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ON THE VALIDITY OF THE PRINCIPLE OF MINIMUM ENTROPY PRODUCTION

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Abstract

The purpose of this analysis is to show the importance of correct performance of picture representation and circumspect interpretation of variation in the application of the minimum entropy production principle. As an incorrect formulation of a variational task, written for the Fourier heat conduction problem has shown, the principle of minimum entropy production *apparently* goes to contradiction with the energy balance equation [1]. This led to further erroneous conclusions. The misunderstanding can be resolved — exceeding far off beyond the actual problem — with Gyarmati's picture representation and variational principle, the *Governing Principle of Dissipative Processes.*

Keywords: the principle of minimum entropy production, picture representation of Gyarmati, Fourier's heat conduction, variational problem of heat conduction, Lagrange density, Euler–Lagrange differential equation, Gyarmati's variational principle, governing principle of dissipative processes.

On the Lampinen Dilemma of the Minimum Entropy Production Principle

We quote from [1] the *ominous formulation* of 'A simple heat conduction problem': 'We consider a one dimensional heat conduction problem where heat is conducted through a plate which is in a stationary state. The thickness of the plate is L and the surfaces are kept at constant temperatures, i. e. the boundary conditions are

$$T(x=0)=T_0, \quad T(x=L)=T_1. \tag{7}$$

The entropy production at each point x is

$$\frac{\mathrm{d}S_{irr}}{\mathrm{d}t}=-\frac{1}{T^2}q\nabla T=\frac{1}{T^2}\lambda(\partial T/\partial x)^2. \tag{8}$$

Here we have used Fourier's model for the heat conduction

$$q=-\lambda\nabla T \tag{9}$$

with a constant heat conductivity λ. Using *Eq.* (8) we get for the entropy production rate

$$\frac{\mathrm{d}S_{irr}}{\mathrm{d}t} = A\lambda \int_0^L (\frac{1}{T^2})(\partial T/\partial x)^2 \mathrm{d}x, \tag{10}$$

where A is the cross sectional area of the plate.

According to the principle of minimum entropy production we formulate the following problem. Find such a temperature distribution $T(x)$ that fulfils the boundary conditions (7) and minimizes the entropy production rate, i. e.

$$\int_0^L (\tfrac{1}{T^2})(\partial T/\partial x)^2 \mathrm{d}x = \min! \qquad (11)\text{'}$$

This quotation will be discussed later.

On the Gyarmati Principle

Gyarmati's *opus magnum*, the GPDP, well known as being regarded the most widely valid and applied integral principle for linear, quasi-linear and certain types of non-linear theories [2, 3] in the thermodynamics of irreversible processes. This principle [4, 5] has already proved its usefulness for the complete regime of transport phenomena [6–32]. The principle tells us, in total generality, that the functional

$$\delta L[\Gamma, J] = \delta \int_0^t \int_V \mathcal{L} \mathrm{d}V \mathrm{d}t = \delta \int_0^t \int_V (\sigma - \Psi - \Phi) \mathrm{d}V \mathrm{d}t = 0 \tag{1}$$

under constraints that the balance equations

$$\rho \dot{a}_i + \nabla J_i = \sigma_i \qquad (i = 1, 2, ...) \tag{2}$$

are satisfied [2]. For the entropy we can write

$$\rho \dot{s} + \nabla J_s = \sigma_s = \sum J_i X_i \geq 0. \tag{3}$$

Now referring to (1), (2) and (3) σ_s is the entropy production; Ψ and Φ are the local dissipation potentials; ρ is the density of the continuum, $\dot{a}$ and $\dot{s}$ denote substantial time derivatives of the specific value of the i-th transport quantity and entropy, respectively; J_i and X_i represent the independent thermodynamic current and thermodynamic forces, respectively. The local dissipation potentials, which were introduced by Rayleigh and Onsager for special cases, always exist in continua. The σ, Ψ and Φ are positive definite bilinear and homogeneous quadratic functions of the thermodynamic forces and currents, respectively. In stationary case the problem is reduced to the following variational task (for details see [2, 5]):

$$\delta \int \Psi \mathrm{d}V = 0. \tag{4}$$

This special case of the GPDP is perfectly equivalent to the earlier formulation of a variational problem developed by Prigogine in 1945, as the principle of minimum production of entropy, since

$$\sigma = 2\Psi = \sum L_{ik} X_i X_k \tag{5}$$

in the linear theory. Hence, instead of (4)

