Periodica Polytechnica
Civil Engineering

62(1), pp. 219–225, 2018
https://doi.org/10.3311/PPci.11398

RESEARCH ARTICLE

A New Compliance-Function-Shape-oriented Robust Approach for Volume-Constrained Continuous Topology Optimization with Uncertain Loading Directions

Anikó Csébfalvi[1*]

Received 28 April, Revised 18 August 2017, Accepted 30 August 2017

Abstract

The paper presents a new compliance-function-shape-oriented robust approach for the volume-constrained continuous topology optimization with uncertain loading directions. The pure set-based algorithm try to rearrange (take away) some amount of the material volume, originally used to minimize the nominal-compliance, to make a more balanced compliance-function-shape on the set of feasible directions which is less sensitive to the directional fluctuation. The objective is the area of the compliance function shape defined on the set of feasible directions. The area-minimal shape searching process is controlled by the maximum allowable increase of the nominal-compliance. The result will be a more robust compliance function shape which can be characterized by a higher nominal-compliance but a smaller curvature about it in any direction. Using the terminology of the classical variational problems, the proposed approach can be classified as a curve-length or surface-area minimizing inner-value problem where the inner condition, namely the maximum allowable increase of the nominal-compliance, expressed as a percentage of the original nominal compliance, the searching domain is defined implicitly as integration limits in the objective formulation and a usual equality relation is used to prescribe the allowable material volume expressed as a percentage of the total material volume. Two examples are presented to demonstrate the viability and efficiency of the proposed robust approach.

Keywords

topology optimization, uncertain parameters, directionally uncertain loads, robust optimization, area minimization

[1] University of Pécs, Hungary
[*] Corresponding author, email: csebfalvi.witch@gmail.com

1 Introduction

Uncertainty is an important consideration in continuous topology optimization to produce robust and reliable solutions. The source of uncertainty may be the variability of applied loads, spatial positions of nodes, material properties, and so on. Various deterministic and stochastic approaches have been developed to account for different types of uncertainty in structural design and optimization methods to get robust and reliable solutions. The interested reader is directed to Bendsøe and Sigmund [1] and Deaton and Grandhi [2] which contain extensive bibliographies on this subject. In this paper, it is assumed that the only source of uncertainty is the variability of the applied load directions and the compliance is used as performance measure in the in the volume-constrained topology optimization. The volume-constrained structural optimization models of continuum structures which are able to take into account the directional uncertainty of the applied loads can be divided into two groups (Ben-Tal et al. [3]): (1) deterministic and (2) stochastic. A critical examination and comparison of the volume-constrained deterministic and stochastic topology optimization models with uncertain loading directions, from engineering point of view, was presented by Csébfalvi and Lógó [4].

The deterministic models which try to minimize the volume-constrained worst compliance on the set of feasible loading directions can be formulated by several different ways. De Gournay et al. [5] presented an approach for shape and topology optimization of the robust compliance via the level set method which minimizes the worst-case compliance using a semi-definite programming method to select the best descent direction in the iteration process. Thore et al. [6] presented a large-scale robust topology optimization method under load-uncertainty where the loads vary in uncertainty sets. The problem can be formulated as a semi-infinite optimization problem, which can be replaced by a non-linear semi-definite problem. A worst-load-direction oriented unified common framework was presented by Csébfalvi [7] for robust optimization of both continuum and truss structures with uncertain load directions, which can be used for volume minimization of continuum structures with compliance constraints and weight minimization of truss structures with displacement and stress constraints.

The stochastic models apply parametric statistical tools to describe the directional uncertainty of the loads. The most popular model minimizes the volume-constrained expected compliance with loading directional uncertainty, where the directional uncertainties are assumed normally distributed and statistically independent. The expected compliance minimization model can be transformed to a standard volume-constrained multi-load compliance-minimization problem by several different ways where the load cases and weights are derived analytically or numerically. However, it is not straightforward how could be select the load cases to ensure that all critical cases are considered in the sampling schema which needed to obtain an accurate approximation of the original continuous problem. Dunning et al. [8] proposed a “pseudo-sample-based” stochastic method for considering loading directional uncertainty in topology optimization in order to produce robust solutions. Carrasco et al. [9] presented a sample-based approach for the volume-constrained expected compliance minimization in topology optimization with stochastic loading directions. In the work of Liu et al. [10] the directional uncertainty of the applied loads is described by directional interval variables which are divided into many small intervals, and then the uncertain small interval variables are approximated by their deterministic midpoints.

