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RESEARCH ARTICLE

A Review of Elasto-Plastic Shakedown Analysis with Limited Plastic Deformations and Displacements

Majid Movahedi Rad[1*]

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Abstract
In classical plasticity the shakedown analysis is among the most important basic problems. The principles of shakedown analysis are counterparts to those of limit analysis in the sense that they provide static and kinematic approaches to the question of whether or not shakedown will occur for a body under multiple variable loading conditions. The principles of limit analysis provide static and kinematic approaches to the question of whether or not the plastic limit state will be reached by a body under proportional loading. The principles of shakedown analysis are, however, considerably more difficult to apply than those of limit analysis. In spite of these difficulties, shakedown analysis is a vital and developing topic in plasticity and a great number of applications have been made. At the application of the plastic analysis and design methods the control of the plastic behaviour of the structures is an important requirement. Since the shakedown analysis provide no information about the magnitude of the plastic deformations and residual displacements accumulated before the adaptation of the structure, therefore for their determination bounding theorems and approximate methods have been proposed.

Keywords
shakedown analysis, complementary strain energy, plastic deformation, residual displacement, probability of failure

1 Department of Structural and Geotechnical Engineering,
Faculty of Architecture, Civil Engineering and Transport Sciences,
Széchenyi István University,
H-9026 Győr, Egyetem tér 1.
* Corresponding author, email: majidmr@sze.hu

1 Introduction

Shakedown for perfectly plastic material in 1932 was first discussed by Bleich [1] and in 1936, Melan [2] extended Bleich's discussion on indeterminate structures and later in 1938 he gave the generalization of the static shakedown theorem to the elastic, plastic continuum [3]. Independent proofs for trusses and frames were presented by Prager [4] and Neal [5]. Until 1956 the essential feature of the basic theorems of shakedown was that no restriction had been imposed on either plastic strains or on the plastic work before the structure reaches the shakedown state. However, shakedown should be related to the amount of work dissipated on plastic deformation up to the shakedown state. In this context, in 1956, Koiter [6] formulated a kinematical shakedown theorem for an elastic-plastic body, to fill what appeared to be a gap in the shakedown theory at that time which is now known as Koiter's shakedown theorem. In 1960, Koiter [7] reviewed the theory of shakedown for quasi-static loading considering an elastic-perfectly plastic material behaviour.

1.1 Melan's statical shakedown theorem

Consider a linearly elastic-perfectly plastic body, with volume V and surface area S, subjected to quasi-static variable multiple loads on the surface S_q. Assume that on the surface $S_u = S - S_q$ the surface displacements are specified to be zero and denote the actual path-dependent stresses and strains caused by the variable loads during the period $0 \le t \le T$ of the loading program by stress $\sigma_{ij}(x_i, t)$ and strain $\varepsilon_{ij}(x_i, t)$, respectively. In addition, to construct the static principal we have to define those path-dependent elastic stresses and strains which would occur during the loading program if the body were perfectly elastic. We will denote them by elastic stress $\sigma_{ij}^e(x_i, t)$ and elastic strain $\varepsilon_{ij}^e(x_i, t) = H_{ijkl}\, \sigma_{kl}^e(x_i, t)$, here H_{ijkl} is fourth-order symmetric tensor .

So long as the yield condition $f(\sigma_{ij}^e, k) \le 0$ is satisfied at every point in the body, the actual stresses are identical with the elastic stresses, i.e. $\sigma_{ij} = \sigma_{ij}^e$. Here k representing the plastic properties of the materials (e.g. the yield stress). When, however, the elastic stresses would violate the yield condition they could not increase any further. Then, plastic deformations will occur which lead to the redistribution of the stresses. In this state, the actual stress field can be expressed as the sum of the elastic stress field and another stress field, called the residual stress field (ρ_{ij})

$$\sigma_{ij}\left(x_i,t\right)=\sigma_{ij}^e\left(x_i,t\right)+\rho_{ij}\left(x_i,t\right). \tag{1}$$

Hence, the residual stresses

$$\rho_{ij}\left(x_i,t\right)=\sigma_{ij}\left(x_i,t\right)-\sigma_{ij}^e\left(x_i,t\right) \tag{2}$$

denote permanent stresses which would remain in the body after unloading under elastic conditions. The residual stresses are generally also path-dependent, but after shakedown has occurred they will not change any more because no more plastic deformations will arise.