$$\delta \int\limits_V \sigma \mathrm{d}V = \delta \int\limits_V \sum_{i,k}^{f} L_{i,k} X_i X_k \mathrm{d}V = 2\delta \int\limits_V \Psi \mathrm{d}V \tag{6}$$

stands, in which we can disregard the multiplying factor 2. According to DE GROOT and MAZUR [3]: 'Stationary non equilibrium states have the important property that, under certain conditions, they are characterised by *a minimum of the entropy production,* compatible with the external constraints imposed on the system. This property is valid only if the *phenomenological coefficients are supposed to be constant.*' This principle — proposed by Prigogine — needs however some more conditions namely the validity of the linear constitutive equations. The entropy production using the *different representational pictures* [2] can be interpreted according to [26] as follows. The temperature scale in the so-called Γ-picture is defined for heat conduction by the transformation

$$\Gamma = \hat{\Gamma}(T). \tag{7}$$

Let us assume that function $\hat{\Gamma}$ is continuously differentiable and that its inverse exists, so we have

$$T = \hat{\Gamma}^{-1}(\Gamma), \tag{8}$$

and

$$\frac{\mathrm{d}\hat{\Gamma}(T)}{\mathrm{d}T} > 0 \quad \text{or} \quad \frac{\mathrm{d}\hat{\Gamma}(T)}{\mathrm{d}T} < 0. \tag{9}$$

Γ shows the strictly monotonic property in the interval $0 < T < \infty$. In the contemporary thermodynamic theory of heat conduction the most frequently used scale transformations are

$$\begin{array}{lcll} \Gamma^{**}(T) & = & T, & \text{Fourier picture,} \\ \Gamma^{*}(T) & = & \ln T, & \text{energy picture,} \\ \Gamma(T) & = & T^{-1}, & \text{entropy picture.} \end{array} \tag{10}$$

It is possible to transform the original linear Fourier law into different pictures postulating the *invariance of heat current* with respect to the scale transformations [2, 26]. We can write

$$J = -L^{**}\nabla T = -\lambda \nabla T = -L^{*}\nabla \ln T = L\nabla T^{-1} = L^{\Gamma}\nabla\Gamma = L^{\Gamma}X^{\Gamma}. \tag{11}$$

Whereas between the coefficiens the relations

$$L^{**} = \lambda = T^{-1}L^{*} = T^{-2}L \tag{12}$$

are valid. The *entropy production* belonging to the heat conduction can be given in different pictures as

$$\sigma = \frac{JX^{**}}{T^2} = \frac{JX^{*}}{T} = JX, \tag{13}$$

where

$$X^{**} = -\nabla T, \quad X^{*} = -\nabla \ln T, \quad X = \nabla T^{-1}, \tag{14}$$

and for the dissipation potentials we have

$$\Psi^{**} = \frac{\lambda}{2}(\nabla T)^2, \quad \Psi^{*} = \frac{L^{*}}{2}(\nabla \ln T)^2, \quad \Psi = \frac{L}{2}(\nabla T^{-1})^2, \tag{15}$$

while among them, the connections

$$\Psi^{**} = T\Psi^{*} = T^2\Psi \tag{16}$$

are valid [2]. The entropy production can be generally written as

$$\sigma = \sum_{i=1}^{f} J_i X_i = \sum_{i=1}^{f} J_i \nabla\Gamma_i \geq 0. \tag{17}$$

The dissipation potentials in the linear theory — when L_{ik} or R_{ik} phenomenological coefficients are constant — are defined as

$$\Psi(X) = \frac{1}{2}\sum_{i,k=1}^{f} L_{ik}\nabla\Gamma_i\nabla\Gamma_k > 0, \tag{18}$$

$$\Phi(J) = \frac{1}{2}\sum_{i,k=1}^{f} R_{ik}J_iJ_k > 0. \tag{19}$$

The most general form of the Fourier equation in the *generalized* Γ-picture is

$$\rho c_v^{\Gamma}\frac{\partial \Gamma}{\partial t} + \nabla L^{\Gamma}\nabla\Gamma = 0. \tag{20}$$

When the transport equations represent quasi-linear partial differential equations, because the conductivities L_{ik} and the resistances R_{ik} depend on the parameters Γ_i, the governing principle remains valid. This is a consequence of the supplementary theorem of Gyarmati [5] which states that 'In the case of quasilinear constitutive equations, the variation of the sum of dissipative potentials with respect to the parameter Γ_i vanishes [5].'

