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Vibration Analysis of Viscoelastic FGM Nanoscale Plate Resting on Viscoelastic Medium Using Higher-order Theory

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Abstract

The present article aims essentially to present an analytical and numerical method which makes it possible to study the damped vibrations of viscoelastic FGM nanoplates resting on viscoelastic foundations. A new model for the higher-order shear deformation plate theory is coupled with the internal Kelvin - Voigt viscoelastic model and the three-parameter viscoelastic foundation model for the purpose of reducing and minimizing the vibration response of the system. It is widely admitted that the mechanical properties of these new functionally gradient materials (FGMs) vary according to the thickness of the plate and depend on its volume fraction. The use of FGM plates seems to be an ideal solution for the study of free vibrations because of their multifunctionality that is fully integrated with the nonlocal Eringen effect. The dynamic response of such a complex system has been investigated by varying the aspect ratio of the plate, the mechanical characteristics of the material used, the internal and external damping and the foundation rigidity. The results obtained, with and without the nonlocal effect, were compared with those of different models of higher-order theories and under various boundary conditions; they were found to be in good agreement with those reported in the literature.

Keywords

higher-order plate theory, FGM materials, viscoelasticity, nonlocal theory, Winkler-Pasternak viscoelastic foundation

1 Introduction

Functionally graded materials (FGMs) were used for the first time by Japanese scientists in the 1980s as high temperature-resistant materials in the area of aerospace construction. Recently, these new materials have been employed in various electrical devices, energy transformation, biomedical technology, and optical systems [1–6]. It is worth indicating that different plate configurations exist today; they are often classified according to their geometry, type of stress experienced, and type of behavior (membrane-flexion), with or without transverse shearing. The plates whose transverse shearing is neglected are called Love-Kirchhoff plates [7]. Love-Kirchhoff's theory applies to thin plates. On the other hand, thick homogeneous plates, for which shear is taken into account, are called Reissner-Mindlin plates [8–9]. The Reissner-Mindlin theory, also referred to as the first-order shear deformation theory (FSDT), is well suited for the analysis of problems linked to bending and vibration of structures. The first-order shear deformation theory (FSDT) of Reissner - Mindlin is more precise than the classical plate theory (CPT) of Love-Kirchhoff. However, Reissner Mindlin's theory requires a shear correction factor and gives a constant distribution of shear stresses across the thickness of the plate, which is not the case here. In order to represent the kinematics of a point of a beam or plate, without the shear correction factor, some higher order shear deformation theories (HSDTs) have been presented in order to describe the behavior of beams and plates under various mechanical loadings. Thus, Levinson [10] and Reddy [11] developed higher-order functions, like the higher-order shear deformation plate theory (HSDT), in terms of thickness in the form of a third-degree polynomial. In ddition, it should be noted that the variation of shear

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stress can be represented using a second degree polynomial as a function of the thickness. Several researchers, such as Touratier [12], Karama et al. [13], Aydogdu [14], Soldatos and Timarci [15], Mechab et al. [16] and Benyamina et al. [17] have developed other types of sinusoidal, hyperbolic or exponential form functions for the mechanical analysis of structures. In order to deal with the problems of static and dynamic analysis of orthotropic plates, Shimpi and Patel [18–19] developed a refined plate theory (RPT) that is classified among higher-order theories. Unlike the first-order shear deformation theory (FSDT) and higher-order shear deformation theory (HSDT), the refined plate theory reduces the calculation time and gives four equilibrium equations instead of five, without correction factor, and with a parabolic variation of shear stress through thickness of the plate. The RPT theory has recently attracted a lot of interest among researchers to solve the problem of vibrations, buckling of isotropic, orthotropic and FGM structures under various loadings [20–22].

Over the last few years, due to the rapid development of technology, particularly in the field of nanostructures that have superior mechanical properties and a large number of applications in technology, the researchers were urged to take into account the effects of scales and atomic forces in order to obtain solutions with acceptable accuracy. Eringen's nonlocal theory is based on this hypothesis which suggests that the stresses at a reference point in the body depend not only on the deformations at that point but also on the deformations at all other points of the body. Consequently, the analysis of nanostructural vibrations has become a subject of major interest for current and future research studies [23–27].

Recently, the nonlocal theory has been used in the analysis of nanobeams and nanoplates made of functionally graded materials (FGMs). For this, Farzad et al. [28] investigated the buckling of FGM nanoplates, subjected to variable thermal, linear and nonlinear loads, resting on a Pasternak-type foundation. They found out that the responses of buckling with the nonlocal effect are weaker than those with a local effect, under various loading conditions. As for Zenkour and Arefi [29], he conducted a static and dynamic analysis of an FGM nanoplate resting on a visco-Pasternak foundation, and subjected to thermo-electromechanical loading. Note that both Eringen's nonlocal elasticity theory and the classical plate theory are used for the determination of the equilibrium equations. The refined higher-order shear deformation theory was used by Żur et al. [30] for the analysis of the vibrations and buckling of FGM structures in terms of the nonlocal parameter, volume fraction index, power law index, mechanical, electrical and magnetic loads, mechanical, electrical, and magnetic loadings, as well as the geometric ratio of the section.

Viscoelastic materials are used in various fields of engineering such as the design of household appliances, automobile, aeronautics and even the vast area of civil engineering. Reducing mechanical vibrations and noise is one of the major concerns in the automotive, naval and aeronautical industries. To remedy this problem, there are anti-vibration sheets, called sandwich sheets, made of a thin layer of viscoelastic material interposed between two steel sheets. The damping capacity can therefore be improved by the viscoelastic material.

On the other hand, Wang and Tsai [31] used the finite element method (FEM) to analyze the quasi-static and dynamic response of the linear viscoelastic plate, where the temperature field is assumed to be constant and homogeneous; here, the relaxation modulus is supposed to be in the Prony series form. Kiasat et al. [32], Pouresmaeeli et al. [33], Liu et al. [34] and Hosseini et al. [35] studied the free vibrations of thin plates made of functionally graded materials and composite materials, using the Love-Kirchhoff theory, also known as the classical plate theory (CPT), resting on visco-Winkler and visco-Winkler-Pasternak foundations, using the Kelvin-Voigt viscoelastic model. As for Ebrahimi and Barati [36] and Arefi and Zenkour [37], they used a refined higher-order plate theory with a trigonometric shear stress function for the purpose of exploring the influence of viscoelastic parameters, due to hygrothermal and piezoelectric charges, on the vibration frequency of FGM nanoplates and viscoelastic sandwich nanoplates with nonlocal effect.

The present work focuses on the viscoelastic study of nanoplates and Winkler-Pasternak type foundations in order to analyze free plate vibrations using the higher-order plate theory with nonlocal effect, as well as a new shape function for the shear-stress distribution through the nanoplate thickness. It is useful to mention that the mechanical properties of the plate vary gradually through its thickness, in accordance with the distribution of the power-law FGM (P FGM). The scale effect, shear deformation, mechanical properties, damping and rigidities of the foundations are taken into account while studying the response of the structure. The results obtained for free vibrations were compared with those of different versions of higher order theories, and under various boundary conditions.

2 Mathematical development

Consider a nanoplate made of FG viscoelastic materials, with length a, width b and thickness h. The properties of the elastic materials of the FGM plate are the Young's modulus $E(z)$ and mass density $\rho(z)$. The plate rests on a viscoelastic foundation; its coordinates are illustrated in Fig. 1.

The Kelvin-Voigt model used consists of an infinite set of springs and dampers connected in parallel; the spring stiffness and damping coefficient are defined respectively by k_w, k_p and c_d. The displacements of any point on the nanoplate can be expressed in terms of average displacement components of the surface. The displacement field is given by: (1)

$$U(x,y,z,t)=u_0(x,y,t)-z\frac{\partial w_b(x,y,t)}{\partial x}+\left(f(z)-z\right)\frac{\partial w_s(x,y,t)}{\partial x},$$
$$V(x,y,z,t)=v_0(x,y,t)-z\frac{\partial w_b(x,y,t)}{\partial y}+\left(f(z)-z\right)\frac{\partial w_s(x,y,t)}{\partial y},$$
$$W(x,y,z,t)=w_b(x,y,t)+w_s(x,y,t).$$

Note that U, V and W are the displacement components along x, y and z, respectively. The fundamental unknowns consist of the four generalized displacements u_0, v_0, w_b, and w_s which are functions of the coordinates x and y. Note that u_0 and v_0 are the displacement components along x and y, and w_b and w_s are the displacement components along z. A new transverse shear deformation shape function is given by the following expression:

$$f(z)=\frac{1}{1-\pi\cdot e^{-\frac{1}{4}}}\left(z-h\cdot e^{\left(\frac{z}{h}\right)^2-\frac{1}{4}}\cdot\sin\left(\frac{z}{h}\pi\right)\right). \tag{2}$$

Different higher-order shear deformation plate theories are summarized in Table 1. According to Soldatos and Timarci [15], the shape function must meet the following three conditions:

Their derivatives should be equal to zero at the point (x, y, $\pm h/2$), on the top and bottom surfaces of the plate (Fig. 2(a)).

$$f'(z)\Big|_{z=\pm\frac{h}{2}}=0 \tag{3}$$

The function $f(z)$ must be an odd function (Fig. 2(b)).

$$\int_{z=-h/2}^{z=h/2} f(z)\,dz=0 \tag{4}$$

The deformation field is expressed in Cartesian coordinates. Taking into account the warping of the straight section of the plate, the refined theory of thick plates can be written in the following form:

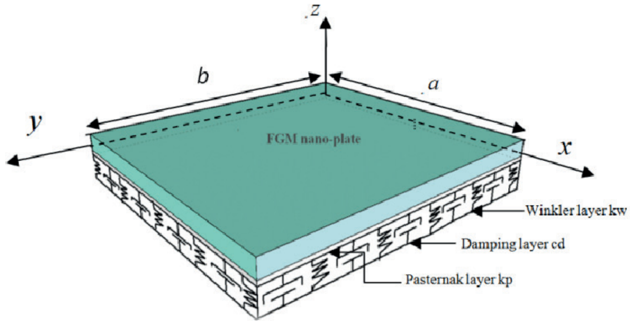

Fig. 1 Geometry of a viscoelastic FG nano-plate resting on a viscoelastic Pasternak foundation

Table 1 The various transverse shear functions used in different plate theories

Theory	Shape function $f(z)$	unknown generalized displacements
CPT(classical plate theory)	0	-
FSDT(first order theory)	z	-
TSDT Aghababaei and Reddy [25]	$z\left(1-\frac{4z^2}{3h^2}\right)$	u_0, v_0, w_0, θ_x, θ_y
SSDT Touratier [12]	$\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)$	u_0, v_0, w_0, θ_x, θ_y
Present model	$\frac{1}{1-\pi e^{-\frac{1}{4}}}\left(z-he^{\left(\frac{z}{h}\right)^2-\frac{1}{4}}\sin\left(\frac{z}{h}\pi\right)\right)-z$	u_0, v_0, w_b, w_s