After this introduction, the static principle of shakedown analysis can be stated as follows:
if any path-independent residual stress field $\rho_{ij}(x_i)$ can be found such that the yield condition
is satisfied at every point in the body for any possible loading

$$f\left[\sigma_{ij}\left(x_i,t\right),k\right]=f\left\{\left[\sigma_{ij}^e\left(x_i,t\right)+\rho_{ij}\left(x_i\right)\right],k\right\}\le 0 \tag{3}$$

case, then during the loading process this residual stress field or another one will develop and the body will shake down. Alternatively, if any path-independent residual stress field can be found such that the corresponding domain of elastic limit loads includes the domain of possible loads then shakedown will occur during the loading process.

1.2 Koiter's kinematical shakedown theorem

Consider a linearly elastic-perfectly plastic body, with volume V and surface area S, subjected to quasi-static multiple loading on the surface S_q and assume that on the surface $S_u = S - S_q$ the displacements are zero. In addition, introduce the kinematically admissible plastic strain rate $\dot{\varepsilon}_{ij}^k\left(x_i,t\right)$ for $0\le t\le T$ which are characterized by the property that for the time interval T the plastic strains $\Delta\dot{\varepsilon}_{ij}^k=\int_0^T\dot{\varepsilon}_{ij}^k\left(x_i,t\right)dt$ constitute a kinematically admissible strain field together with displacement field $\Delta u_i^k\int_0^T\dot{u}_i^k dt,$ which at the same time, satisfies the boundary condition $\Delta\dot{u}_i^k = 0$ on S_u. Then the kinematic principal of shakedown analysis states that if any kinematically admissible plastic strain rate and velocity field can be found such that

$$\int_0^T dt\int_{S_q} q_i\dot{u}_i^k dS>\int_0^T dt\int_V\sigma_{ij}^k\dot{\varepsilon}_{ij}^k dV=\int_0^T dt\int_V d(\dot{\varepsilon}_{ij}^k)dV \tag{4}$$

then the body will not shake down during the loading procedure in the time interval $0\le t\le T$. Alternatively, shakedown will occur during the loading procedure in the time interval $0\le t\le T$ if for all possible kinematically admissible strain rate and velocity fields

$$\int_0^T dt\int_{S_q} q_i\dot{u}_i^k dS\le\int_0^T dt\int_V\sigma_{ij}^k\dot{\varepsilon}_{ij}^k dV=\int_0^T dt\int_V d(\dot{\varepsilon}_{ij}^k)dV. \tag{5}$$

In expressions (4) and (5) the stress σ_{ij}^k is associated with the plastic strain rate $\dot{\varepsilon}_{ij}^k$ through the constitutive equations. The proof of the kinematic principal can be found in the literature (see e.g. Koiter [7], Martin [8]).

1.3 Plastic hinge application in shakedown analysis

One of the most successful applications of shakedown analysis is proposed in plastic hinge theory. The basic ideas were first recognized and applied to the steel beams by Kazinczy [9] in 1914. He proved that at a point of a beam plastic hinge develops when the moment reaches the plastic moment. In a plastic hinge unrestricted plastic rotations can develop only in the sense of the moment, while after unloading residual deformations remain. Hence the plastic hinges naturally cannot be considered as actual hinges. For a more concise historic development of plastic hinge theory the reader is referred to Kaliszky et al. [10] and Kaliszky [11].