The Lagrangian formulation of GPDP is [2]

$$\delta\int_V \mathcal{L}\mathrm{d}V = 0, \tag{21}$$

$$\mathcal{L} = \sum_{i=1}^{f}(\rho\dot{a}_i - \sigma_i)\Gamma_i - [\Psi(\nabla\Gamma,\nabla\Gamma) + \Phi(J,J)]. \tag{22}$$

In case of simultaneous — but independent — variation of (21) with respect to f independent scalars Γ_i and f independent vectors $\mathbf{J}_i$, we get the transport equations

$$\frac{\partial\mathcal{L}}{\partial\Gamma_i} - \sum_{\alpha=1}^{f}\frac{\partial}{\partial x_\alpha}\frac{\partial\mathcal{L}}{\partial\frac{(\partial\Gamma_i)}{\partial x_\alpha}} = 0, \qquad (i = 1, 2 ..., f), \tag{23}$$

and the constitutive equation

$$\frac{\partial\mathcal{L}}{\partial J_i} = 0, \tag{24}$$

as Euler–Lagrange equations.

The Lagrangian Density and Euler–Lagrange Differential Equation in Case of Fourier Heat Conduction Process according to Gyarmati

The Lagrangian density *in entropy picture* is

$$\mathcal{L}_{\frac{1}{T}} = \mathcal{L}_{\frac{1}{T}}\left[\frac{1}{T},\nabla\frac{1}{T};\frac{\partial T}{\partial t}\right] = \rho\frac{c_v}{T}\frac{\partial T}{\partial t} - \frac{L}{2}\left(\nabla\frac{1}{T}\right)^2, \tag{25}$$

and here the index of the Lagrangian density refers to the varied parameter. The equation

$$\frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial \frac{1}{T}} - \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial (\frac{\partial \frac{1}{T}}{\partial x_\alpha})} = 0 \tag{26}$$

just comes from (21) as Euler–Lagrange equation. One can see from the variational problem (21), too, that the entropy picture as *rara avis* leads from *entropy production* directly to the *Lagrangian density.* This cannot be said in cases of the Fourier and the energy pictures, for in these the entropy production σ has to be multiplied by T^2 and T respectively, to get the correct Lagrangian densities. To determine the correct Lagrangian density we have to take the *generalized picture* as a model according to the product $J\,\nabla\Gamma$. This is the *'causa sine qua non'* of proper picture formation [2, 26]. We must not forget that the time derivatives are not varied in a proper application of the Gyarmati principle.

The Lagrangian density in the *Fourier picture* is

$$\mathcal{L}_T = \mathcal{L}_T[T, \nabla T; \frac{\partial T}{\partial t}] = \rho_{c_v} T \frac{\partial T}{\partial t} + \frac{\lambda}{2}(\nabla T)^2. \tag{27}$$

Here the index T refers to the varied parameter. Hence the Euler–Lagrange differential equation of (27) is

$$\frac{\partial \mathcal{L}_T}{\partial T} - \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \frac{\partial \mathcal{L}_T}{\partial (\frac{\partial T}{\partial x_\alpha})} = 0, \tag{28}$$

in the Fourier picture.

As Lagrangian density in the *energy picture we get*

$$\mathcal{L}_{\ln T} = \mathcal{L}_{\ln T}[\ln T, \nabla \ln T; \frac{\partial T}{\partial t}] = \rho_{c_v} \ln T \frac{\partial T}{\partial t} - \frac{L^*}{2}(\nabla \ln T)^2, \tag{29}$$

and for the Euler–Lagrange differential equation yields

$$\frac{\partial \mathcal{L}_{\ln T}}{\partial \ln T} - \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \frac{\partial \mathcal{L}_{\ln T}}{\partial (\frac{\partial \ln T}{\partial x_\alpha})} = 0. \tag{30}$$

The Proper Results for the Lagrange Densities in Different Pictures

Entropy Picture

$$\mathcal{L}_{\frac{1}{T}} = L\frac{(\nabla T)^2}{T^4} = L(\nabla T^{-1})^2. \tag{31}$$