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{Bmatrix}-z\begin{Bmatrix}\frac{\partial^2 w_b}{\partial x^2}\\ \frac{\partial^2 w_b}{\partial y^2}\\ 2\frac{\partial^2 w_b}{\partial x\partial y}\end{Bmatrix}+f(z)\begin{Bmatrix}\frac{\partial^2 w_s}{\partial x^2}\\ \frac{\partial^2 w_s}{\partial y^2}\\ 2\frac{\partial^2 w_s}{\partial x\partial y}\end{Bmatrix}, \tag{5}$$

$$\varepsilon_z=0,\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\frac{\partial f(z)}{\partial z}\begin{Bmatrix}\frac{\partial w_s}{\partial x}\\ \frac{\partial w_s}{\partial y}\end{Bmatrix}.$$

The strain and stress fields in a medium are linked by constitutive laws; these laws characterize the mechanical behavior of the medium. Consequently, the linear elastic relationship for an FGM plate can be written in the following matrix form:

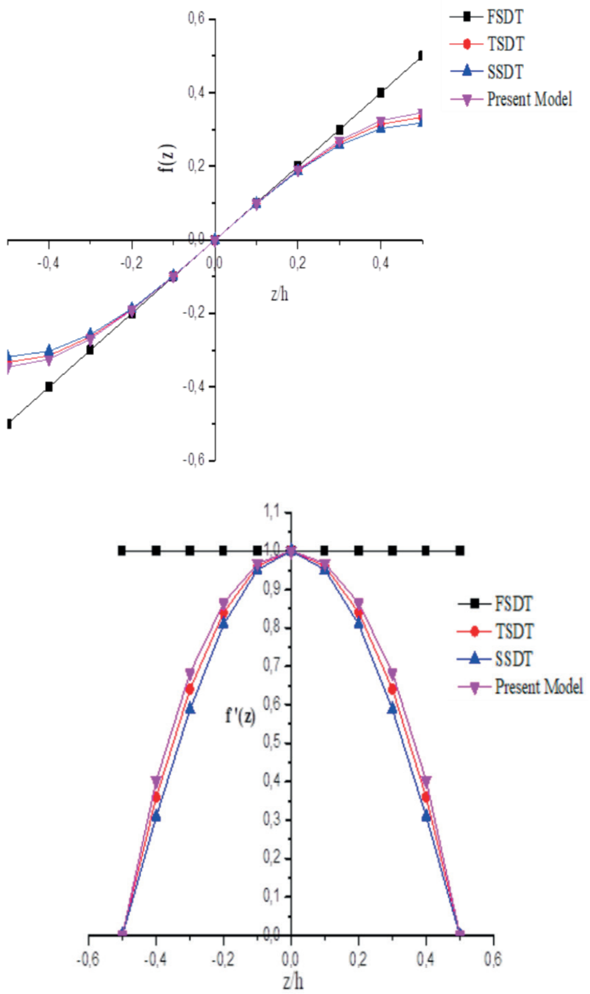

Fig. 2 Convergence of the proposed functionand its derivative with those of (FSDT), (TSDT), (SSDT) and present models

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix}=\begin{bmatrix}C_{11}(z) & C_{12}(z) & 0 & 0 & C_{16}(z)\\ C_{12}(z) & C_{22}(z) & 0 & 0 & C_{26}(z)\\ 0 & 0 & C_{44}(z) & C_{45}(z) & 0\\ 0 & 0 & C_{45}(z) & C_{55}(z) & 0\\ C_{16}(z) & C_{26}(z) & 0 & 0 & C_{66}(z)\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}. \quad (6)$$

The elasticity relationships are generally expressed as a function of the rigidity constants which themselves are expressed in terms of the elasticity modulus that are determined by mechanical tests in which the material used is subjected to a particular stress and deformation state. The terms $C_{ij}(z)$ represent the stiffness constants which depend on the constituents of the FGM material.

$$\begin{aligned}&C_{11}(z)=C_{22}(z)=\frac{E(z)}{1-\nu^2}, C_{12}(z)=\frac{\nu E(z)}{1-\nu^2}\\ &C_{44}(z)=C_{55}(z)=C_{66}(z)=\frac{E(z)}{2(1+\nu)}\\ &C_{16}(z)=C_{26}(z)=C_{45}(z)=0\end{aligned} \quad (7)$$

Furthermore, the parameters $E(z)$ and ν are the Young's modulus and Poisson's ratio of the material; they depend on the characteristics of the FGM plate. The mechanical properties of the FGM plate containing ceramic and metal, which are uniformly distributed, are given by the power law hypothesis which is written as the general mixing rule under the following form:

$$P(z)=(P_c-P_m)\left(\frac{z}{h}+\frac{1}{2}\right)^p+P_m. \quad (8)$$

where: P denotes the effective material property, and the subscripts c and m stand for ceramic and metal, respectively.

In addition, the equations expressing Young's modulus $E(z)$ and density $\rho(z)$ of the material of a functionally graded plate can be written as follows:

$$\begin{aligned}&E(z)=(E_c-E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^p+E_m,\\ &\rho(z)=(\rho_c-\rho_m)\left(\frac{z}{h}+\frac{1}{2}\right)^p+\rho_m,\end{aligned} \quad (9)$$

where: E_c, E_m, ρ_c and ρ_m are Young's modulus and volume densities corresponding to ceramic and metal, respectively; p represents the exponent of the volume fraction which takes only values greater or equal to zero ($0 \le p \le \infty$). The value zero corresponds to a ceramic plate.

In our study, the Poisson's ratio is assumed to be constant.

3 Non-local viscoelastic theory

3.1 Nonlocal elasticity theory

Given the importance of the intermolecular attractions of the material, the theory of nonlocal effect developed by Eringen suggests that the stresses at a reference point x in the body depend not only on the deformations at x but also on the deformations at all points of the body (scale effect). The constitutive relation of the elastic constitutive law of a nanosolid is expressed by the following relation: (10)

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}-\mu\nabla^2\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}=$$

$$\begin{bmatrix}C_{11}(z) & C_{12}(z) & 0 & 0 & C_{16}(z)\\ C_{12}(z) & C_{22}(z) & 0 & 0 & C_{26}(z)\\ 0 & 0 & C_{44}(z) & C_{45}(z) & 0\\ 0 & 0 & C_{45}(z) & C_{55}(z) & 0\\ C_{16}(z) & C_{26}(z) & 0 & 0 & C_{66}(z)\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}$$

The term ∇^2 is the Laplace operator in two-dimensional Cartesian coordinates. It is expressed by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The parameter $\mu = (e_0 a)^2$ is the nonlocal parameter which depends on the material constant e_0 and the internal characteristic a (lattice parameter, crack length or molecular diameters).

3.2 Theory of viscoelasticity

The theory of viscoelasticity can take into account materials capable of storing and dissipating mechanical energy. On the basis of the Kelvin-Voigt model on elastic materials with viscoelastic structural damping coefficient η, the rigidities C_{ij} which depend on Young's modulus E and shear modulus G are replaced by the operators, $C_{ij}\left(1+\eta\frac{\partial}{\partial t}\right)$. Therefore, Eq. (6) can be written again as follows: (11)

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix} = \left(1+\eta\frac{\partial}{\partial t}\right)\begin{bmatrix} C_{11}(z) & C_{12}(z) & 0 & 0 & C_{16}(z)\\ C_{12}(z) & C_{22}(z) & 0 & 0 & C_{26}(z)\\ 0 & 0 & C_{44}(z) & C_{45}(z) & 0\\ 0 & 0 & C_{45}(z) & C_{55}(z) & 0\\ C_{16}(z) & C_{26}(z) & 0 & 0 & C_{66}(z)\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}$$

The theory of the nonlocal viscoelasticity principle, previously developed by several researchers, initially assumed a combination of the models of nonlocal elasticity and viscoelasticity. Therefore, for nonlocal viscoelastic plates, the nonlocal viscoelastic stress field can be expressed in the following form: (12)

$$(1-\mu\nabla^2)\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix} = \left(1+\eta\frac{\partial}{\partial t}\right)\begin{bmatrix} C_{11}(z) & C_{12}(z) & 0 & 0 & C_{16}(z)\\ C_{12}(z) & C_{22}(z) & 0 & 0 & C_{26}(z)\\ 0 & 0 & C_{44}(z) & C_{45}(z) & 0\\ 0 & 0 & C_{45}(z) & C_{55}(z) & 0\\ C_{16}(z) & C_{26}(z) & 0 & 0 & C_{66}(z)\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}$$

4 Equilibrium equations

To establish the equations governing the equilibrium of an FGM viscoelastic nanoplate resting on the Winkler-Pasternak viscoelastic foundation, the principle of virtual work (Hamilton's principle) can be used in the following form:

$$\int_{t_0}^{t_1}\Pi\,dt = \int_{t_0}^{t_1}\left(\delta U - \delta K - \delta W_{fe}\right)dt = 0\,. \tag{13}$$

In this case, the virtual strain energy may be expressed as:

$$\delta U = \iiint_V\left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz}\right)dV\,. \tag{14}$$

By substituting Eq. (5) into Eq. (13), the virtual strain energy becomes:

$$\delta U = \iint_A \begin{bmatrix} N_{xx}\delta\left(\frac{\partial u_0}{\partial x}\right) - M_{xx}\delta\left(\frac{\partial^2 w_b}{\partial x^2}\right) + S_{xx}\delta\left(\frac{\partial w_s}{\partial x^2}\right)\\ +N_{yy}\delta\left(\frac{\partial v_0}{\partial y}\right) - M_{yy}\delta\left(\frac{\partial^2 w_b}{\partial y^2}\right) + S_{yy}\delta\left(\frac{\partial w_s}{\partial y^2}\right) +\\ N_{xx}\delta\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2M_{xy}\delta\left(\frac{\partial^2 w_b}{\partial x\partial y}\right) + S_{xy}\delta\left(\frac{\partial^2 w_s}{\partial x\partial y}\right)\\ +Q_{xz}\delta\left(\frac{\partial w_s}{\partial x}\right) + Q_{yz}\delta\left(\frac{\partial w_s}{\partial y}\right)\end{bmatrix} dA, \tag{15}$$

where: N_{ij}, M_{ij} and Q_{ij} are the normal forces, bending moments and shear forces, respectively. The unusual term S_{ij} has the dimension of a moment that is induced by $(f(z)–z)$ in the displacement field; it is defined as:

$$\left(N_{ij}, M_{ij}, S_{ij}, Q_{ij}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{ij}\left(1, z, f(z)-z, \frac{\partial f(z)}{\partial z}\right)dz\,. \tag{16}$$