This paper presents a novel compliance-function-shape-oriented robustness measure and a robust algorithm using this measure for the volume-constrained continuous topology optimization with uncertain loading directions. The pure set-based algorithm try to rearrange (take away) some amount of the material volume, originally used to minimize the nominal-compliance, to make a more balanced compliance-function-shape on the set of feasible directions which is less sensitive to the directional fluctuation. The essence of the proposed new approach is shown in Figure 1. The objective function is the surface area of the compliance function shape defined on the set of feasible loading directions. The area-minimal shape searching process is controlled by the maximum allowable increase of the nominal-compliance. The result will be a more robust compliance function shape which can be characterized by a higher nominal-compliance but a smaller curvature about it in any direction.

In Figure 1, the directional uncertainty of the applied load is denoted by a symmetric set $\dot{\alpha}-\overline{\alpha}\le\alpha\le\dot{\alpha}+\overline{\alpha}$ about the nominal direction $\dot{\alpha}$. The nominal and the novel robust shape-oriented directional compliance function denoted by $\boldsymbol{nc}(\alpha)$ and $\boldsymbol{sc}(\alpha)$, respectively.

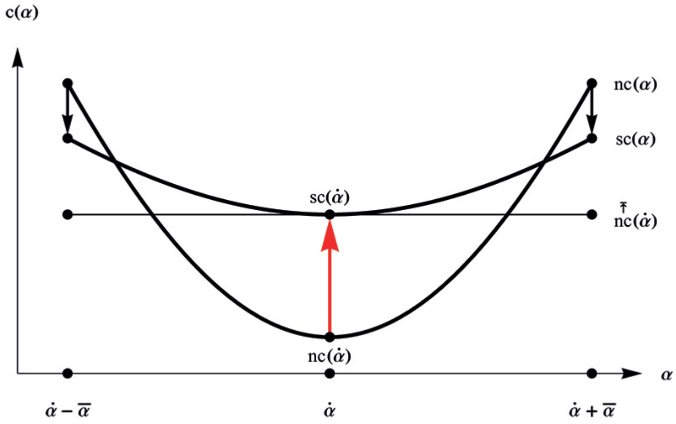

Fig. 1 The visualization of the proposed new approach as a curve-length-minimization problem with an inner-value-condition

Using the terminology of the classical variational problems, the proposed approach can be classified as a curve-length or surface-area minimizing inner-value problem where the inner condition, namely the maximum allowable increase of the nominal-compliance, expressed as a percentage of the original (deterministic) nominal compliance, the searching domain is defined implicitly as integration limits in the objective function formulation and the usual equality relation is used to prescribe the allowable material volume expressed as a percentage of the total material volume.

The paper is organized as follows. Section 2 focuses on the mathematical formulation and the numerical algorithm of the considered problem. The examples used to illustrate the proposed approach are presented in Section 3. Finally, some concluding remarks are presented in Section 4.

2 Model and algorithm

In this paper, for sake of simplicity but without loss of generality the theoretical model of the compliance-function-shape-oriented robust approach will be formulated only for 2D structures with two directionally uncertain point loads. The model, when there is only one directionally uncertain load, can be given by straightforward simplifications from this model. It will be shown at the end of this section that in the case of one directionally uncertain load there is special symmetric problem type where the objective and gradient functions can be generated symbolically using symbolic manipulation software.

A directionally uncertain point load with magnitude f_i, $i\in\{1,2\}$ can be written in terms of two orthogonal loads. We construct a four dimensional load vector $\boldsymbol{f}(\boldsymbol{\alpha})$ such that odd entries of the vector correspond to horizontal loads and even entries to vertical loads:

$$\boldsymbol{f}(\boldsymbol{\alpha}) = [\mathrm{f}_1\boldsymbol{cos}(\alpha_1),\ \mathrm{f}_1\boldsymbol{sin}(\alpha_1),\ \mathrm{f}_1\boldsymbol{cos}(\alpha_1),\ \mathrm{f}_1\boldsymbol{sin}(\alpha_1)] \qquad (1)$$