Energy picture

$$\mathcal{L}_{\ln T} = L^*(\nabla \ln T)^2. \tag{32}$$

Fourier picture

$$\mathcal{L}_T = \lambda(\nabla T)^2. \tag{33}$$

The connections between the Lagrange densities

$$\begin{aligned} \sigma &= \mathcal{L}_{\frac{1}{T}} && \text{in the entropy picture,} \\ \sigma &= \frac{\mathcal{L}_{\ln T}}{T} && \text{in the energy picture,} \\ \sigma &= \frac{\mathcal{L}_T}{T^2} && \text{in the Fourier picture,} \end{aligned} \tag{34}$$

i. e.

$$\mathcal{L}_{\frac{1}{T}} = \frac{\mathcal{L}_{\ln T}}{T} = \frac{\mathcal{L}_T}{T^2}. \tag{35}$$

Here we can show the temperature dependency of the Lagrange densities if they enter a foreign or inappropriate picture. Thus

$$\begin{aligned} \mathcal{L}_T &= \mathcal{L}_{\frac{1}{T}}(T)T^2 = \mathcal{L}_{\ln T}(T)T, \\ \mathcal{L}_{\ln T} &= \mathcal{L}_{\frac{1}{T}}(T)T = \mathcal{L}_T(T)T^{-1}, \\ \mathcal{L}_{\frac{1}{T}} &= \mathcal{L}_{\ln T}(T)T^{-1} = \mathcal{L}_T(T)T^{-2}, \end{aligned} \tag{36}$$

and so with these expressions one can avoid the *nonlinearities* for the Lagrange densities, too. The connections and the procedures are the same as for the phenomenological coefficients vice versa (see later *Eq.* (49)).

The Proper Results for Euler–Lagrange Equations

Entropy picture, where $\mathcal{L}_{\frac{1}{T}} = L(\nabla T^{-1})^2$,

$$\frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial \frac{1}{T}} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial (\frac{\partial \frac{1}{T}}{\partial x})} = 0, \tag{37}$$

so

$$\frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial \frac{1}{T}} = 0, \quad \frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial (\nabla \frac{1}{T})} = L2\nabla \frac{1}{T}, \quad \nabla \frac{\partial \mathcal{L}_{\frac{1}{T}}}{\partial (\nabla \frac{1}{T})} = L\nabla 2\nabla \frac{1}{T} = L2\Delta \frac{1}{T} = 0, \tag{38}$$

hence

$$-L2\Delta \frac{1}{T} = 0, \tag{38a}$$

and because in stationary state div $J_q = 0$, therefore as

$$J_q = L(\nabla T^{-1}), \ \mathrm{div} J_q = \nabla\left(L(\nabla T^{-1})\right) = L\Delta \frac{1}{T} = 0, \tag{39}$$

we have got the good result. Our result can be transformed to the well known Fourier picture with the next steps

$$\mathrm{div} J_q = \nabla(L(\nabla T^{-1})) = \nabla(\lambda T^2(\nabla T^{-1})) = \nabla(-\lambda T^2 \frac{\nabla T}{T^2}) = -\lambda \Delta T = 0, \tag{40}$$

where L and λ are constants in the entropy and Fourier pictures, respectively. Now the 'key' is the fact that div $J_q = 0$, in all pictures.

Energy picture, where $\mathcal{L}_{\ln T} = L^*(\nabla \ln T)^2$,

$$\frac{\partial \mathcal{L}_{\ln T}}{\partial \ln T} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}_{\ln T}}{\partial (\frac{\partial \ln T}{\partial x})} = 0, \tag{41}$$

so

$$\begin{aligned} \frac{\partial \mathcal{L}_{\ln T}}{\partial \ln T} &= 0, \\ \frac{\partial \mathcal{L}_{\ln T}}{\partial (\nabla \ln T)} &= 2L^*(\nabla \ln T), \\ \nabla \frac{\partial \mathcal{L}_{\ln T}}{\partial (\nabla \ln T)} &= L^*(\nabla 2(\nabla \ln T) = L^* 2\Delta \ln T = 0, \end{aligned} \tag{42}$$

hence

$$-L^{*}\Delta \ln T = 0 \tag{42a}$$

and because in stationary state div $J_q = 0$, therefore as

$$J_q = -L^{*}(\nabla \ln T), \quad \mathrm{div} J_q = \nabla(-L^{*}(\Delta \ln T)) = -L^{*}\Delta \ln T = 0, \tag{43}$$

we have got the good result, which can be transformed to the Fourier picture as follows

$$\mathrm{div} J_q = \nabla(-L^{*}(\nabla \ln T)) = \nabla(-\lambda T(\nabla \ln T)) = \nabla(-\lambda T \frac{\nabla T}{T}) = -\lambda \Delta T = 0, \tag{44}$$

where L^{*} and λ are constants in energy and Fourier pictures, respectively, as div $J_q = 0$, in all pictures due to the stationary state in case of energy balance.