In addition, the expressions of local elastic forces and moments are given by: (17)

$$\begin{Bmatrix}N_{xx}\\ N_{yy}\\ N_{xy}\\ M_{xx}\\ M_{yy}\\ M_{xy}\\ S_{xx}\\ S_{yy}\\ S_{xy}\end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & B_{11}^f & B_{12}^f & B_{16}^f\\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & B_{12}^f & B_{22}^f & B_{26}^f\\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & B_{16}^f & B_{26}^f & B_{66}^f\\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & D_{11}^f & D_{12}^f & D_{16}^f\\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & D_{12}^f & D_{22}^f & D_{26}^f\\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & D_{16}^f & D_{26}^f & D_{66}^f\\ B_{11}^f & B_{12}^f & B_{16}^f & D_{11}^f & D_{12}^f & D_{16}^f & F_{11}^f & F_{12}^f & F_{16}^f\\ B_{12}^f & B_{22}^f & B_{26}^f & D_{12}^f & D_{22}^f & D_{26}^f & F_{12}^f & F_{22}^f & F_{26}^f\\ B_{16}^f & B_{26}^f & B_{66}^f & D_{16}^f & D_{26}^f & D_{66}^f & F_{16}^f & F_{26}^f & F_{66}^f\end{bmatrix}\begin{Bmatrix}u_{0,x}\\ v_{0,y}\\ u_{0,y}+v_{0,x}\\ -w_{s,xx}\\ -w_{b,yy}\\ -2w_{b,xy}\\ w_{s,x}\\ w_{s,y}\\ 2w_{s,xy}\end{Bmatrix}.$$

$$\begin{Bmatrix}Q_{xz}\\ Q_{yz}\end{Bmatrix} = \begin{bmatrix}A_{44}^f & A_{45}^f\\ A_{45}^f & A_{55}^f\end{bmatrix}\begin{Bmatrix}w_{s,x}\\ w_{s,y}\end{Bmatrix}. \tag{18}$$

Similarly, the elements of the matrix given by Eqs. (17) and (18) can be expressed as: (19)

$$\left\{A_{ij}, B_{ij}, D_{ij}, B_{ij}^f\right\} = \int C_{ij}(z)\left\{1, z, z^2, f(z)-z\right\}dz \quad i,j = 1, 2, 6,$$

$$\left\{D_{ij}^f, F_{ij}^f\right\} = \int C_{ij}(z)\left\{z\left(f(z)-z\right), \left(f(z)-z\right)^2\right\}dz \quad i,j = 1, 2, 6,$$

$$\left\{A_{ij}^f\right\} = \int C_{ij}(z)\left\{\left(\frac{\partial f(z)}{\partial z}\right)^2\right\}dz \quad i,j = 4, 5.$$

On the other hand, the kinetic energy is defined as follows:

$$K = \frac{1}{2}\iiint_V \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial t}\right)^2 \right] dV \tag{20}$$

When Eq. (1) is substituted into Eq. (20), the virtual kinetic energy becomes:

$$\begin{aligned}
\delta K = \iint_A \Bigg[& \left(I_1 \frac{\partial u_0}{\partial t} - I_2 \frac{\partial^2 w_b}{\partial x \partial t} + I_4 \frac{\partial w_s}{\partial x \partial t} \right) \delta \left(\frac{\partial u_0}{\partial t} \right) \\
& - \left(I_2 \frac{\partial u_0}{\partial t} - \frac{\partial^2 w_0}{\partial x \partial t} + I_5 \frac{\partial w_s}{\partial x \partial t} \right) \delta \left(\frac{\partial^2 w_b}{\partial x \partial t} \right) + \\
& \left(I_4 \frac{\partial u_0}{\partial t} - I_5 \frac{\partial^2 w_b}{\partial x \partial t} + I_6 \frac{\partial^2 w_s}{\partial x \partial t} \right) \delta \left(\frac{\partial^2 w_s}{\partial x \partial t} \right) + \\
& \left(I_1 \frac{\partial v_0}{\partial t} - I_2 \frac{\partial^2 w_b}{\partial y \partial t} + I_4 \frac{\partial^2 w_s}{\partial y \partial t} \right) \delta \left(\frac{\partial v_0}{\partial t} \right) - \\
& \left(I_2 \frac{\partial v_0}{\partial t} - \frac{\partial^2 w_b}{\partial y \partial t} + I_5 \frac{\partial^2 w_s}{\partial y \partial t} \right) \delta \left(\frac{\partial^2 w_b}{\partial y \partial t} \right) + \\
& \left(I_4 \frac{\partial v_0}{\partial t} - I_5 \frac{\partial^2 w_b}{\partial v \partial t} + I_6 \frac{\partial^2 w_s}{\partial y \partial t} \right) \delta \left(\frac{\partial^2 w_s}{\partial y \partial t} \right) + \\
& I_1 \left(\frac{\partial w_b}{\partial t} + \frac{\partial w_s}{\partial t} \right) \delta \left(\frac{\partial w_b}{\partial t} + \frac{\partial w_s}{\partial t} \right) \Bigg] dV.
\end{aligned} \tag{21}$$

Where: I_1, I_2, I_3, I_4, I_5 and I_6 are the mass inertias which are defined by: (22)

$$\{I_i\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z) \left\{ 1, z, z^2, f(z) - z, z\left(f(z) - z\right), \left(f(z) - z\right)^2 \right\} dz, i = 1..6.$$

The virtual work done by the viscoelastic Winkler-Pasternak foundation is given by:

$$\begin{aligned}
\delta W_{fe} &= \iint_A f_e(x, y) \delta W dA \\
\delta W_{fe} &= \iint_A \left(k_w W - k_p \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) W + c_d \frac{\partial W}{\partial t} \right) \delta W dA,
\end{aligned} \tag{23}$$

where: k_w, k_p and c_d are, respectively, the Winkler coefficient, Pasternak coefficient and damping parameter.

By performing integration by parts of Eq. (15), Eq. (21) and Eq. (23), one obtains the equations of motion used in the refined FGM plate theory. These equations may be applied to homogeneous thin or thick laminated plates. They take into account the transverse shearing effect. Therefore:

$$\begin{aligned}
&\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 \frac{\partial^2 u_0}{\partial t^2} - I_2 \frac{\partial^3 w_b}{\partial x \partial t^2} + I_4 \frac{\partial^3 w_s}{\partial x \partial t^2}, \\
&\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_1 \frac{\partial^2 v_0}{\partial t^2} - I_2 \frac{\partial^3 w_b}{\partial y \partial t^2} + I_4 \frac{\partial^3 w_s}{\partial y \partial t^2}, \\
&\delta w_b : \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = f_e + I_1 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) \\
&+ I_2 \left(\frac{\partial^3 u_0}{\partial x \partial t^2} + \frac{\partial^3 v_0}{\partial y \partial t^2} \right) - I_3 \left(\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \frac{\partial^4 w_b}{\partial y^2 \partial t^2} \right) + \\
&I_5 \left(\frac{\partial^4 w_s}{\partial x^2 \partial t^2} + \frac{\partial^4 w_s}{\partial y^2 \partial t^2} \right), \\
&\delta w_s : \frac{\partial^2 S_{xx}}{\partial x^2} + 2\frac{\partial^2 S_{xy}}{\partial x \partial y} + \frac{\partial^2 S_{yy}}{\partial y^2} - \frac{\partial Q_{xz}}{\partial x} - \frac{\partial Q_{yz}}{\partial y} = f_e + \\
&I_1 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) + I_4 \left(\frac{\partial^3 u_0}{\partial x \partial t^2} + \frac{\partial^3 v_0}{\partial y \partial t^2} \right) - \\
&I_5 \left(\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \frac{\partial^4 w_b}{\partial y^2 \partial t^2} \right) + I_6 \left(\frac{\partial^4 w_s}{\partial x^2 \partial t^2} + \frac{\partial^4 w_s}{\partial y^2 \partial t^2} \right).
\end{aligned} \tag{24}$$

$$(1 - \mu \nabla^2) \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = \left(1 + \eta \frac{\partial}{\partial t} \right) \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & B_{11}^f & B_{12}^f & B_{16}^f \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & B_{12}^f & B_{22}^f & B_{26}^f \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & B_{16}^f & B_{26}^f & B_{66}^f \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & D_{11}^f & D_{12}^f & D_{16}^f \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & D_{12}^f & D_{22}^f & D_{26}^f \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & D_{16}^f & D_{26}^f & D_{66}^f \\ B_{11}^f & B_{12}^f & B_{16}^f & D_{11}^f & D_{12}^f & D_{16}^f & F_{11}^f & F_{12}^f & F_{16}^f \\ B_{12}^f & B_{22}^f & B_{26}^f & D_{12}^f & D_{22}^f & D_{26}^f & F_{12}^f & F_{22}^f & F_{26}^f \\ B_{16}^f & B_{26}^f & B_{66}^f & D_{16}^f & D_{26}^f & D_{66}^f & F_{16}^f & F_{26}^f & F_{66}^f \end{bmatrix} \begin{Bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \\ -w_{s,xx} \\ -w_{b,yy} \\ -2w_{b,xy} \\ w_{s,x} \\ w_{s,y} \\ 2w_{s,xy} \end{Bmatrix} \tag{25}$$

The internal forces with viscoelastic and nonlocal parameters can be expressed in the following form:

$$\begin{Bmatrix} Q_{xz} \\ Q_{yz} \end{Bmatrix} - \mu\nabla^2 \begin{Bmatrix} Q_{xz} \\ Q_{yz} \end{Bmatrix} = \left(1+\eta\frac{\partial}{\partial t}\right)\begin{bmatrix} A_{44}^f & A_{45}^f \\ A_{45}^f & A_{55}^f \end{bmatrix}\begin{Bmatrix} w_{s,x} \\ w_{s,y} \end{Bmatrix}. \quad (26)$$