2 Control of plastic deformations

By the use of plastic analysis and design methods, significant saving in material can be obtained. However, as a result of this benefit excessive plastic deformations and large residual displacements might develop, which in turn might lead to unserviceability and collapse of the structure. The most important tool for controlling the plastic behaviour of structures is the application of the static and kinematic theorems of shakedown proposed by Melan [3] and Koiter [7], respectively. These two theorems have been successfully applied to the solution of a large number of problems (see e.g. Maier [12]; Polizzotto [13]; König [14]; Kaliszky [15] Kaliszky and Lógó [16-19]; Weichert and Maier [20]; Levy et al. [21] and Simon and Weichert [22]). The applications of shakedown theorems gives no information on the magnitude of plastic deformations and residual displacements accumulated before the adaptation of the structure. Therefore during the past decades several bounding theorems have been proposed for the approximate determination of the plastic deformations and residual displacements developing during the loading history (see e.g. Ponter [23]; Corradi [24]; Capurso et al. [25]; Kaneko and Maier [26]; Polizzotto [13]; Tin-Loi [27]; Weichert and Maier [20] and Liepa et al. [28]).

Kaliszky and Lógó [29-30]; Movahedi and Lógó [31]; Lógó et al. [32] and Movahedi [33-34] proposed that the complementary strain energy of the residual forces could be considered an overall measure of the plastic performance of structures and the plastic deformations should be controlled by introducing a limit for the magnitude of this energy:

$$\frac{1}{2}\sum_{i=1}^{n}\boldsymbol{Q}_i^r F_i\boldsymbol{Q}_i^r\le W_{p0} \tag{6}$$

Here W_{p0} is an assumed bound for the complementary strain energy of the residual forces and $\boldsymbol{Q}^r$ residual internal forces.

This constraint can be expressed in terms of the residual moments $M^r_{i,a}$ and $M^r_{i,b}$ acting at the ends (a and b) of the finite elements as:

$$\frac{1}{6E}\sum_{i=1}^{n}\frac{l_i}{I_i}\left[\left(M^r_{i,a}\right)^2+\left(M^r_{i,a}\right)\left(M^r_{i,b}\right)+\left(M^r_{i,b}\right)^2\right]\le W_{p0} \qquad (7)$$

By the use of Eq. (7) a limit state function can be constructed:

$$g\left(W_{p0},M^r\right)=W_{p0}-\frac{1}{6E}\sum_{i=1}^{n}\frac{l_i}{I_i}\left[\left(M^r_{i,a}\right)^2+\left(M^r_{i,a}\right)\left(M^r_{i,b}\right)+\left(M^r_{i,b}\right)^2\right] \qquad (8)$$

The plastic deformations are controlled while the bound for the magnitude of the complementary strain energy of the residual forces exceeds the calculated value of the complementary strain energy of the residual forces. On similar way a limit state function can be determined in case of axially loaded structures.

2.1 Reliability based control of the plastic deformations

In engineering the problem parameters (geometrical, material, strength, and manufacturing) are given or considered with uncertainties. The obtained analysis and/or design task is more complex and can lead to reliability analysis and design. Instead of variables influencing performance of the structure (manufacturing, strength, geometrical) only one bound modelling resistance scatter can be applied.

Reliability methods aim at evaluating the probability of failure of a system whose modelling takes into account randomness. Classically, the system is decomposed into components and the system failure is defined by various scenarios about the joint failure of components. Thus the determination of the probability of failure of each component is paramount importance.

Here the bound on the complementary strain energy of the residual forces controlling the plastic behaviour of the structure. Introducing the basic concepts of the reliability analysis and using the force method the failure of the structure can be defined as follows:

$$g\left(X_R,X_S\right)=X_R-X_S\le 0; \qquad (9)$$

where X_R indicates the bound for the statically admissible forces X_S. The probability of failure is given by

$$P_f=F_g\left(0\right) \qquad (10.a)$$

and can be calculated as

$$P_f=\int f\left(X\right)dx. \qquad (10.b)$$

Let assumed that due to the uncertainties the bound for the magnitude of the complementary strain energy of the residual forces is given randomly and for sake of simplicity it follows the Gaussian distribution with given mean value $\bar{W}_{p0}$ and standard deviation σ_w. Due to the number of the probabilistic variables (here only single) the probability of the failure event can be expressed in a closed integral form:

$$P_{f,calc}=\int f(\bar{W}_{p0},\sigma_w)dx. \qquad (10.c)$$

By the use of the strict reliability index a reliability condition can be formed:

$$\beta_{target}-\beta_{calc}\le 0; \qquad (10.d)$$

where β_{target} and β_{calc} are calculated as follows:

$$\beta_{target}=-\Phi^{-1}\left(P_{f,target}\right); \qquad (10.e)$$

$$\beta_{calc}=-\Phi^{-1}\left(P_{f,calc}\right). \qquad (10.f)$$

Here Φ^{-1}: inverse cumulative distribution function (so called probit function) of the Gaussian distribution. (Due to the simplicity of the present case the integral formulation is not needed, since the probability of failure can be described easily with the distribution function of the normal distribution of the stochastic bound W_{po}).

The numerical analyses shows that the given mean values and different expected probability on the bound of the complementary strain energy of the residual forces can influence significantly the magnitude of the plastic limit load.

3 Shakedown analysis with limited residual strain energy capacity

The mechanical model can be given by the following conditions: determine the maximum load multiplier and cross-sectional dimensions under the conditions that

(i) the structure with given layout is strong enough to carry the (dead loads + live loads),
(ii) satisfies the constraints on the self-equilibrated residual forces and limited strength capacities,
(iii) satisfies the constraints on plastic deformations and residual displacements,
(iv) safe enough and the required amount of material does not exceed a given limit. The design solution

3.1 Deterministic problems

Applying the static principle, the statically admissible bending moment fields M_j should be considered. These can easily be obtained on the statically determinate released structure if we choose the magnitudes of the redundant forces arbitrarily and determine the bending moment distribution of the structure by the use of equilibrium equations. Then, a statically admissible stable shakedown load multiplier m_{sh} can be obtained from the condition that even the maximum bending moment does not exceed the fully plastic moment, i.e. $\max|M_j|\le M_p$.

The solution method based on the static theorem of shakedown analysis is formulated as follows:

$$\text{Maximize } m_{sh} \tag{11.a}$$

subject to

$$\boldsymbol{G}^* M^r = 0; \tag{11.b}$$

$$M^e = \boldsymbol{F}^{-1}\boldsymbol{G}\boldsymbol{K}^{-1} m_{sh}\boldsymbol{Q}_h; \tag{11.c}$$

$$-M^p \le M^r + maxM^e \le M^p; \tag{11.d}$$

$$\frac{1}{6E}\sum_{i=1}^{n}\frac{l_i}{I_i}\left[\left(M_{i,a}^r\right)^2 + \left(M_{i,a}^r\right)\left(M_{i,b}^r\right) + \left(M_{i,b}^r\right)^2\right] \le W_{p0}; \tag{11.e}$$

$$\sum A_i l_i - V_0 \le 0. \tag{11.f}$$

Here $\boldsymbol{F}$, $\boldsymbol{K}$, $\boldsymbol{G}$, $\boldsymbol{G}^*$: flexibility, stiffness, geometrical and equilibrium matrices, respectively. Eq. (11.b) is an equilibrium equation for the residual moment, M^r. Eq. (11.c) express the calculations of the elastic fictitious moments, M^e. Eq. (11.d) is used as yield conditions. Eq. (11.e) is used to control the plastic deformations. The material redistribution is controlled by Eq. (11.f). This is a mathematical programming problem which can be solved by the use of nonlinear algorithm. Selecting one of the loading combination $\boldsymbol{Q}_h$; (h = 1, 2, ..., 5); a shakedown load multiplier m_{sh} can be determined, then the limit curve can be constructed (Fig. 1).

