A PROBABILISTIC PROOF OF INEQUALITIES OF SOME MEANS

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Note: In this journal several articles have been published on the physical role, the generalization and the inequalities of the different means [1], [2]. The following paper is the continuation of these investigations from another aspect.

In this paper we prove inequalities of some means using a probabilistic method and we also deal with a classical inequality.

Theorem. If a and b are positive and $a < b$, then

$$\frac{1}{e} \cdot \frac{b^{\frac{b}{b-a}}}{a^{\frac{a}{b-a}}} < \frac{a+b}{2} \tag{1}$$

$$\frac{b-a}{\ln b - \ln a} < \frac{1}{e} \cdot \frac{b^{\frac{b}{b-a}}}{a^{\frac{a}{b-a}}} \tag{2}$$

Proof. Let $x_1, x_2, x_3, \ldots$ be independent random variables uniformly distributed over the interval (a, b). By the strong law of large numbers $\lim\limits_{n\to\infty} \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{a+b}{2}$ and $\lim\limits_{n\to\infty} \frac{\ln x_1 + \ln x_2 + \ln x_3 + \ldots + \ln x_n}{n} =$

$$= \frac{b \cdot \ln b - a \cdot \ln a}{b-a} - 1 \quad \text{i.e.} \quad \lim_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \ldots x_n} = \frac{b^{\frac{b}{b-a}}}{a^{\frac{a}{b-a}}} \cdot \frac{1}{e}$$

with probability one, therefore (1) is a consequence of the arithmetic-geometric mean inequality.

Similarly, (2) follows from the geometric-harmonic mean inequality.

Remarks

1. If $f_i(x)$, $x \in (a, b)$, $i = 1, 2$ are continuous and strictly monotonic functions, $p_k > 0$, $k = 1, 2, \ldots, n$

$$m_i(x_1, x_2, \ldots, x_n) = f_i^{-1}\left(\frac{p_1 \cdot f_i(x_1) + p_2 \cdot f_i(x_2) + \ldots + p_n \cdot f_i(x_n)}{p_1 + p_2 + \ldots + p_n}\right)$$

and f_2 is increasing, then a necessary and sufficient condition that $m_1 \leq m_2$ for all $x_1, x_2, \ldots, x_n$ and $p_1, p_2, \ldots, p_n$ is that $f_2 \cdot f_1^{-1}$ should be convex ([3] p. 75). If we know that $m_1 \leq m_2$ (e.g. because of the here mentioned theorem) and x_k, $k = 1, 2, \ldots$ are independent random variables depending on some parameters, then using our probabilistic method we obtain great many new inequalities.

2. Let us consider the Carleman's inequality: if $x_1, x_2 \ldots$ are nonnegative and not all zero, then

$$\sum_{n=1}^{\infty} (x_1 x_2 \ldots x_n)^{1/n} < e \sum_{n=1}^{\infty} x_n .$$

The constant e is the best possible [3]. It is also known [4] that if λ_n is the best constant for the inequality

$$\sum_{m=1}^{n} (x_1 x_2 \ldots x_m)^{1/m} < \lambda_n \sum_{m=1}^{n} x_m ,$$

then

$$\lambda_n = e - \frac{2\pi^2 e}{(\log n)^2} + 0\left(\frac{1}{(\log n)^3}\right), \quad (n \to \infty).$$

At the same time it is evident that if x_k, $k = 1, 2, \ldots$ are independent random variables having the same uniform distribution e.g. over $(0, 1)$ then

$$\lim_{n \to \infty} \frac{1}{n}\left[\sum_{m=1}^{n} (x_1 x_2 \ldots x_m)^{1/m} - \lambda \cdot \sum_{m=1}^{n} x_m\right] = \frac{1}{e} - \frac{\lambda}{2}$$

with probability one and the right side is negative if and only if $\lambda > \frac{2}{e}$, therefore in this sense $\frac{2}{e}$ is the best possible constant. Many other inequalities (Hardy's inequality etc.) can be handled in the same way.

3. Using the notations:

$$M_{ar} = \frac{a+b}{2}; \qquad M_{\exp} = \frac{1}{e} \cdot \frac{b^{\frac{b}{b-a}}}{a^{\frac{a}{b-a}}}; \qquad M_{\log} = \frac{b-a}{\ln b - \ln a};$$

$$M_{\text{geom}} = \sqrt{ab}; \qquad M_{\text{harm}} = 2\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} \qquad (a \neq b, \quad a > 0, \quad b > 0)$$

we have proved:

$$M_{\log} < M_{\exp} < M_{\text{ar}}.$$

It is also true that:

$$M_{\text{harm}} < M_{\text{geom}} < M_{\log} < M_{\exp} < M_{\text{ar}} . \tag{3}$$

The proof of the relation:

$$M_{\text{geom}} < M_{\log} \tag{3a}$$

can be found in [1].

The inequality:

$$M_{\log} < M_{\text{ar}} \tag{3b}$$

is also proved in [1] and [5].

Summary

The main conclusion of the paper is the following sequence of inequalities:

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{1}{e} \frac{b^{\frac{b}{b-a}}}{a^{\frac{a}{b-a}}} < \frac{a+b}{2}$$

where $a > 0, b > 0$ and $a \neq b$.

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