The equations governing FGM viscoelastic nanoplates resting on a viscoelastic medium having three parameters in the case of free vibrations, in terms of displacements, are obtained by introducing the constitutive Eq. (25) and Eq. (26) in the equations of equilibrium (24). This makes it possible to obtain the four fundamental relations of the refined theory of nonlocal viscoelastic plates of FGM materials:

$$\begin{aligned}
&\delta u_0 : \mu\frac{\partial^2}{\partial x^2}\left(I_1\frac{\partial^2 u_0}{\partial t^2} - I_2\frac{\partial^3 w_b}{\partial x\partial t^2} + I_4\frac{\partial^3 w_s}{\partial x\partial t^2}\right) \\
&+\mu\frac{\partial^2}{\partial y^2}\left(I_1\frac{\partial^2 u_0}{\partial t^2} - I_2\frac{\partial^3 w_b}{\partial x\partial t^2} + I_4\frac{\partial^3 w_s}{\partial x\partial t^2}\right) + \\
&\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_b}{\partial x^3} + B_{11}^f\frac{\partial^3 w_s}{\partial x^3} + \\ & A_{12}\frac{\partial^2 v_0}{\partial x\partial y} - B_{12}\frac{\partial^3 w_b}{\partial y^2\partial x} + B_{12}^f\frac{\partial^3 w_s}{\partial y^2\partial x}\end{aligned}\right) + \\
&\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & A_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x\partial y}\right) + 2B_{66}\frac{\partial^3 w_b}{\partial x\partial y^2} + \\ & 2B_{66}^f\frac{\partial^3 w_s}{\partial x\partial y^2}\end{aligned}\right) \\
&= I_1\frac{\partial^2 u_0}{\partial t^2} - I_2\frac{\partial^3 w_b}{\partial x\partial t^2} + I_4\frac{\partial^3 w_s}{\partial x\partial t^2}
\end{aligned} \quad (27)$$

$$\begin{aligned}
&\delta v_0 : \mu\frac{\partial^2}{\partial x^2}\left(I_1\frac{\partial^2 v_0}{\partial t^2} - I_2\frac{\partial^3 w_b}{\partial y\partial t^2} + I_4\frac{\partial^3 w_s}{\partial y\partial t^2}\right) \\
&+\mu\frac{\partial^2}{\partial y^2}\left(I_1\frac{\partial^2 v_0}{\partial t^2} - I_2\frac{\partial^3 w_b}{\partial y\partial t^2} + I_4\frac{\partial^3 w_s}{\partial y\partial t^2}\right) + \\
&\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & A_{12}\frac{\partial^2 u_0}{\partial x\partial y} - B_{12}\frac{\partial^3 w_b}{\partial x^2\partial y} + B_{12}^f\frac{\partial^3 w_s}{\partial x^2\partial y} \\ & +A_{22}\frac{\partial^2 v_0}{\partial y^2} - B_{22}\frac{\partial^3 w_b}{\partial y^3} + B_{22}^f\frac{\partial^3 w_s}{\partial y^3}\end{aligned}\right) + \\
&\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & A_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y} + \frac{\partial^2 v_0}{\partial x^2}\right) - 2B_{66}\frac{\partial^3 w_b}{\partial x^2\partial y} + \\ & 2B_{66}^f\frac{\partial^3 w_s}{\partial x^2\partial y}\end{aligned}\right) \\
&= I_1\frac{\partial^2 v_0}{\partial t^2} - I_2\frac{\partial^3 w_b}{\partial y\partial t^2} + I_4\frac{\partial^3 w_s}{\partial y\partial t^2}
\end{aligned} \quad (28)$$

$$\begin{aligned}
&\delta w_b : \frac{\partial^2}{\partial x^2}\left(\begin{aligned} & f_e + I_1\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + I_2\left(\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{\partial^3 v_0}{\partial y\partial t^2}\right) \\ & -I_3\left(\frac{\partial^4 w_b}{\partial x^2\partial t^2} + \frac{\partial^4 w_b}{\partial y^2\partial t^2}\right) + I_5\left(\frac{\partial^4 w_s}{\partial x^2\partial t^2} + \frac{\partial^4 w_s}{\partial y^2\partial t^2}\right)\end{aligned}\right) + \\
&\frac{\partial^2}{\partial y^2}\left(\begin{aligned} & f_e + I_1\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + I_2\left(\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{\partial^3 v_0}{\partial y\partial t^2}\right) \\ & -I_3\left(\frac{\partial^4 w_b}{\partial x^2\partial t^2} + \frac{\partial^4 w_b}{\partial y^2\partial t^2}\right) + I_5\left(\frac{\partial^4 w_s}{\partial x^2\partial t^2} + \frac{\partial^4 w_s}{\partial y^2\partial t^2}\right)\end{aligned}\right) + \\
&\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_b}{\partial x^4} + D_{11}^f\frac{\partial^4 w_s}{\partial x^4} + B_{12}\frac{\partial^3 v_0}{\partial x^2\partial y} \\ & -D_{12}\frac{\partial^4 w_b}{\partial x^2\partial y^2} + D_{12}^f\frac{\partial^4 w_s}{\partial x^2\partial y^2}\end{aligned}\right) \\
&+\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & B_{12}\frac{\partial^3 u_0}{\partial x^2\partial y} - D_{12}\frac{\partial^4 w_b}{\partial x^2\partial y^2} + D_{12}^f\frac{\partial^4 w_s}{\partial x^2\partial y^2} + \\ & B_{22}\frac{\partial^3 v_0}{\partial y^3} - D_{22}\frac{\partial^4 w_b}{\partial y^4} + D_{22}^f\frac{\partial^4 w_s}{\partial y^4}\end{aligned}\right) + \\
&2\left(1+\frac{\partial}{\partial t}\right)\left(B_{66}\left(\frac{\partial^3 u_0}{\partial x\partial y^2} + \frac{\partial^3 v_0}{\partial x^2\partial y}\right) - 2D_{66}\frac{\partial^3 w_b}{\partial x^2\partial y^2} + 2D_{66}^f\frac{\partial^3 w_s}{\partial x^2\partial y^2}\right) \\
&= f_e + I_1\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + I_2\left(\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{\partial^3 v_0}{\partial y\partial t^2}\right) - \\
&I_3\left(\frac{\partial^4 w_b}{\partial x^2\partial t^2} + \frac{\partial^4 w_b}{\partial y^2\partial t^2}\right) + I_5\left(\frac{\partial^4 w_s}{\partial x^2\partial t^2} + \frac{\partial^4 w_s}{\partial y^2\partial t^2}\right)
\end{aligned} \quad (29)$$

$$\begin{aligned}
&\delta w_s : \frac{\partial^2}{\partial x^2}\left(\begin{aligned} & f_e + I_1\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + I_4\left(\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{\partial^3 v_0}{\partial y\partial t^2}\right) - \\ & I_5\left(\frac{\partial^4 w_b}{\partial x^2\partial t^2} + \frac{\partial^4 w_b}{\partial y^2\partial t^2}\right) + I_6\left(\frac{\partial^4 w_s}{\partial x^2\partial t^2} + \frac{\partial^4 w_s}{\partial y^2\partial t^2}\right)\end{aligned}\right) + \\
&\frac{\partial^2}{\partial y^2}\left(\begin{aligned} & f_e + I_1\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + I_4\left(\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{\partial^3 v_0}{\partial y\partial t^2}\right) \\ & -I_5\left(\frac{\partial^4 w_b}{\partial x^2\partial t^2} + \frac{\partial^4 w_b}{\partial y^2\partial t^2}\right) + I_6\left(\frac{\partial^4 w_s}{\partial x^2\partial t^2} + \frac{\partial^4 w_s}{\partial y^2\partial t^2}\right)\end{aligned}\right) + \\
&\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & B_{11}^f\frac{\partial^3 u_0}{\partial x^3} - D_{11}^f\frac{\partial^4 w_b}{\partial x^4} + F_{11}^f\frac{\partial^4 w_s}{\partial x^4} + \\ & B_{12}^f\frac{\partial^3 v_0}{\partial x^2\partial y} - D_{12}^f\frac{\partial^4 w_b}{\partial x^2\partial y^2} + F_{12}^f\frac{\partial^4 w_s}{\partial x^2\partial y^2}\end{aligned}\right) + \\
&+\left(1+\frac{\partial}{\partial t}\right)\left(\begin{aligned} & B_{12}^f\frac{\partial^3 u_0}{\partial y^2\partial x} - D_{12}^f\frac{\partial^4 w_b}{\partial x^2\partial y^2} + F_{12}^f\frac{\partial^4 w_s}{\partial x^2\partial y^2} + A_{55}^f\frac{\partial^2 w_s}{\partial x^2} \\ & +B_{22}^f\frac{\partial^3 v_0}{\partial y^3} - D_{22}^f\frac{\partial^4 w_b}{\partial y^4} + F_{22}^f\frac{\partial^4 w_s}{\partial y^4} + A_{55}^f\frac{\partial^2 w_s}{\partial y^2}\end{aligned}\right) + \\
&2\left(1+\frac{\partial}{\partial t}\right)\left(B_{66}^f\left(\frac{\partial^3 u_0}{\partial x\partial y^2} + \frac{\partial^3 v_0}{\partial x^2\partial y}\right) - 2D_{66}^f\frac{\partial^3 w_b}{\partial x^2\partial y^2} + 2F_{66}^f\frac{\partial^3 w_s}{\partial x^2\partial y^2}\right) \\
&= f_e + I_1\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + I_4\left(\frac{\partial^3 u_0}{\partial x\partial t^2} + \frac{\partial^3 v_0}{\partial y\partial t^2}\right) \\
&-I_5\left(\frac{\partial^4 w_b}{\partial x^2\partial t^2} + \frac{\partial^4 w_b}{\partial y^2\partial t^2}\right) + I_6\left(\frac{\partial^4 w_s}{\partial x^2\partial t^2} + \frac{\partial^4 w_s}{\partial y^2\partial t^2}\right)
\end{aligned} \quad (30)$$

5 Analytical solution

In general, rectangular plates are classified according to the type of support used. The goal is to find the analytical solutions of Eqs. (27) to (30) under various boundary conditions imposed on the lateral edges, as shown in Table 2. The edges of the plate are either embedded or simply supported. The boundary conditions used are:

Simply supported plate (S) along direction x:

$$\begin{aligned}
& at \quad x = 0, a: \\
& \bullet v_0 = w_b = \frac{\partial w_s}{\partial y} = 0 \\
& \bullet N_{xx} = A_{11}\frac{\partial u_0}{\partial x} + A_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - B_{11}\frac{\partial^2 w_s}{\partial x^2} + B_{11}^f \frac{\partial w_s}{\partial x} \\
& + A_{12}\frac{\partial v_0}{\partial y} - B_{12}\frac{\partial^2 w_b}{\partial y^2} - 2B_{16}\frac{\partial^2 w_b}{\partial y \partial x} + B_{12}^f \frac{\partial w_s}{\partial y} + 2B_{16}^f \frac{\partial w_s}{\partial y \partial x} = 0 \\
& \bullet M_{xx} = B_{11}\frac{\partial u_0}{\partial x} + B_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{11}\frac{\partial^2 w_s}{\partial x^2} + D_{11}^f \frac{\partial w_s}{\partial x} + \\
& B_{12}\frac{\partial v_0}{\partial y} - D_{12}\frac{\partial^2 w_b}{\partial y^2} - 2D_{16}\frac{\partial^2 w_b}{\partial y \partial x} + D_{12}^f \frac{\partial w_s}{\partial y} + 2D_{16}^f \frac{\partial w_s}{\partial y \partial x} = 0 \\
& \bullet S_{xx} = B_{11}^f\frac{\partial u_0}{\partial x} + B_{16}^f\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{11}^f\frac{\partial^2 w_s}{\partial x^2} + F_{11}^f \frac{\partial w_s}{\partial x} + \\
& B_{12}^f\frac{\partial v_0}{\partial y} - D_{12}^f\frac{\partial^2 w_b}{\partial y^2} - 2D_{16}^f\frac{\partial^2 w_b}{\partial y \partial x} + F_{12}^f \frac{\partial w_s}{\partial y} + 2F_{16}^f \frac{\partial w_s}{\partial y \partial x} = 0
\end{aligned} \tag{31}$$