Exploiting the fact that load vector $\boldsymbol{F}$ has maximum four nonzero entries the surface-area of compliance function $\boldsymbol{sc}(\boldsymbol{x})$ can be described as follows (see, for example, Olver [11]):

$$sc(x)=\int_{\dot{\alpha}_1-\bar{\alpha}_1}^{\dot{\alpha}_1+\bar{\alpha}_1}\int_{\dot{\alpha}_2-\bar{\alpha}_2}^{\dot{\alpha}_2+\bar{\alpha}_2}\sqrt{1+\left(\frac{\partial c(\alpha)}{\partial\alpha_1}\right)^2+\left(\frac{\partial c(\alpha)}{\partial\alpha_2}\right)^2}\,d\alpha_2 d\alpha_1 \quad (2)$$

where

$$c(\alpha)=f(\alpha)\,Qf(\alpha)^t \quad (3)$$

is the directional compliance function and Q is a 4 × 4 symmetric matrix consisting of such entries of K^{-1} which are needed in the directional compliance computation. It worth to note, that the selected invers elements can be computed without inverting matrix K as a whole.

The volume-constrained surface-area-minimization model can be described in the following form:

$$sc(x)\rightarrow min \quad (4)$$

$$mv(x)=\varphi V_0 \quad (5)$$

$$KU=F \quad (6)$$

$$nc(x)=\tau C_0 \quad (7)$$

$$0\le x\le 1 \quad (8)$$

where x is the vector of design variables (the element densities), $sc(x)$ is the surface-are of the compliance function on the set of the feasible load directions, U and F are the displacement and load vectors, respectively, K is the global stiffness matrix, $mv(x)$ and V_0 are the material volume and design domain volume, respectively, φ is the prescribed volume fraction, $nc(x)$ is the nominal compliance function, C_0 is the compliance-minimal compliance, and $\tau > 1$ is the allowed maximum nominal compliance increase factor.

In the case of 2D topology optimization problems, the design domain is assumed to be rectangular and discretized with $n = e^x \times e^y$ square elements discretized with four nodes per element and two degrees of freedoms (DOFs) per node. Both nodes and elements are numbered column-wise from left to right.

The algorithm of the new robust approach has been developed in Matlab language under Windows 7 operation system as a variant of a very efficient 88 line Matlab code developed by Andreassen et al. [12] for the traditional deterministic SIMP-type volume-constrained compliance-minimization problem, starting from the famous 99 line code which was originally developed by Sigmund [13]. To solve the large and nonlinear optimization problem the **`fmincon`** solver from the Matlab environment was used with numerically generated objective and gradient function values. It is an open and very challenging question that what would be the most efficient numerical algorithm which could be solve the problem within a more reasonable time.

In the case of one directionally uncertain load there is special symmetric problem type where the objective and gradient functions can be generated symbolically using symbolic manipulation software. Namely, when the directionally uncertain loads is acting symmetrically about the symmetry axis of the problem, then the 2×2 symmetric Q matrix will be diagonal with $Q_{1,1}$, $Q_{2,2}$ diagonal entries, therefore, an appropriate symbolic manipulation software can be able to cope with the symbolic integration and differentiation problems. The result will be an exactly defined objective $sc(x)$ and gradient $\partial sc(x) / \partial x_i$, $i \in \{1,2 ..., n\}$ which decrease significantly the computation time of the optimization process.

In the totally symmetric case, assuming that the structure is rotated such a way that the nominal angle will be $\dot{\alpha}=0$, the objective function will be the following:

$$sc(x)=\int_{-\bar{\alpha}}^{+\bar{\alpha}}\sqrt{1+\left(\frac{\partial c(\alpha)}{\partial\alpha}\right)^2}\,d\alpha \quad (9)$$

where

$$f(\alpha)=[f\cos(\alpha), f\sin(\alpha)] \quad (10)$$

and

$$c(\alpha)=f(\alpha)Q\,f(\alpha)^t \quad (11)$$

After symbolic differentiation and integration the objective function $sc(x)$ will be the following:

$$sc(x)=EllipticE\left(2\bar{\alpha},-f^4\left(Q_{1,1}-Q_{2,2}\right)^2\right) \quad (12)$$

where is the elliptic integral of the second kind, which is a callable function in each "state-of-the-art" scientific program developing environment.