Fourier picture, where $\mathcal{L}_T = \lambda(\nabla T)^2$,

$$\frac{\partial \mathcal{L}_T}{\partial T} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}_T}{\partial(\frac{\partial T}{\partial x})} = 0 \tag{45}$$

now

$$\begin{aligned} \frac{\partial \mathcal{L}_T}{\partial T} &= 0, \\ \frac{\partial \mathcal{L}_T}{\partial(\nabla T)} &= 2\lambda(\nabla T), \\ \nabla \frac{\partial \mathcal{L}_T}{\partial(\nabla T)} &= \lambda \nabla 2(\nabla T) = \lambda 2\Delta T = 0, \end{aligned} \tag{46}$$

so

$$-\lambda 2\Delta T = 0 \tag{46a}$$

and as for the stationary state div $J_q = 0$, hence

$$J_q = -\lambda(\nabla T), \qquad \mathrm{div} J_q = \nabla(-\lambda(\nabla T)) = -\lambda \Delta T = 0 \tag{47}$$

we have got the right result.

Resolution of the Lampinen Dilemma

First we have to analyse the picture constants generally. According to (12) we have the connections for the picture constants

$$\begin{aligned} \lambda &= T^{-1}L^{*} = T^{-2}L, && \text{Fourier picture,} \\ L^{*} &= T\lambda = T^{-1}L, && \text{energy picture,} \\ L &= T^{2}\lambda = TL^{*}, && \text{entropy picture.} \end{aligned} \tag{48}$$

We see that λ, L^* and L can be replaced by the appropriate products, too. The phenomenological coefficients are constant near equilibrium. This can be proved if one takes the entropy production from the various pictures. But in the stationary case of heat conduction we must take care of the phenomenological coefficients. They are constant only in their own picture and become temperature dependent if they enter a foreign picture, i. e.

$$\begin{aligned} \lambda &= T^{-1}L^*_\lambda(T) = T^{-2}L_\lambda(T), && \text{Fourier picture,} \\ L^* &= T\lambda_{L^*}(T) = T^{-1}L_{L^*}(T), && \text{energy picture} \\ L &= T^2\lambda_L(T) = TL^*_L(T), && \text{entropy picture.} \end{aligned} \tag{49}$$

In a foreign representation picture the conductivity factors will be

$$\begin{aligned} \lambda &\to \lambda_{L^*}(T) \quad \text{or} \quad \lambda \to \lambda_L(T), \\ L^* &\to L^*_\lambda(T) \quad \text{or} \quad L^* \to L^*_L(T), \\ L &\to L_\lambda(T) \quad \text{or} \quad L \to L_{L^*}(T), \end{aligned} \tag{50}$$

if we take them from their own picture where they were constant. Now we see the different *nonlinearities*. We can cancel the nonlinearities with the following substitutions

$$\begin{aligned} &\lambda_{L^*}(T) = T^{-1}L^* && \text{and } \lambda_L(T) = T^{-2}L && \text{nonlinear } \lambda \text{ in } L^* \text{ or } L \text{ picture}, \\ &L^*_\lambda(T) = T\lambda && \text{and } L^*_L(T) = T^{-1}L && \text{nonlinear } L^* \text{ in } \lambda \text{ or } L \text{ picture}, \\ &L_\lambda(T) = T\lambda && \text{and } L^*_L(T) = TL^* && \text{nonlinear } L \text{ in } \lambda \text{ or } L^* \text{ picture}. \end{aligned} \tag{51}$$

And now let us consider the resolution of the particular problem of the ominous Lampinen dilemma:

Case 1

According to the formulation of [1] the force in the expression of the entropy production is represented in the entropy picture. Despite this representation, the flux is inserted in the Fourier picture. Now this was a step which caused nonlinearity because a constant phenomenological coefficient from its own picture entered a foreign picture where it became temperature dependent. But with the insertion of expression (51) this illusory nonlinearity can be cancelled, i. e. $\lambda \to \lambda_L(T) = T^{-2}L$ gives us the right form of the Lagrange density in the entropy picture, which is the linear one

$$\frac{\lambda(\nabla T)^2}{T^2} \to \lambda_L(T)\frac{(\nabla T)^2}{T^2} = L\frac{(\nabla T)^2}{T^4}, \tag{52}$$

as we know it.