Simply supported plate (S) along direction y:

$$\begin{aligned}
& at \quad y = 0, b: \\
& \bullet u_0 = w_b = \frac{\partial w_s}{\partial x} = 0 \\
& \bullet N_{yy} = A_{12}\frac{\partial u_0}{\partial x} + A_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - B_{12}\frac{\partial^2 w_s}{\partial x^2} + B_{12}^f \frac{\partial w_s}{\partial x} \\
& + A_{22}\frac{\partial v_0}{\partial y} - B_{22}\frac{\partial^2 w_b}{\partial y^2} - 2B_{26}\frac{\partial^2 w_b}{\partial y \partial x} + B_{22}^f \frac{\partial w_s}{\partial y} + 2B_{26}^f \frac{\partial w_s}{\partial y \partial x} = 0 \\
& \bullet M_{yy} = B_{12}\frac{\partial u_0}{\partial x} + B_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{12}\frac{\partial^2 w_s}{\partial x^2} + D_{12}^f \frac{\partial w_s}{\partial x} + \\
& B_{22}\frac{\partial v_0}{\partial y} - D_{22}\frac{\partial^2 w_b}{\partial y^2} - 2D_{26}\frac{\partial^2 w_b}{\partial y \partial x} + D_{22}^f \frac{\partial w_s}{\partial y} + 2D_{26}^f \frac{\partial w_s}{\partial y \partial x} = 0 \\
& \bullet S_{yy} = B_{12}^f\frac{\partial u_0}{\partial x} + B_{26}^f\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{12}^f\frac{\partial^2 w_s}{\partial x^2} + F_{12}^f \frac{\partial w_s}{\partial x} + \\
& B_{22}^f\frac{\partial v_0}{\partial y} - D_{22}^f\frac{\partial^2 w_b}{\partial y^2} - 2D_{26}^f\frac{\partial^2 w_b}{\partial y \partial x} + F_{22}^f \frac{\partial w_s}{\partial y} + 2F_{26}^f \frac{\partial w_s}{\partial y \partial x} = 0
\end{aligned} \tag{32}$$

Embedded plate (C) (33)

$$at \quad x = 0, a, y = 0, b \qquad , u_0 = v_0 = w_b = \frac{\partial w_s}{\partial x} = \frac{\partial w_s}{\partial y} = 0$$

Table 2 The admissible functions $X_i(x)$ and $Y_i(y)$ as given by Kiasat et al. [32] and Sobhy [38]

BC	$X_i(x)$	$Y_i(y)$
SSSS	$\sin(\alpha x)$	$\sin(\beta y)$
CCCC	$1-\cos(2\alpha x)$	$1-\cos(2\beta y)$
CCSS	$1-\cos(2\alpha x)$	$\sin(\beta y)$
CSCS	$\cos\left(\frac{3\alpha x}{2}\right)-\cos\left(\frac{\alpha x}{2}\right)$	$\cos\left(\frac{3}{2}\beta y\right)-\cos\left(\frac{1}{2}\beta y\right)$
CSSS	$\cos\left(\frac{3}{2}\alpha x\right)-\cos\left(\frac{1}{2}\alpha x\right)$	$\sin(\beta y)$
CCSC	$1-\cos(2\alpha x)$	$\cos\left(\frac{3}{2}\beta y\right)-\cos\left(\frac{1}{2}\beta y\right)$

The Navier method is used under the specified boundary conditions to find the analytical solution to Eqs. (27)–(29) and (30). The displacement functions, which satisfy the boundary conditions, are expressed as the following Fourier series:

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \begin{Bmatrix} U_{ij}\dfrac{\partial X_i(x)}{\partial x}Y_j(y) \\ V_{ij}X_i(x)\dfrac{\partial Y_j(y)}{\partial y} \\ W_{bij}X_i(x)Y_j(y) \\ W_{sij}X_i(x)Y_j(y) \end{Bmatrix} e^{\lambda t}. \tag{34}$$

The quantities U_{ij}, V_{ij}, W_{bij} and W_{sij} refer to the amplitudes, and λ corresponds to the complex eigen-frequency of the (ith, jth) vibrational eigen mode, with $\alpha = i\pi/a$, $\beta = i\pi/b$

Using Eq. (34) in the equations of motion Eqs. (27) to (30) gives, for any fixed values of i and j, in the case of free vibrations, the eigenvalue equations can expressed as:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{Bmatrix} U_{ij} \\ V_{ij} \\ W_{bij} \\ W_{sij} \end{Bmatrix} = \{0\}. \tag{35}$$

It is worth noting that $[U_{ij}, V_{ij}, W_{bij}, W_{sij}]$ is the displacement, and $[L]$ is the global matrix of the system. The elements $L_{ij} = L_{ji}$ of the coefficients matrix are given in the Appendix. For non-trivial solutions of Eqs. (27) to (30), the following determinants should be set equal to zero to find the eigenfrequency λ:

$$Det\left(\left[L\right]\right)=0 \tag{36}$$

Note also that the frequency and parameters used in this study are dimensionless and can be written in the following form:

$$K_w=\frac{k_w a^4}{D_c},\ K_p=\frac{k_p a^2}{D_c},\ \psi=\frac{\eta}{a^2}\sqrt{\frac{D_c}{\rho_c h}},$$
$$C_d=\frac{c_d a^2}{\sqrt{\rho_c h D_c}},\ D_c=\frac{E_c h^3}{12(1-\nu^2)}. \tag{37}$$

Finally, the dimensionless eigenfrequency may be expressed as follows:

$$\varpi=-\Omega\left(\xi\pm i\sqrt{1-\xi^2}\right),\Omega=\lambda a^2\sqrt{\frac{\rho_c h}{D_C}} \tag{38}$$

Here Ω and ξ are respectively the dimensionless undamped frequency and damping ratio.

Regarding the expression under the square root $\xi\sqrt{1-\xi^2}$, three different cases may be considered:

$0<\xi<1$, in the case of underdamped vibration of nanoplate,

$\xi=1$, in the case of critically damped vibration of nanoplate,

$\xi>1$, in the case of overdamped vibration of nanoplate.

6 Comparative studies

In order to validate the suggested model, the authors decided to compare the results obtained with those previously found by several other researchers. The first example involves three natural frequencies, given by the present model for various values of the nonlocal parameter (μ), which are compared with those given by the nonlocal higher-order theory for simply supported plates using Reddy's third-order shear deformation theory (TSDT) [25], as presented in Table 3.

The approach followed here is pretty straightforward. Indeed, it was revealed that for the first vibrational mode, (Ω_{11}) second vibrational mode (Ω_{22}), and third vibrational mode (Ω_{33}), and for any nonlocal parameter value, this method gives very precise results. Moreover, it is important to mention that the findings obtained by this method agree very well with those obtained by the theory developed by Aghababaei and Reddy (Higher-order shear deformation theory or TSDT) [25].

On the other hand, the first natural frequencies of the exponentially graded material (EGM) sandwich plate lying on elastic foundations, studied in the present article, under different boundary conditions and for different length-to-thickness ratio values, are compared with those suggested by Sobhy [38], as displayed in Table 4. It is worth noting that the values obtained were for inhomogeneous plates ($p=0.5$ and $p=3.5$) and homogeneous plates ($p=0$) resting on elastic foundations, under any boundary conditions. It is important to mention that the results obtained are quite precise.

Table 3 Comparison of non-dimensional frequenciesof simply supported plate between fundamental frequencies given by present model and those of Aghababaei and Reddy (TSDT) [25] ($E=30\times10^6$, $\nu=0.3$, $a=1$, $a/b=1$, $a/h=10$)

Frequencies	μ	TSDT	FSDT	Classical	Present study
Ω_{11}	0	0.094	0.093	0.096	0.093
	1	0.085	0.085	0.088	0.085
	2	0.079	0.079	0.082	0.079
	3	0.074	0.074	0.076	0.074
	4	0.070	0.070	0.072	0.070
	5	0.066	0.066	0.068	0.066
Ω_{22}	0	0.346	0.341	0.385	0.341
	1	0.259	0.255	0.288	0.255
	2	0.215	0.213	0.240	0.212
	3	0.188	0.186	0.210	0.185
	4	0.170	0.167	0.189	0.167
	5	0.156	0.154	0.173	0.153
Ω_{33}	0	0.702	0.689	0.867	0.684
	1	0.421	0.413	0.520	0.410
	2	0.329	0.323	0.406	0.320
	3	0.279	0.274	0.345	0.271
	4	0.247	0.242	0.305	0.239
	5	0.223	0.219	0.276	0.217

7 Numerical results

For the purpose of investigating the effect of the viscosity parameter, nonlocal parameter, damping coefficient and material properties on the vibration response of an FGM viscoelastic nanoplate (metal and ceramic: (Al / Al2O3) lying on a viscoelastic foundation, a parametric study was performed on a square plate made of the following material and having the geometric characteristics given below:

Ceramic: alumina: $E_c=380$ GPa, $\rho_c=3800$ kg/m^3, $\nu=0.3$

Metal: aluminum: $E_m=380$ GPa, $\rho_m=3800$ kg/m^3, $\nu=0.3$

Fig. 3 displays the variation of the damping ratio ξ as a function of the nonlocal parameter μ for these damping coefficients of the viscoelastic foundation $C_d=0$ and 60 and for $K_w=50$ and $K_p=10$, with a material index $p=0{,}1$ and 5 under the boundary conditions (*SSSS*, *CCSS* and *CCCC*). Note that for the three boundary conditions and

Table 4 Comparison of natural frequencyof EGM sandwich plate with those of Sobhy [38] ($b/a = 2$, $K_w = 50$ and $K_p = 10$)