The symbolically generated gradient function entries are the following:

$$\partial sc(x)/\partial x_i=(E-F)/(Q_{1,1}-Q_{2,2})(D_{1,1}-D_{2,2}) \quad (13)$$

where

$$E=EllipticE\left(2\bar{\alpha},-f^4\left(Q_{1,1}-Q_{2,2}\right)^2\right)$$

$$F=EllipticF\left(2\bar{\alpha},-f^4\left(Q_{1,1}-Q_{2,2}\right)^2\right)$$

and

$$(D_{1,1}-D_{2,2})=(\partial Q_{1,1}/\partial x_i-\partial Q_{2,2}/\partial x_i),$$
$$i\in\{1,2,...,n\}.$$

EllipticF(*,*)is the elliptic integral of the first kind which is also a callable function in each standard scientific program developing environment. It is worth to note that $\partial Q_{1,1}/\partial x_i$ and $\partial Q_{2,2}/\partial x_i$,$i\in\{1,2,...,n\}$ can be generated by simple matrix manipulations (see Csébfalvi [14]).

3 Examples

In this section two design problems will be presented to demonstrate the efficiency and viability of the new robust solution searching approach. Using the terminology of the classical variational problems, the first example is curve-length-minimization problem with an inner-value-condition whereas the second one is surface-area-minimization problem with the same inner-value-condition. The presented examples with reproducible numerical results as a benchmark problems may be used for testing the quality of exact and heuristic solution procedures to be developed in the future for robust topology optimization.

3.1 Example 1

The first design example is a simple symmetric "academic" problem with only one directionally uncertain point load for which, in the case of trusses as a function of side loads, the analytical solutions are known.

The example, shown in Figure 2, is beam with a ground structure of 30*mm* × 60*mm* × 1*mm*. An external point load f acts in the end-middle position of the beam and its value is $f = -1$ and its nominal direction is $\dot{\alpha} = 0$. The directional uncertainty of the point load is described by a symmetric angle set about the nominal direction: $\dot{\alpha} - \bar{\alpha} \le \alpha \le \dot{\alpha} + \bar{\alpha}$, where $\bar{\alpha} = \pi / 6$. According to the symmetricity of the angle set, it cannot destroy the structural symmetry therefore in the numerical solution of the curve-length-minimization process symbolically generated objective and gradient functions can be used.

The symbolically generated objective function can be given by straightforward simplifications from the general case (10):

$$\boldsymbol{sc}(\boldsymbol{x}) = \boldsymbol{EllipticE}\left(\pi / 3, -\left(Q_{1,1} - Q_{2,2}\right)^2\right) \tag{14}$$

The $\partial \boldsymbol{sc}(\boldsymbol{x}) / \partial x_i,\ i \in \{1,2,...,n\}$ gradient can be given with similar simplification from the general case (11).

The Young's modulus is $E_0 = 1$, the Poisson's ratio is $v = 0.3$ and the fixed volume fraction is $\varphi = 3$. The penalization power is $p = 3$ and we applied density filtering with filter radius $r_{min} = 3$.

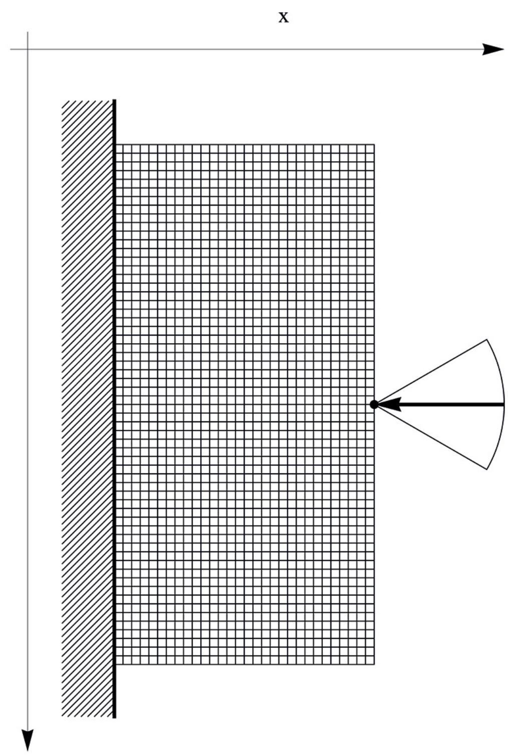

Fig. 2 The design domain, boundary conditions, and the applied external point load with ±π/6 directional uncertainty of a cantilever beam

The nominal-compliance-minimal design with $\dot{\alpha} = 0$ is shown in Figure 3.

