SOME NOTES ON THE PROBABILISTIC PROBLEM A, B, C, D OF A. S. EDDINGTON

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1. Preliminaries

In the year 1934 A. S. EDDINGTON, the famous English astronomer, had raised the following probabilistic problem [1]. *If the persons A, B, C and D each speak the truth once in 3 times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth?*

This problem and its solution given by EDDINGTON were the starting points of a series of discussions. We quote the well-known book of F. M. REZA [2] in which we find a reference to the American mathematician M. GARDNER, who remarks in his paper [3] that the problem with its solution given by EDDINGTON shows the confusion ruled in probabilistic thinking before the set theoretical methods, that is, before the theory of KOLMOGOROV.

"The solution of EDDINGTON" — reads in Gardner's paper — "raised a vivid dispute that has not been settled until today."

REZA remarks in his book that the main difficulty is how we interpret and set up the problem. We can avoid the difficulties, as REZA remarks, by making use of the following assumptions: (1) Each of the four persons is telling a declaration. (2) Each of A, B, C is telling a declaration that affirms or denies the declaration told by the following person (that is, B, C and D).

Let us remark that a solution of this problem is found in the excellent book by E. PARZEN [4].

2. Assertion—negation sequences with true or false values

In what follows it is attempted to give a possible solution of the problem set by EDDINGTON differing from the solutions by both PARZEN and EDDINGTON. Referring to the propositions of REZA quoted above, we start from the following assumptions:

(1) *Persons A_1, A_2, A_3, . . . , A_{n-1} each are telling a declaration; this declaration is an assertion or a negation concerning the declaration of the imme-*

diately following person. So we have an assertion—negation sequence consisting f the declarations of the persons $A_1, A_2, A_3, \ldots, A_{n-1}$.

(2) *The declaration of each of the persons* $A_1, A_2, A_3, \ldots, A_{n-1}$ *has a probability* $0 \leqq p \leqq 1$ *true (the person tells the truth) and a probability* $1-p = q$ *false (the person lies). The persons* $A_1, A_2, A_3, \ldots, A_{n-1}$ *each are telling the truth or a lie independently from the others. If for example the declaration of any person* A_i *is true (false), then this event is totally independent from the true or false characters of the declarations of the other persons* $A_1, A_2, A_3, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n-1}$.

(3) *The declaration of the person* A_{n-1} *refers to the true or false character of the declaration of the last person* A_n, *but it is completely irrelevant whether the declaration of* A_n *is an assertion or a negation or something else; it is only essential whether* A_n *tells the truth or lies. Concerning the probability of the event that the last person* A_n *tells the truth or lies, no assumption is made, it will result from the assertion—negation sequence as well as the true or false characters of the declarations of* $A_1, A_2, \ldots, A_{n-1}$.

To show how a given assertion—negation sequence acts, let us regard a simple example including only $n = 3$ persons:

If A_1 *asserts that* A_2 *denies that* A_3 *is a liar, what is the probability that* A_3 *is telling the truth?*

(In this case the assertion—negation sequence has the form:

A_1 — A_2

assertion negation; at the end stands that A_3 is a liar. Each assertion—negation sequence ends with the closure: the last person is either a liar or he is telling the truth.)

Solution of our example:

Case I: A_1 *tells the truth.* In this case A_2 denies indeed that A_3 is a liar' or what is the same, A_2 asserts that A_3 tells the truth. If the assertion of A_2 is true (if the person A_2 tells the truth), then A_3 tells the truth, but if A_2 lies, then A_3 lies too. In this case A_3 tells the truth with probability $pp = p^2$.

Case II: A_1 *lies.* Now A_2 does not deny but asserts that A_3 lies. If the declaration of A_2 is true, then A_3 lies indeed. If A_2 lies, then A_3 tells the truth. In this case A_3 tells the truth with probability $qq = q^2$.

Consequently in our assertion—negation sequence A_3 tells the truth with probability $p^2 + q^2$.