BCs	Sobhy [38]	$p = 0$			$p = 0.5$			$p = 3.5$		
		$a/h = 5$	10	20	$a/h = 5$	10	20	$a/h = 5$	10	20
SSSS	FSDT	1.5386	1.5942	1.6113	1.3971	1.4304	1.4399	1.3361	1.3589	1.3653
	TSDT	1.5387	1.5947	1.6113	1.3989	1.4310	1.4401	1.3375	1.3594	1.3655
	SSDT	1.5388	1.5947	1.6113	1.3990	1.4310	1.4401	1.3376	1.3594	1.3655
	Present	1.5387	1.5947	1.6113	1.3989	1.4310	1.4401	1.3374	1.3594	1.3654
CCSS	FSDT	2.2578	2.5112	2.5997	1.9476	2.0880	2.1341	1.7472	1.8273	1.8519
	TSDT	2.2590	2.5114	2.5997	1.9610	2.0930	2.1355	1.7622	1.8320	1.8532
	SSDT	2.2596	2.5115	2.5998	1.9614	2.0932	2.1356	1.7630	1.8323	1.8533
	Present	2.2590	2.5114	2.5997	1.9608	2.0930	2.1355	1.7617	1.8318	1.8531
CCCC	FSDT	2.3359	2.6010	2.6940	2.0111	2.1586	2.2072	1.8022	1.8847	1.9107
	TSDT	2.3372	2.6012	2.6940	2.0250	2.1638	2.2087	1.8157	1.8895	1.9121
	SSDT	2.3379	2.6013	2.6941	2.0255	2.1640	2.2088	1.8165	1.8898	1.9121
	Present	2.3372	2.6012	2.6941	2.0248	2.1638	2.2087	1.8151	1.8893	1.9120
CSSS	FSDT	2.2176	2.3921	2.4491	1.9319	2.0302	2.0610	1.7653	1.8242	1.8421
	TSDT	2.2181	2.3922	2.4493	1.9395	2.0330	2.0618	1.7720	1.8267	1.8428
	SSDT	2.2185	2.3923	2.4493	1.9397	2.0331	2.0618	1.7724	1.8268	1.8428
	Present	2.2182	2.3922	2.4493	1.9394	2.0329	2.0618	1.7717	1.8266	1.8428
CSCS	FSDT	2.3323	2.5262	2.5904	2.0265	2.1364	2.1712	1.8456	1.9117	1.9319
	TSDT	2.3328	2.5262	2.5904	2.0349	2.1395	2.1720	1.8531	1.9145	1.9327
	SSDT	2.3332	2.5263	2.5905	2.0352	2.1396	2.1721	1.8535	1.9146	1.9327
	Present	2.3329	2.5263	2.5904	2.0348	2.1395	2.1720	1.8528	1.9144	1.9327

for $C_d = 0$, the damping ratio ξ decreases as the nonlocal parameter µand the index p go up. For $p = 0$, i.e. for an isotropic ceramic material, ξ is equal to 0.0737, 0.1198 and 0.1533, respectively, for the boundary conditions (*SSSS*, *CCSS* and *CCCC*) with a value of $\mu = 0$. However, it is important to mention that ξ decreases and can take the values 0.0525, 0.0813 and 0.1001, respectively, when the coefficient $\mu = 3$.

For $C_d = 60$, the damping ratio ξ increases proportionally with the growth of the nonlocal parameter μ and index p, under different boundary conditions. In general, for a value of C_d equal to zero, there is a subcritical damping. However, for $C_d = 60$, the damping becomes over-critical except for an isotropic plate embedded on all four sides (*CCCC*).

Fig. 4 shows the variation of the damping ratio ξ of a plate (SSSS) supported by a viscoelastic foundation ($C_d = 10$, $K_w = 50$ and $K_p = 10$) as a function of the nonlocal parameter μ, for the values 0, 1, 5 and 10 of the material index p and damping coefficients of the viscoelastic material $\psi = 0$ and 0.01. It is noted that the damping ratio ξ increases proportionally with the nonlocal parameter μ, the material index p, and the damping coefficients of the viscoelastic material ψ. Here the system oscillates, and the amplitude of oscillation gradually decays to zero. In this case ($\xi < 1$), the system is called underdamped.

Moreover, the results confirm the advantage of using compositional gradient materials. It is worth noting that as the material index p increases ($p = 10$), like in the case of a pure metal, the damping ratio increases too ($\xi = 0.3422$); however, it is the reverse case when the material index decreases ($p = 0$), like in the case of pure ceramics, where the damping ratio decreases ($\xi = 0.2754$), with $\mu = 1$ and $\psi = 0.01$, as can be seen in Fig. 4(b). Consequently, a low damping factor indicates a decreased capacity to dissipate energy.

Figs. 5, 6 and 7 illustrate the effect of the damping coefficient of the foundation on the free vibration of the FGM square nanoplate simply supported (*SSSS*), for different Winkler ($K_w = 10, 50, 100, 150$) and Pasternak ($K_p = 0, 5, 10$) rigidities values of the foundation. In addition, the curves (a), (b) and (c) represent the variation of the imaginary frequency, real frequency and damping ratio, respectively. This variation increases progressively with the foundation parameters up to a critical point (the system goes back to its equilibrium position very quickly, without oscillating). As illustrated in Fig. 6, the vibrating system is damped for $K_p = 5$ and $K_w = 10$, and the damping coefficient C_d of the foundation is equal to 27.669. However, in Fig. 7, for $K_p = 10$, the damping coefficient C_d of the foundation is equal to 33.085, with $K_w = 10$. Note that the increasing value of the damping coefficient is due to the increasing stiffness of the foundation.

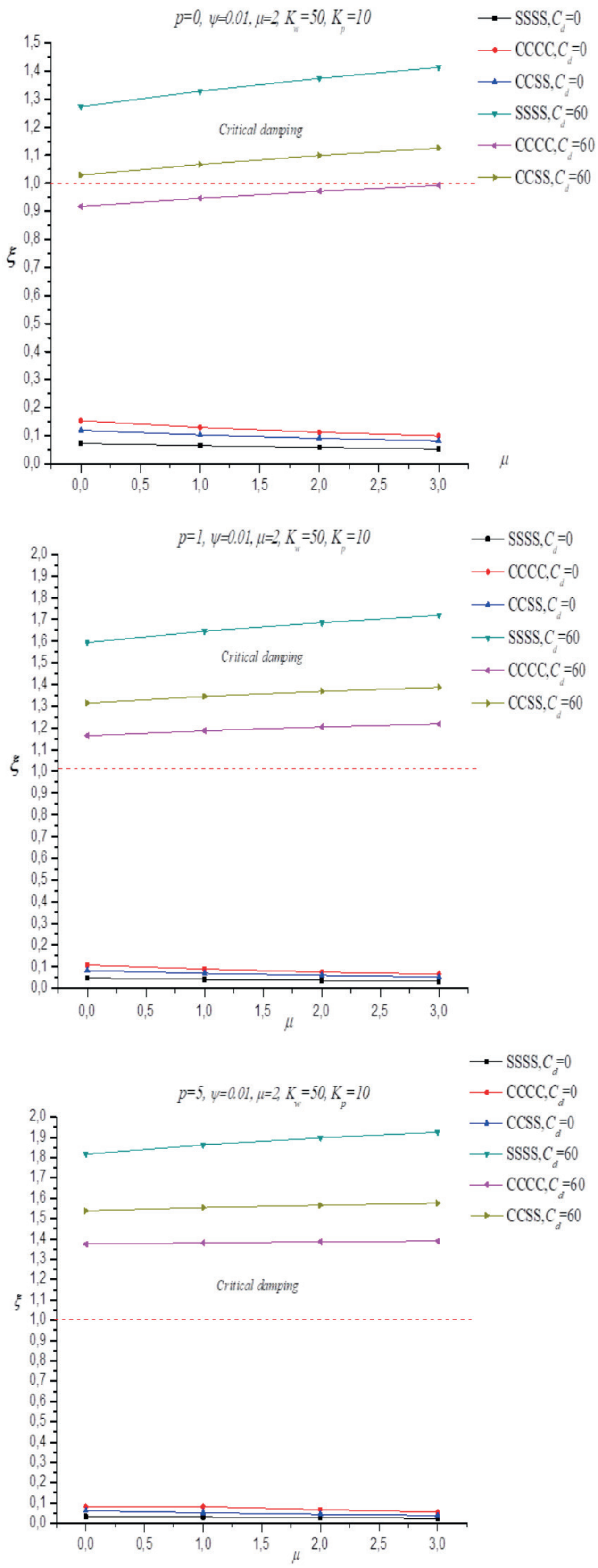

Fig. 3 Variation of the damping ratio ξ as a function of the nonlocal parameter μ, for different values of the damping coefficient C_d of the viscoelastic foundation, of the material index p, under different boundary conditions

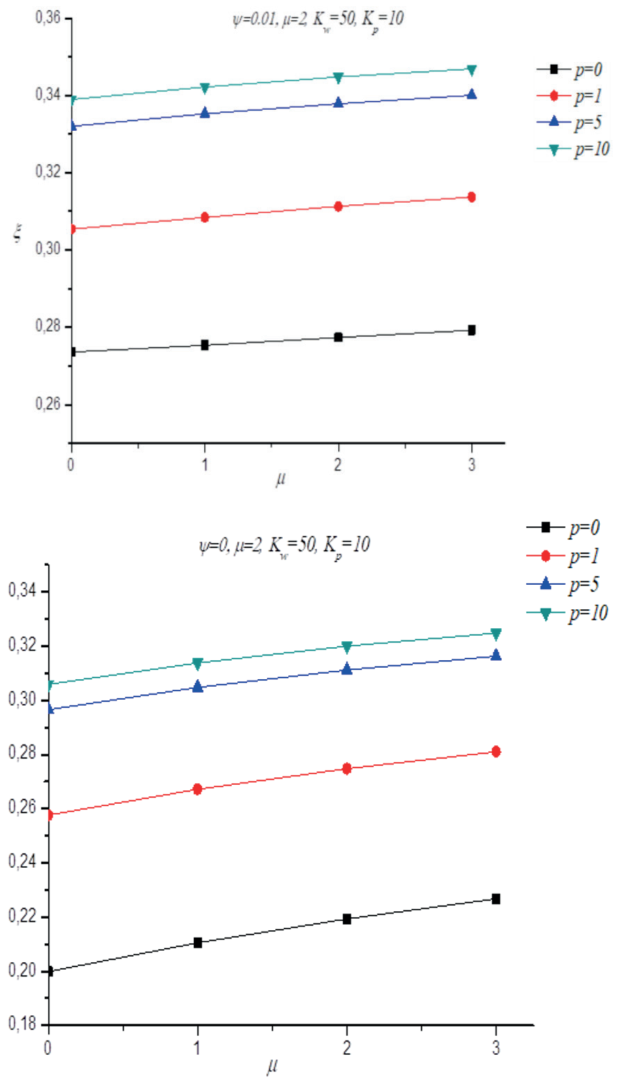

Fig. 4 Variation of the damping ratio ξ of a plate (*SSSS*) as a function of the nonlocal parameter μ for different values of the volume fraction index p and the damping coefficient Ψ of the viscoelastic material

Beyond this point ($\xi > 1$), the real part of the frequency increases and admits two solutions for the same value of each of the three parameters C_d, K_p, and K_w, and the damping becomes over-critical; the imaginary part of the frequency tends towards zero . In this case, the plate oscillates more slowly towards the equilibrium position than it does in the case of critical damping.