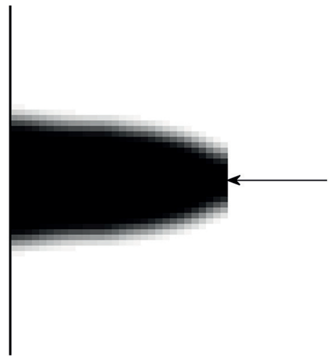

Fig. 3 The nominal-compliance-minimal design with $\dot{\alpha} = 0$

The optimal shapes for $\tau \in \{1.25, 1.50, 2.00\}$ are presented in Figure 4–6.

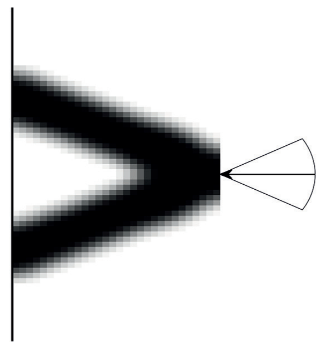

Fig. 4 Compliance-function-shape-optimized design with $\tau = 1.25$

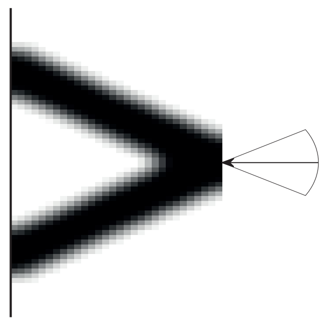

Fig. 5 Compliance-function-shape-optimized design with $\tau = 1.50$

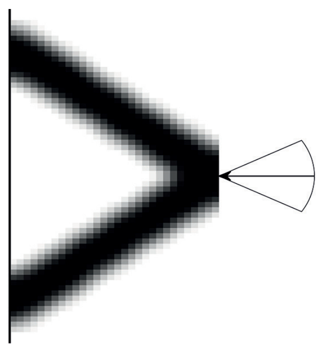

Fig. 6 Compliance-function-shape-optimized design with $\tau = 2.00$

The nominal compliance function shape and the curve-length-minimized compliance function shapes for different inner-value-conditions with $\tau \in \{1.25, 1.51, 2.00\}$ on the set of feasible loading directions are presented in Figure 7.

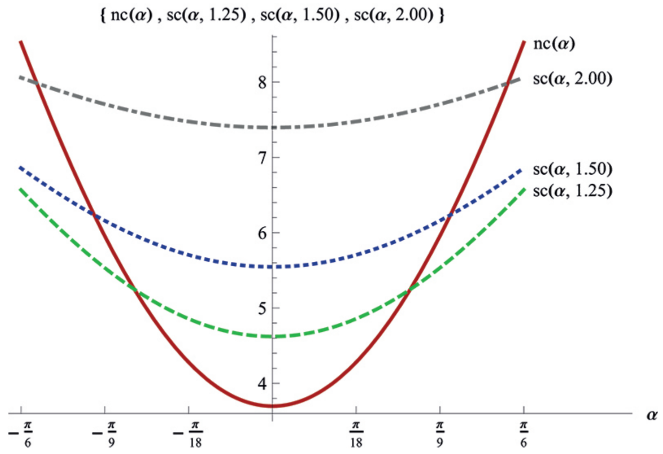

Fig. 7 The nominal compliance function shape and the curve-length-minimized compliance function shapes for different inner-value-conditions with $\tau \in \{1.25, 1.51, 2.00\}$ on the set of feasible loading directions

The performance measures of the nominal and the curve-length-minimized compliance functions are presented in Table 1, in which each row describes an optimization process where the optimal objective function value is presented as a bold number in a light grey cell and the corresponding column label defines the currently used objective function. Table 1 well illustrates the fact that higher the allowed nominal compliance increase the smaller the compliance fluctuation about the current nominal compliance value using curve-length-minimization model.