Case 2

On the other hand, $\sigma = \frac{\lambda(\nabla T)^2}{T^2}$ can be understood as the entropy production in the Fourier picture, too. From this one can obtain the Lagrange density by multiplying it by T^2. This multiplication has not been done, therefore σ has gone in [1] into the variation as Lagrange density. But for the Fourier picture this was the wrong step. Thus we could not get the right form of the Euler–Lagrange linear differential equation of the Fourier heat conduction process.

Case 3

Let us see the consequence of the foreign flux in [1]:

As in [1] the flux was represented in Fourier picture, but the force in entropy picture, therefore *the conjugation* according to Fourier heat conduction was violated. In expression (13) we showed the right conjugations of the forces and the fluxes for Fourier heat conduction. Without applying the above substitution of expressions (51) there remains a flux *'out of conjugation'* type in the problem of [1] for the entropy production and Lagrange density. Now the Lagrange density is the same as in the left hand side of expression (52). Because of this remaining foreign flux we apply another substitution from the equations of the expression (49) namely $\lambda = T^{-2}L_\lambda(T)$. Therefore $\lambda\frac{(\nabla T)^2}{T^2}$ can be written as

$$L_\lambda(T)\frac{(\nabla T)^2}{T^4}, \tag{53}$$

which is a nonlinear expression. But in this situation we have to emphasize that the flux coming from the Fourier picture (the index indicates this fact) into the entropy picture causes a new problem, namely that 'the process' is not the Fourier heat conduction process any more because of the violation of the conjugation rule of expression (13) for the forces and the fluxes. In the expressions in [1] a force in the entropy picture 'drives' a current in Fourier picture. This is an embarrassing situation. So because of a foreign current the author of [1] 'left' unconsciously the problem of the classical Fourier heat conduction process causing a nonlinear situation with nonlinear Lagrange density. With this Lagrange density was formed the Euler–Lagrange differential equation showing nonlinear character. Now we can create a system of out-of-conjugation type Lagrange densities in the different pictures. We can see in *Table 1* these cases according to

Table 1

λ	enters the entropy picture:	$\lambda\frac{(\nabla T)^2}{T^2}$	$=L_\lambda(T)\frac{(\nabla T)^2}{T^4}$,
λ	enters the energy picture:	$\lambda\frac{(\nabla T)^2}{T}$	$=L_\lambda^*(T)\frac{(\nabla T)^2}{T^2}$,
L^*	enters the entropy picture:	$L^*\frac{(\nabla T)^2}{T^3}$	$=L_{L^*}(T)\frac{(\nabla T)^2}{T^4}$,
L^*	enters the Fourier picture:	$L^*\frac{(\nabla T)^2}{T}$	$=\lambda_{L^*}(T)(\nabla T)^2$,
L	enters the energy picture:	$L\frac{(\nabla T)^2}{T^3}$	$=L_L^*(T)\frac{(\nabla T)^2}{T^2}$,
L	enters the Fourier picture:	$L\frac{(\nabla T)^2}{T^2}$	$=\lambda_L(T)(\nabla T)^2$.

the actual foreign currents and representational pictures with a convenient substitution of (49):

In the first column of *Table 1* we find the linear phenomenological coeffcients and in the second one we can find the nonlinear temperature dependent phenomenological coefficients, showing the nonlinear Lagrange densities relative to linear shaped Lagrange densities.

We build the nonlinear Euler–Lagrange differential equations from the nonlinear Lagrange densities in the first column of *Table 1*. As to the variational disposal we can set out from the second column of *Table 1* where, according to the nonlinear phenomenological coefficient, there are two possibilities. On the one hand, the variation can go according to the phenomenological coefficient (i. e. according to the force as *case a*) and on the other hand, according to the index (i. e. according to the flux as *case b*). Therefore we list according to the lines (from 1 to 6) of *Table 1* the nonlinear Euler–Lagrange differential equations in *Table 2* with constant phenomenological coefficients:

Table 2

1_b	and	6_a :$(\nabla T)^2 - T\Delta T = 0$,
2_b	and	4_a :$(\nabla T)^2 - 2T\Delta T = 0$,
3_b	and	5_a :$2(\nabla T)^2 - T\Delta T = 0$,
4_b	and	2_a :$-T\Delta T = 0$,
5_b	and	3_a: $\Delta T/T = 0$,
6_b	and	1_a :$\Delta T = 0$.