Similarly, Figs. 8 and 9 show the evolution of the non-dimensional frequencies of free vibrations for different values of the nonlocal parameter μ of square FGM nanoplates simply supported SSSS with a geometric ratio a/h respectively equal to 5 and 20, with stiffnesses of the foundation $K_w = 50$ and $K_p = 10$, as a function of the damping coefficient of the foundation C_d. It has been observed that the real part of the non-dimensional natural frequency

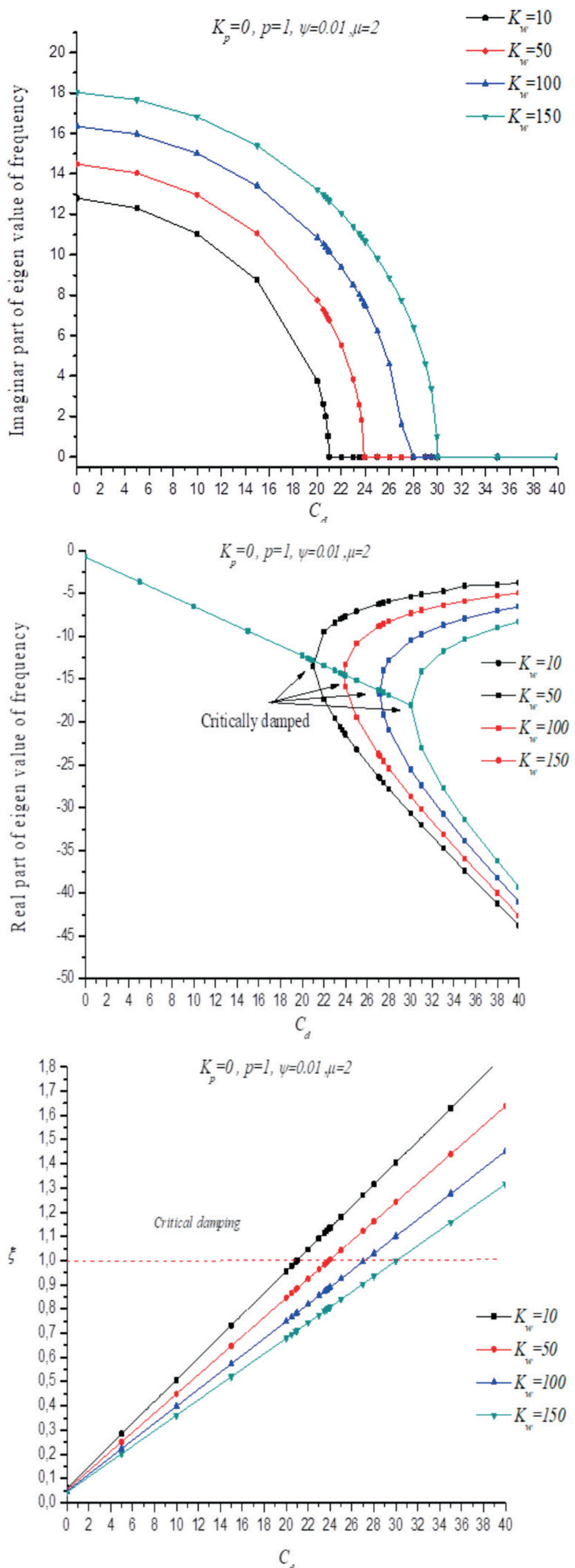

Fig. 5 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates *SSSS*, for different Winkler stiffness coefficients K_w; (a) Imaginary part, (b) Real part and (c) Damping ratio

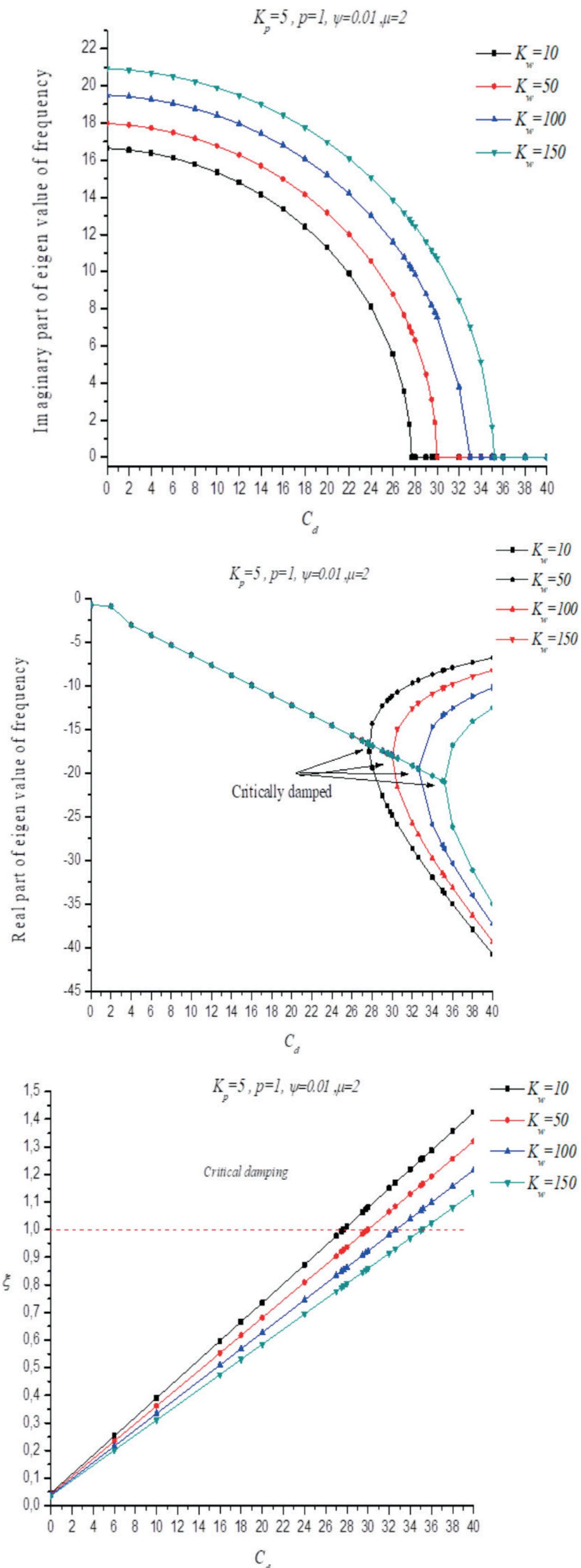

Fig. 6 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates *SSSS*, for different values of the Winkler stiffness coefficient K_w; (a) Imaginary part, (b) Real part and (c) Damping ratio

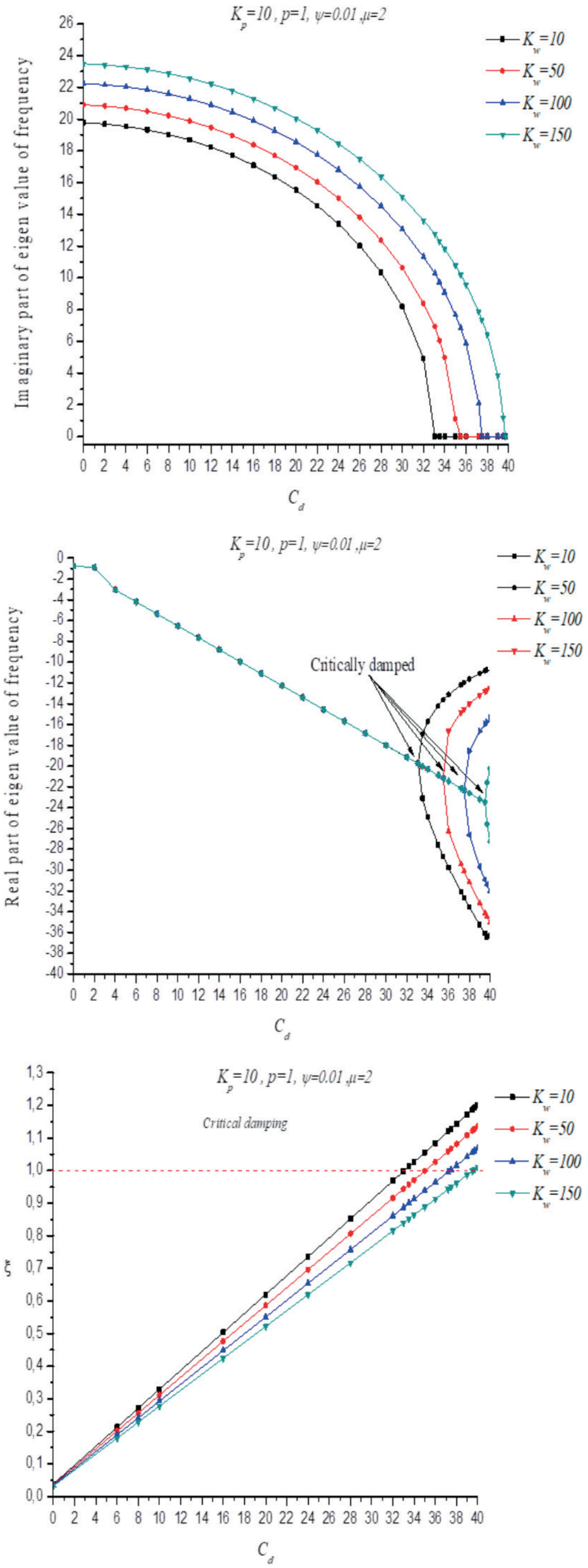

Fig. 7 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates SSSS, for different values of the Winkler stiffness coefficient K_w; (a) Imaginary part, (b) Real part and (c) Damping ratio

increases proportionally with the nonlocal parameter μ; however, the imaginary part is inversely proportional to this parameter. The additional damping reduces the nanoplate stiffness. For a geometric ratio $a/h = 20$, the values of the non-dimensional frequencies are too close for the different values of the nonlocal parameter μ, since the effect of the transverse deformation on these frequencies is negligible (Love-Kirchhoff theory). On the other hand, for a geometric ratio $a/h = 5$ (thick plate), the effect of shearing is significant which causes an increase in the non-dimensional frequency that is represented by a significant difference between the curves as a function of C_d up to the critical point. For example, in Fig. 8(b), the ($a/h = 5$) for $\mu = 4$, the critical point is reached for a value of Cd equal to 31.646, and in Fig. 9(b) ($a/h = 20$) for $\mu = 4$, the critical point is reached for a value of $C_d = 35.845$.