Table 1 Performance measures of the nominal and the curve-length-minimized compliance functions

τ	$nc(x)$	$sc(x)$	$\vec{c}(x)$	$\overleftarrow{c}(x)$	$\overleftrightarrow{c}(x)$
	3.70	9.80	8.50	3.70	4.80
1.25	4.60	4.10	6.60	4.60	1.90
1.50	5.50	2.90	6.90	5.50	1.30
2.00	7.40	1.80	8.10	7.40	0.66

3.2 Example 2

In this section, we present a little bit larger example with two directionally uncertain point loads, to show the essence of the presented robust surface-area-minimal solution searching process in a non-symmetric case. The example, presented in Figure 8, is a cantilever beam, with a ground structure of $30mm \times 60mm \times 1mm$ and two unit loads denoted by $f_i = -1$, $i \in \{1,2\}$ acting in the bottom-middle and bottom-end positions, from left to right in the given order. For each point load the nominal load direction is $\dot{\alpha} = 3\pi/2$ which can be perturbed by maximum $\bar{\alpha} = \pi/6$ in any directions: $4\pi/3 \le \alpha_i \le 5\pi/3, i \in \{1,2\}$. The Young's modulus is $E_0 = 1$, the Poisson's ratio is $v = 3$ and the fixed volume fraction is $\varphi = 0.3$. The penalization power is $p = 3$ and we applied density filtering with filter radius $r_{min} = 3$.

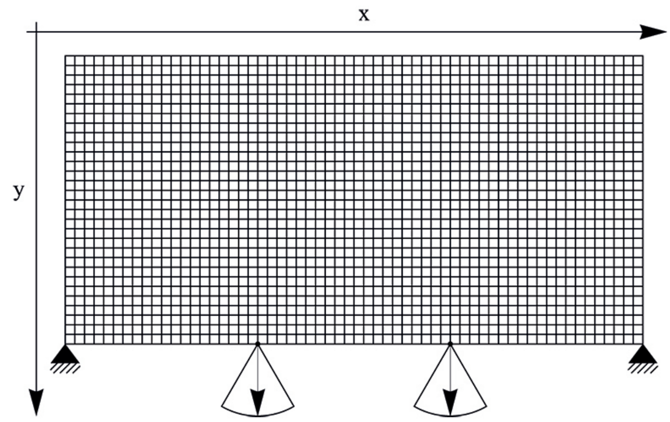

Fig. 8 The design domain, boundary conditions, and the applied external point loads with $\pm\pi/6$ directional uncertainty of a cantilever beam

In this example, we assume that in the surface-area-minimization the coefficient of the inner-value-condition is $\tau = 1.2$, to demonstrate that this relatively small nominal-compliance increasing possibility (20%) is able to change drastically the compliance-function-shape on the set of feasible loading directions. The nominal-compliance-minimal design and the corresponding directional-compliance-function-shape are presented in Figure 9–10, respectively. The surface-area-minimal design and the corresponding directional-compliance-function-shape are presented in Figure 11–12. The common plot of the nominal-compliance-minimal and surface-area-minimal directional-compliance-function-shapes, shown in Figure 13, is a good indication of the strength of the proposed "easy-to-understand" robust approach. It is easy to see that the surface-area-minimal compliance model drastically outperform the nominal-compliance model because its shape is nearly parallel to the $\propto_1 - \propto_2$ axes with a very small curvature over the set of feasible loading directions.