Let us see, as a second example, the problem by EDDINGTON. Before beginning the analysis of this problem let us remark that from the viewpoint of the assertion—negation sequences it is only necessary to give the probability of the truth (or falsehood) of the declarations of the persons A, B and C, but as to the last person D, the probability of his true or false declaration will result from the declarations of the previous persons. Hence we shall solve a (in this sense) modified problem.

The problem reads as follows: *If A asserts that B denies that C asserts that D lies, what is the probability that D is telling the truth?*

Solution:

Case I: *A tells the truth.* Now B denies indeed that C asserts that D lies.

I/a: *B tells the truth.* In this case C does not assert, but denies that D lies. Hence C asserts that D is telling the truth. Well, if C lies, then D lies too, but if C tells the truth, then D also tells the truth.

Hence in the case I/a D tells the truth with probability

$$ppp = p^3.$$

I/b: *B lies.* Hence C asserts indeed that D lies. If the assertion of C is true, then D lies. But if C lies, then D tells the truth. In this case we have the probability for D telling the truth:

$$pqq = pq^2.$$

Case II: *A lies.* In this case B does not deny, but asserts that C asserts that D is a liar.

II/a: *B tells the truth.* Hence C asserts indeed that D is a liar. If C tells the truth, then D lies and if C lies, then D tells the truth. In this case (II/a) D tells the truth with probability

$$qpq = pq^2.$$

II/b: *B lies.* Hence C does not assert but denies that D is a liar, that is, C asserts that D tells the truth. If the assertion of C is true, then D tells the truth and if C lies, then D lies too. In the case II/b D tells the truth with probability

$$qqp = pq^2.$$

The cases I/a, I/b, II/a, II/b being mutually exclusive, D is telling the truth in the assertion—negation sequence of the EDDINGTON problem with probability

$$P = p^3 + pq^2 + pq^2 + pq^2 = p^3 + 3pq^2.$$

In the original EDDINGTON problem there were $p = \frac{1}{3}$ and $q = \frac{2}{3}$, so we have

$$P = \frac{1}{27} + 3 \cdot \frac{1}{3} \cdot \frac{4}{9} = \frac{13}{27}.$$

If $p = q = \frac{1}{2}$, then $P = \frac{1}{8} + 3 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{4}{8} = \frac{1}{2}$.

If $p = \frac{2}{3}$, $q = \frac{1}{3}$, we have

$$P = \frac{8}{27} + 3 \cdot \frac{2}{3} \cdot \frac{1}{9} = \frac{14}{27}.$$

It is clear that by modifying the formulation of this problem, we obtain a probability for D telling the truth that may be different from the previous one. For example, if we put up the problem:

If A asserts that B asserts that C asserts that D is a liar, what is the probability that D tells the truth?

This problem can be solved in the same way as before and we obtain for D telling the truth the probability

$$q^3 + 3p^2q.$$

After our previous discussions there is a clear picture of the structure of the assertion—negation sequences belonging to the various possible formulations of the EDDINGTON problem. All can be solved by the same analysis as before.

Finally let us remark that our model can be applied in several practical cases. If for example the broadcasting stations A, B, and C each independently from the others are telling the truth with probability $0 \leq p \leq 1$ and are sending false communications with probability $1-p = q$, and A asserts that B denies that C asserts that in the land D occurred an earthquake, what is the probability that in fact, in the land D there occurred an earthquake?

3. The probability sample space of the problem

To each assertion—negation sequence with given persons $A_1, A_2, \ldots, A_{n-1}, A_n$ we will construct a unique sample space.

For sake of simplicity let us regard first the case of only $n = 3$ persons: A_1 and A_2 are telling assertions or negations and the declaration of A_2 refers to the true or false declaration of the last person A_3. The assertions or negations of the persons A_1 and A_2 as well as their true or false characters have been seen to perfectly determine the probability of A_3 telling the truth (or A_3 telling a lie). Our discussion of the assertion—negation sequence of these persons A_1, A_2 and A_3 in item 2 has shown that we have only to analyse the declarations of A_1 and A_2 concerning their true or false character and we obtained in this manner the probability of the event that A_3 is telling the truth. Hence it is enough to take into account all the pairs:

(true, true); (true, false); (false, true); (false, false),

where the first word denotes the value of the declaration of A_1 and the second word the value of the declaration of A_2.