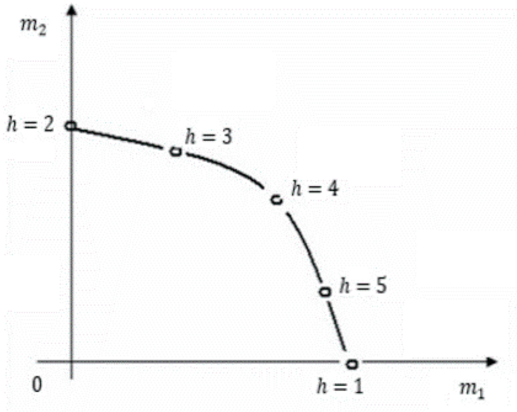

Fig. 1 Limit curve and safe domain

3.1.1 Formulations of axially loaded structures

The design solution method based on the static theorem of shakedown analysis is formulated as below:

$$\text{Maximize } m_{sh} \tag{12.a}$$

subject to

$$\boldsymbol{G}^* N^r = 0; \tag{12.b}$$

$$N^e = \boldsymbol{F}^{-1}\boldsymbol{G}\boldsymbol{K}^{-1} m_{sh}\boldsymbol{Q}_h; \tag{12.c}$$

$$-N^p \le N^r + maxN^e \le N^p; \tag{12.d}$$

$$\frac{1}{2}\sum_{i=1}^{n}\boldsymbol{Q}_i^r F_i \boldsymbol{Q}_i^r \le W_{p0}. \tag{12.e}$$

$$\sum A_i l_i - V_0 \le 0. \tag{12.f}$$

3.2 Probabilistic problems

Here all the equations have the same meanings as in Eqs. (11.a–d) and (11.f) while Eq.(13.e) is the reliability condition which controls the plastic behaviour of the structure by use of the residual strain energy.

Basic reliability problems are formulated as below:

$$\text{Maximize } m_{sh} \tag{13.a}$$

subject to

$$\boldsymbol{G}^* M^r = 0; \tag{13.b}$$

$$M^e = \boldsymbol{F}^{-1}\boldsymbol{G}\boldsymbol{K}^{-1} m_{sh}\boldsymbol{Q}_h; \tag{13.c}$$

$$-M^p \le M^r + maxM^e \le M^p; \tag{13.d}$$

$$\beta_{target} - \beta_{calc} \le 0; \tag{13.e}$$

$$\sum A_i l_i - V_0 \le 0. \tag{13.f}$$

3.2.1 Alternative design formulations

The classical minimum volume design model can be created by interchanging the objective function Eq.(13.a) and the last constraint Eq.(13.f) an alternative design formulation can be formulated as follows:

$$\text{Minimize } V = \sum A_i l_i \tag{14.a}$$

$$\boldsymbol{G}^* M^r = 0; \tag{14.b}$$

$$M^e = \boldsymbol{F}^{-1}\boldsymbol{G}\boldsymbol{K}^{-1} m_{sh}\boldsymbol{Q}_h; \tag{14.c}$$

$$-M^p \le M^r + maxM^e \le M^p; \tag{14.d}$$

$$\beta_{target} - \beta_{calc} \le 0; \tag{14.e}$$

$$m_{sh} - m_0 \le 0. \tag{14.f}$$

Here all the equations have the same meanings as they had before in Eqs.(13.b-e) while Eq.(14.f) gives an upper bound for the external loads. The application of shakedown analysis and design methods with limited residual strain energy capacity for deterministic and probabilistic problems (pile foundations and skeletal frame structures) are proposed by Movahedi and lógó [31]; Lógó et al. [32] and Movahedi [35].

4 Conclusions

In this paper a review about the application of shakedown analysis with limited plastic deformations and displacements of elasto-plastic were presented. Shakedown analysis is a vital and developing topic in plasticity and a great number of applications have been made. At the application of the plastic analysis and design methods the control of the plastic behaviour of the structures is an important requirement. Since the shakedown analysis provide no information about the magnitude of the plastic deformations and residual displacements accumulated before the adaptation of the structure, therefore for their determination bounding theorems and approximate methods have been proposed. The numerical analysis shows that the bounds can influence significantly the results of shakedown analysis.

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