From (49), with substitution, these equations can be expressed with the temperature dependent phenomenological coefficients. The identities in *Table 2* refer to the similarity of Euler–Lagrange differential equations. This *shadow or phantom* type nonlinear differential equation system of the Fourier heat conduction problem can be instructive and gives an interesting picture of nonlinear heat conduction with temperature dependent phenomenological coefficients. This article clearing the problem is not only *argumentum ad nominem* for [1], as time to time come to light similar ideas for the Fourier heat conduction problem in connection with the Prigogine principle. The nonlinear differential equation as end result from [1] is the same as the 1_b type nonlinear differential equation in *Table 2.* Symmetries exist because of the two interpretations of the variational disposal in the different pictures.

Summary

The Fourier heat conduction process is one where the flux depends only on the conjugated thermodynamic force which appears in the entropy production, i. e. which can be determined by the phenomenological coefficient L_{jj} of the main diagonal of a tensor, so this is a simple or direct irreversible process. Stationary states, states in which the properties of the system are time independent, play an important role in applications of non-equilibrium thermodynamics. Stationary non-equilibrium states have an important feature and this is a special one: under certain conditions they are characterized by a minimum rate of entropy production which is compatible with the external constraints imposed on the system. This feature manifests itself under the conditions of constant phenomenological coefficients. But this is generally not valid in real systems, so the above statement means that the overall gradients of the thermodynamic properties throughout the entire system must be small enough for the assumption of constant phenomenological coefficients to be justified. This can be approximately justified by the constancy of the conductivity coefficients in Fourier heat conduction, too. 'Under certain conditions' means that the Onsager conditions are fulfilled, i. e.

the linear laws,
the reciprocity relations and
the constancy of phenomenological coefficients

are valid. During the evolution of a system from its initial state to the stationary state the rate of entropy production constantly diminishes and in a stationary state the change in rate of entropy production is stopped. But if the Onsager conditions are not fulfilled this cannot be proved in a general

form. The constancy of phenomenological coefficients is one of the Onsager conditions. We can speak only in this case about *linearity* in a clear sense. If the phenomenological coefficients have changes, there are two cases of nonlinearity. First we speak about *quasilinearity* if the phenomenological coefficients depend on the local equilibrium state variables. If they depend on the thermodynamic forces, then we can regard them rigorously as *nonlinear* and speak about *nonlinearity.* These problems were also cleared by Gyarmati in his cited work.

As to the representational pictures and the phenomenological coefficients the linearity and nonlinearity must be considered from the own picture of point of view of the phenomenological coefficients. So we have to speak about *real* linearity or nonlinearity only if a phenomenological coefficient is in its own picture. Therefore the conscious use of the picture representation is a significant obligation. In the short communication [1] the author violated the principle of minimum entropy production by the arbitrary use of picture representation among others. This led to a nonlinear differential equation instead of a linear one. In this manner was born the *causeless criticism of the principle of minimum entropy production.* Finally it must be emphasized that the picture representation concept needs not only vertical but also lateral thinking. This is one feature which is also important if one makes approaches to non-equilibrium thermodynamic concepts. We always have to meet this requirement. As we sum up our investigations we can say that neither a right Lagrange density nor a right variational comprehension is identical with an inappropriate flux in a representational picture. So the original problem becomes of the out-of-conjugation type, i. e. nonlinearity will exist. During the calculation process one must follow the same representation picture and so the discussion of the problem must be made according to one of the three dissipation potentials or Lagrange densities as follows:

$$\begin{aligned} \sigma T^2 &= J X^{**}, && \text{Fourier picture,} \\ \sigma T &= J X^{*}, && \text{energy picture,} \\ \sigma &= J X, && \text{entropy picture.} \end{aligned} \tag{54}$$

The classical Fourier heat conduction problem is a stationary type process near the thermodynamic equilibrium for which the Prigogine principle, the principle of minimum entropy production is valid. In short communication [1], because of a wrongly interpreted Lagrange density or incorrect variational disposal, the classical Fourier heat conduction problem was not discussed. *Quidquid agis prudenter agas et respice finem.*

