOV, S. – GRANCHAROVA, A. – GANI, R. (1993): System Modelling Control, Zakopane, Poland, May, 1993, pp. 181–185.
2. LASDON, L. – MITTER, S. – WAREN, A. (1967): *IEEE Trans. Aut. Cont.*, AC–12, pp. 132–138.
3. HSU, E. – BACHER, S. – KAUFMAN, A. (1972): *AIChE Journal*, Vol. 18, pp. 1133–1139.
4. HICKS, G. – RAY, W. (1971): *Canadian J. Chem. Eng.*, Vol. 49, pp. 522–528.