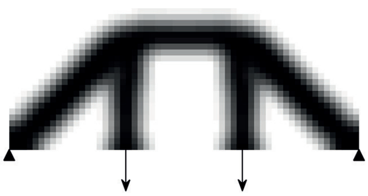

Fig. 9 The nominal-compliance-minimal design

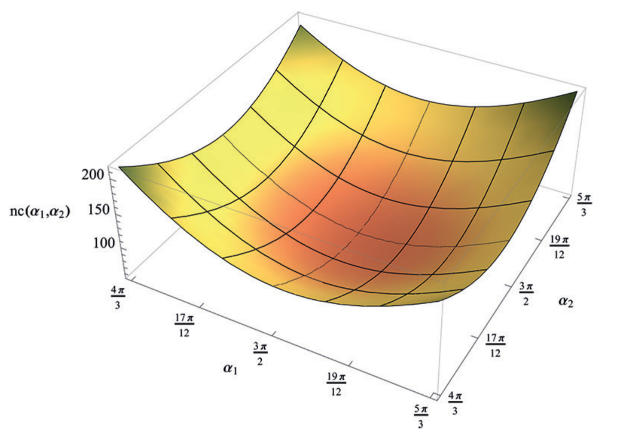

Fig. 10 directional-compliance-function-shape in the nominal case

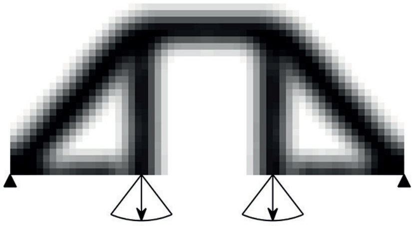

Fig. 11 The surface-area-minimal design

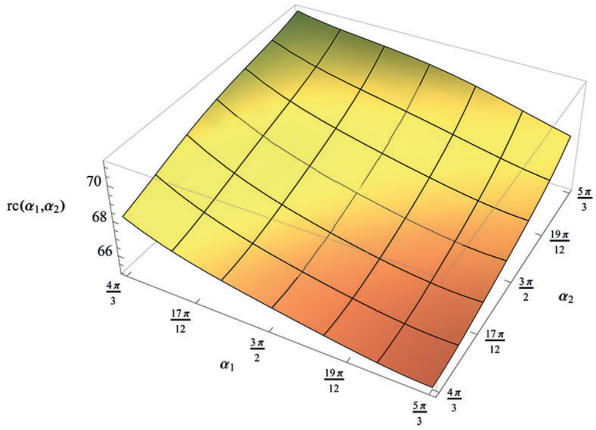

Fig. 12 directional-compliance-function-shape in the surface-area-minimal case

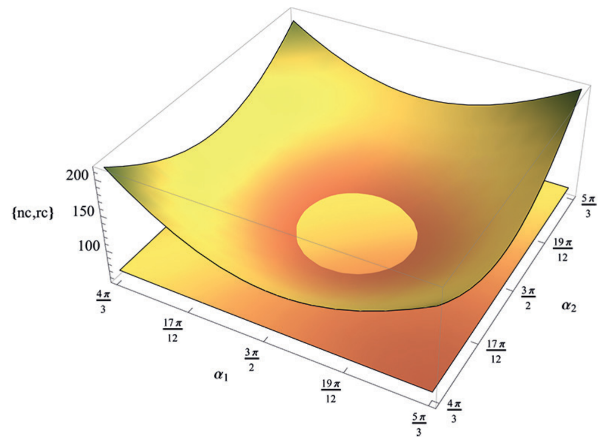

Fig. 13 Common plot of the nominal-compliance-minimal and surface-area-minimal directional-compliance-function-shapes

The performance measures of the nominal and the surface-area-minimized compliance functions are presented in Table 2, in which each row describes an optimization process where the optimal objective function value is presented as a bold number in a light grey cell and the corresponding column label defines the currently used objective function. Table 2 is an extremely good demonstration of the efficiency of the shape-oriented approach because, according to its very small range value, the directional compliance function is practically invariant to the directional uncertainty.

Table 2 Performance measures of the nominal and the surface-area-minimized compliance functions

τ	$nc(x)$	$sc(x)$	$\vec{c}(x)$	$\overleftarrow{c}(x)$	$\overleftrightarrow{c}(x)$
	55.93	155.70	214.30	55.68	158.60
1.2	67.14	4.07	71.23	65.04	6.19

4 Conclusions

In this paper, a novel compliance-function-shape-oriented robustness measure and a robust algorithm for the volume-constrained continuous topology optimization with uncertain loading directions based on this measure have been presented. The nonparametric pure set-based algorithm try to rearrange (take away) some amount of the material volume, originally used to minimize the nominal-compliance, to make a more balanced compliance-function-shape on the set of uncertain loading directions which is less sensitive to the directional fluctuation. The objective is the area of the compliance function shape defined on the set of feasible directions. The area-minimal shape searching process is controlled by the maximum allowable increase of the nominal-compliance. The result will be a more robust compliance function shape which can be characterized by a higher nominal-compliance but a smaller curvature about it in any direction. Using the terminology of the classical variational problems, the proposed approach can be classified as a curve-length or surface-area minimizing inner-value problem where the inner condition, namely the maximum allowable increase of the nominal-compliance is expressed as a percentage of the original nominal compliance and described by an inequality relation, the searching domain is defined implicitly as integration limits in the objective function formulation and the usual linear equality relation is used to prescribe the allowable material volume expressed as a percentage of the total material volume. The author hopes that the presented "easy to define but hard to solve" approach, which means a new direction in the robust topology optimization, will encourage further efforts to improve the numerical treatment of the problem and to extend it to other uncertainty cases.