References

1. LAMPINEN, M. J.: Short Communication. A problem of the Principle of Minimum Entropy Production. *J. Non-Equilib. Thermodyn.*, Vol. 15. No. 4 (1990).
2. GYARMATI, I.: Non-Equilibrium Thermodynamics. Field Theory and Variational Principles. Springer, Berlin-Heidelberg-New-York (1970).
3. DE GROOT, S. R. – MAZUR, P.: Non-Equilibrium Thermodynamics. North-Holland Publishing Co., Amsterdam (1962).
4. GYARMATI, I.: *Zh. Fiz. Khim.* Vol. 39, p. 1489 (1965).
5. GYARMATI, I.: *Ann. Phys.* Vol. 7, p. 23, p. 353 (1969).
6. SÁNDOR, J.: *Zh. Fiz. Khim.* Vol. 44, p. 2727 (1970).
7. SÁNDOR, J.: *Acta Chim.* (Budapest) Vol. 67, p. 303 (1971).
8. SÁNDOR, J.: *Electrochim. Acta*, Vol. 17, p. 673 (1972).
9. VERHÁS, J.: *Z. Phys. Chem.* Vol. 249, p. 119 (1972).
10. VINCZE, GY.: *Ann. Phys.* Vol. 7, p. 225 (1971).
11. VINCZE, GY. : *Acta Chim.* (Budapest) Vol. 75, p. 33 (1972).
12. VINCZE, GY.: Ann.Phys. Vol. 7, p. 30, p. 55 (1973).
13. FARKAS, H. – NOSZTICZIUS, Z.: *Ann. Phys.* Vol. 7, p. 341 (1971).
14. FARKAS, H.: *Int. J. Engng. Sci.* Vol. 13, p. 1035 (1975).
15. SINGH, P.: *Int. J. Heat Mass Transfer*, Vol. 19, p. 581 (1976).
16. SINGH, P.: *J. Non-Equilib. Thermodyn.* Vol. 1, p. 105, (1976).
17. MÁRKUS, F. – GAMBÁR, K.: *J. Non-Equilib. Thermodyn.* Vol. 16, pp. 27-31 (1991).
18. NYIRI, B.: *J. Non-Equilib. Thermodyn.* Vol. 16, pp. 39-55 (1991).
19. VERHÁS, J.: *Ann. Phys.* Vol. 20, p. 90 (1967).
20. VERHÁS, J.: *Int. J. Non-Equilib. Thermodyn.* Vol. 8, p. 201 (1983).
21. RUSZIN, É. – VÁN, P.: *Ukranszkij Fizicseszkij Zsurnal*, Vol. 5, p. 36, pp. 743-750 (1991).
22. BHATTACHARYA, D.: *Ann. Phys.* Vol. 39, p. 325, (1982).
23. SINGH, P. – BHATTACHARYA, D.: *Acta Mechanica*, Vol. 30, p. 137 (1978).
24. SINGH, P. – BHATTACHARYA, D.: *Indian J. Pure Appl. Math.*, Vol. 10(a) p. 1092 (1979).
25. BHATTACHARYA, D.: Wärme- und Stoffübertragung, Vol. 17, p. 26 (1982).
26. FARKAS, H.: *Internat. J. Engng. Sci.* Vol. 13, p. 1035 (1975).
27. BIOT, M. A.: Variational Principles in Heat Transfer, Oxford Press (1970).
28. GLANSDORFF, P. – PRIGOGINE, I.: *Physica*, Vol. 20, p. 773 (1954).
29. BHATTACHARYA, D.K.: The Application of Governing Principle of Dissipative Processes to Incompressible Fluid Flow of Heat Transfer. Thesis. Kharagpur. (India), 1978, *Ann.d. Phys*, Vol. 7, 39 (1982), pp. 325-332. *Acta Mechanica*, Vol. 47 (1983) pp. 87-94.
30. VERHÁS, J.: Gyarmati's Variational Principle and Some Applications. Accademia Peloritana Dei Pericolanti Messina (1991).
31. VÁN, P. – RUSZIN, É.: Derivation of the Basic Equations of MHD from the Governing Principle of Dissipative Processes. *Acta Physica Hungarica* Vol. 68 (3-4), pp. 227-239 (1990).
32. FEKETE, D.– SYSTEMATIC, A.: Application of Gyarmati's Wawe Theory of Thermodynamics to Thermal Waves in Solids. *Phys. Stat. Sol.(b)* Vol. 105, p. 161 (1981), Budapest.