The probability measures of the pairs are:

$P(\text{true, true}) = pp$; $P(\text{true, false}) = pq$; $P(\text{false, true}) = qp$; $P(\text{false, false}) = qq$, where $P(E)$ denotes the probability of the event E, as usually. So we have altogether the probability

$$p^2 + 2pq + q^2 = (p + q)^2 = 1.$$

Thus the set of our pairs forms a sample space, the sample space of our problem for the case of $n = 3$ persons.

In our simple example of 3 persons we have already calculated the probability for the event that the person A_3 tells the truth in a given assertion—negation sequence. We see now that this event is formed by the elementary events: (true, true) and (false, false).

Each possible formulation of the 3-persons problem, that is, each possible assertion—negation sequence of the case of $n = 3$ persons has the same sample space introduced above.

In the case of $n = 4$ persons the sample space consists of all the 3-tuples (these are the elementary events of the sample space):

(true, true, true); (true, true, false); (true, false, true);
(false, true, true); (true, false, false); (false, true, false);
(false, false, true); (false, false, false).

Our sample space contains now $2^3 = 8$ elementary events; the probability of an elementary event has the form

$$p^j q^k,$$

where j and k denote the number of the words "true" and "false", respectively, and $j+k = 3$. The sum of the probabilities of the elementary events is

$$\sum_{j=0}^{3} \binom{3}{j} p^j q^k = (p + q)^3 = 1 .$$

In the EDDINGTON problem all those elementary events are seen to form the random event "D is telling the truth", in which the word "true" occurs only once and the word "false" twice, and beside them the single elementary event:

(true, true, true).

In general, if we have to do with n persons, the related sample space is the set of all possible $(n-1)$-tuples, each consisting of the words "true" and "false". In an elementary event the word "true" occurs j-times and the word "false" k-times, where $0 \leq j \leq n-1$, $0 \leq k \leq n-1$ and $j+k = n-1$, and we have to examine all possible arrangements of these words. The probability of such an elementary event is

$$p^j q^k.$$

Our sample space contains now altogether 2^{n-1} elementary events with the total probability mass

$$\sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^k = (p+q)^{n-1} = 1\,.$$

If there is a given assertion—negation sequence, i.e. a generalized EDDINGTON-problem of n persons $A_1, A_2, \ldots, A_{n-1}, A_n$, it can be solved by the same analysis as performed in the case of 4 persons to obtain the probability of A_n telling the truth (in the given assertion—negation sequence).

Notice furthermore that by counting the probabilities of A_n telling the truth to all possible assertion—negation sequences, in general we obtain different values. Namely, those elementary events, the collection of which forms the random event "A_n tells the truth" in the given assertion—negation sequence, are different in general from sequence to sequence. Moreover, these probabilities are *conditional probabilities*, that is, in each given assertion—negation sequence the probability of the following event is counted: "A_n tells the truth with the condition that from the assertions and negations of the persons $A_1, A_2, \ldots, A_{n-1}$ a definite assertion—negation sequence has already been formed". Thinking so, we should have to construct a more general sample space, where the elementary events of the previous sample space are associated with the possible assertion—negation sequences.*

But proceeding in this way and constructing this new, more general sample space no new results are achieved after all.

* Among the different assertion—negation sequences there are several ones with the same logical meaning, i.e. logically equivalent. For example, in the case of $n = 3$ persons the following two sequences are logically equivalent: (1) A_1 asserts that A_2 asserts that A_3 tells the truth, and (2) A_1 asserts that A_2 denies that A_3 lies.

Summary

In this paper a solution is given to the famous probabilistic problem of A. S. EDDINGTON. This solution differs from the solutions given by EDDINGTON and by PARZEN, respectively.

References

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