Acknowledgement

The present study was supported by the Hungarian National Scientific and Research Foundation (OTKA) (grant K 119440)

References

[1] Bendsøe, M. P., Sigmund, O. "*Topology Optimization: Theory, Methods and Applications*". 370 p. Springer-Verlag, Berlin, Heidelberg. 2004. https://doi.org/10.1007/978-3-662-05086-6

[2] Deaton, J. D., Grandhi, R. V. "A survey of structural and multidisciplinary continuum topology optimization: post 2000". *Structural and Multidisciplinary Optimization*, 49(1), pp. 1–38. 2014. https://doi.org/10.1007/s00158-013-0956-z

[3] Ben-Tal, A., El Ghaoui, L., Nemirovski, A. S. "*Robust Optimization*". 576 p. Princeton Series in Applied Mathematics. Princeton University Press. 2009.

[4] Csébfalvi, A., Lógó, J. "Critical Examination of Volume-Constrained Topology Optimization for Uncertain Load Magnitude and Direction". In: Proceedings of ICCSEEC 15. Proceedings of The Fifteenth International Conference on Civil, Structural and Environmental Engineering Computing. Civil-Comp Press, Stirlingshire, UK. 2015. https://doi.org/10.4203/ccp.108.189

[5] De Gournay, F., Allaire, G., Jouve, F. "Shape and topology optimization of the robust compliance via the level set method". *ESAIM: Control, Optimization and Calculus of Variations*, 14(1), pp. 43–70. 2008. https://doi.org/10.1051/cocv:2007048

[6] Thore, C. J., Holmberg, E., Klarbring, A. "Large-scale robust topology optimization under load-uncertainty". In: *Proceedings of 11th World Congress on Structural and Multidisciplinary Optimization*. Sydney, Australia, June 7–12, 2015. http://web.aeromech.usyd.edu.au/WCSMO2015/papers/1096_paper.pdf

[7] Csébfalvi, A. "Structural optimization under uncertainty in loading directions: benchmark results". *Advances in Engineering Software*, In press, 2016. https://doi.org/10.1016/j.advengsoft.2016.02.006

[8] Dunning, P. D., Kim, H. A., Mullineux, G. "Introducing Loading Uncertainty in Topology Optimization". *AIAA Journal*, 49(4), pp. 760–768. 2011. https://doi.org/10.2514/1.J050670

[9] Carrasco, M., Ivorra, B., Lecaros, R., Ramos, A. M. "An expected compliance model based on topology optimization for designing structures submitted to random loads". *Differential Equations & Applications*, 4(1), pp. 111–120. 2012. http://files.ele-math.com/articles/dea-04-07.pdf

[10] Liu, J., Wen, G. "Topology optimization of continuum structures with uncertainty in loading direction". In: *Proc of M2D*, Ponta Delgada, Portugal, 2015.

[11] Olver, P. J. "Chapter 21: The Calculus of Variations". *Applied Mathematics Lecture Notes*, pp. 1148, 2012.

[12] Andreassen, E., Clausen, A., Schevenels, M., Lazarov, B.S., Sigmund, O. „Efficient topology optimization in MATLAB using 88 lines of code". *Structural and Multidisciplinary Optimization*, 43(1), pp. 1–16. 2011. https://doi.org/10.1007/s00158-010-0594-7

[13] Sigmund, O. "A 99 line topology optimization code written in Matlab". *Structural and Multidisciplinary Optimization,* 21(2), pp. 120–127. 2001. https://doi.org/10.1007/s001580050176

[14] Csébfalvi, A. "Robust Topology Optimization: A NewAlgorithm for Volume-constrained Expected Compliance Minimization with Probabilistic Loading Directions using Exact Analytical Objective and Gradient". *Periodica Polytechnica Civil Engineering*, 61(1), pp. 154–163, 2017. https://doi.org/10.3311/PPci.10214