Figs. 10, 11 and 12 show the effect of the damping coefficient C_d of the foundation and the viscosity ψ of the material on the non-dimensional free vibrations of simply supported square FGM nanoplates, for different volume fraction indices p. Curves (a), (b) and (c) respectively show the variation of the dimensionless imaginary frequency, dimensionless real frequency and damping ratio. The viscosity ψ of the material induces a considerable reduction in the real and imaginary parts of the dimensionless frequencies until a critical point is reached, where the dimensionless imaginary part of the frequency becomes zero. At this point ($\xi = 1$), the nanoplate is critically damped and does not oscillate.

For example, in Fig. 10(c), for an isotropic material (ceramic), the viscosity value is $\psi = 0.075$, and the critical point is reached for $C_d = 25.75$. Note also that the critical point can be attained when $C_d = 42.934$, for the viscosity value $\psi = 0.01$. This behavior is certainly due to the decrease in the viscoelasticity of the material ψ.

With regard to Fig. 13, it shows the variation of the damping ratio ξ as a function of the damping coefficient C_d of the viscoelastic foundation, for an FGM nanoplate, under various boundary conditions. It is easy to notice also that for a simply supported plate (*SSSS*), the damping is subcritical for a viscoelastic coefficient $C_d < 35$. On the other hand, for plates under CCSS and CSSS boundary conditions, these viscoelastic coefficient values are greater and may be equal to 42. For more rigid plates, under *CSCS* and *CCCC* boundary conditions, the damping remains sub-critical for $C_d < 48.34$.

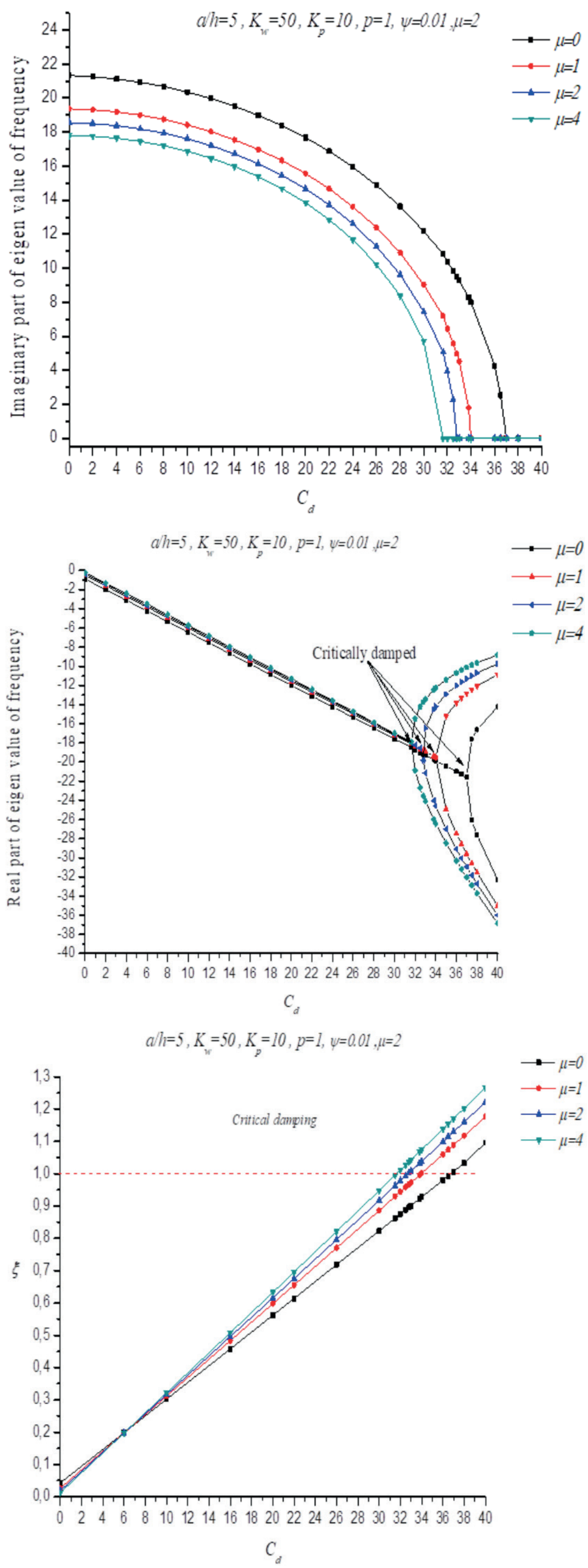

Fig. 8 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates *SSSS* for different values of the nonlocal parameter μ; (a) Imaginary part, (b) Real part and (c) Damping ratio

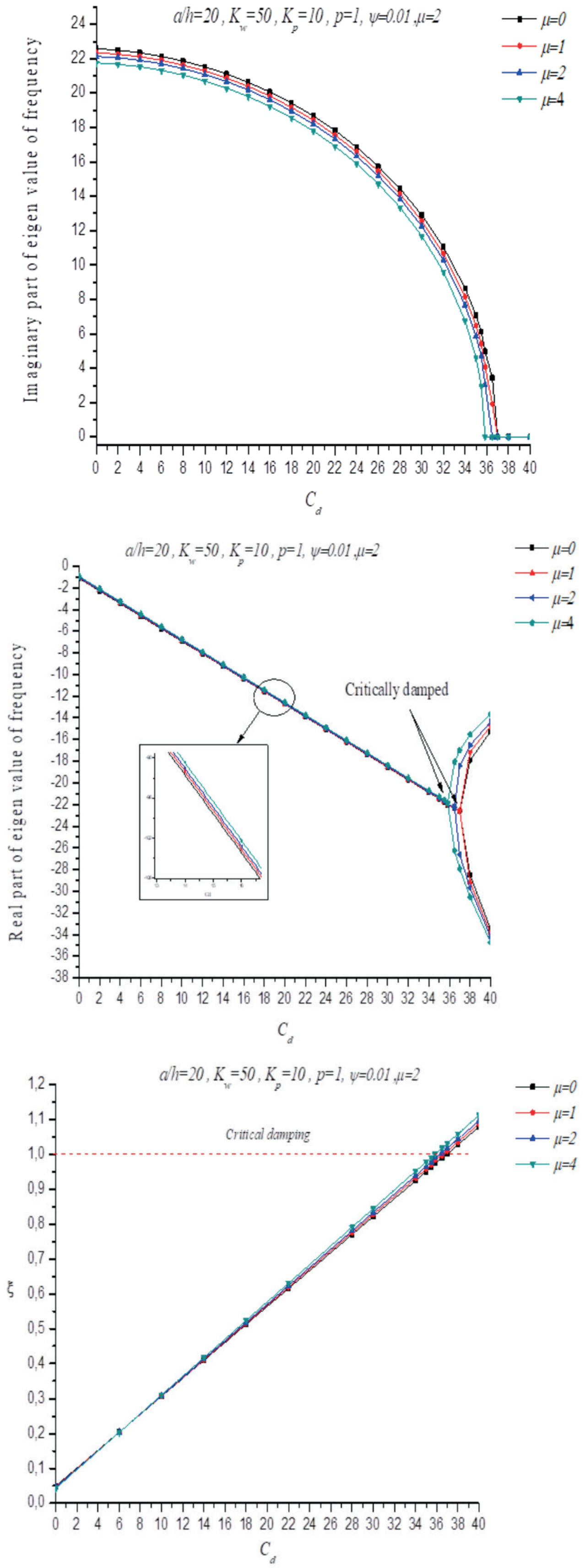

Fig. 9 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates *SSSS* for different values of the nonlocal parameter μ; (a) Imaginary part, (b) Real part and (c) Damping ratio

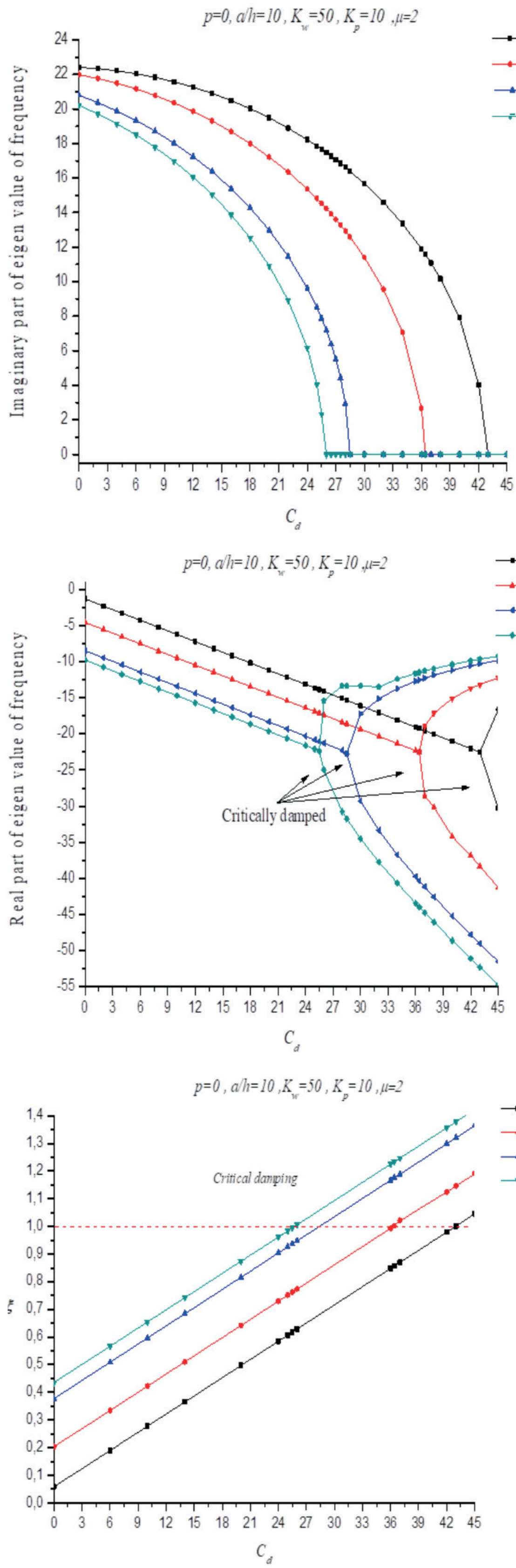

Fig. 10 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates *SSSS* for different values of the viscoelastic parameter Ψ of the material that constitutes the plate; (a) Imaginary part, (b) Real part and (c) Damping ratio

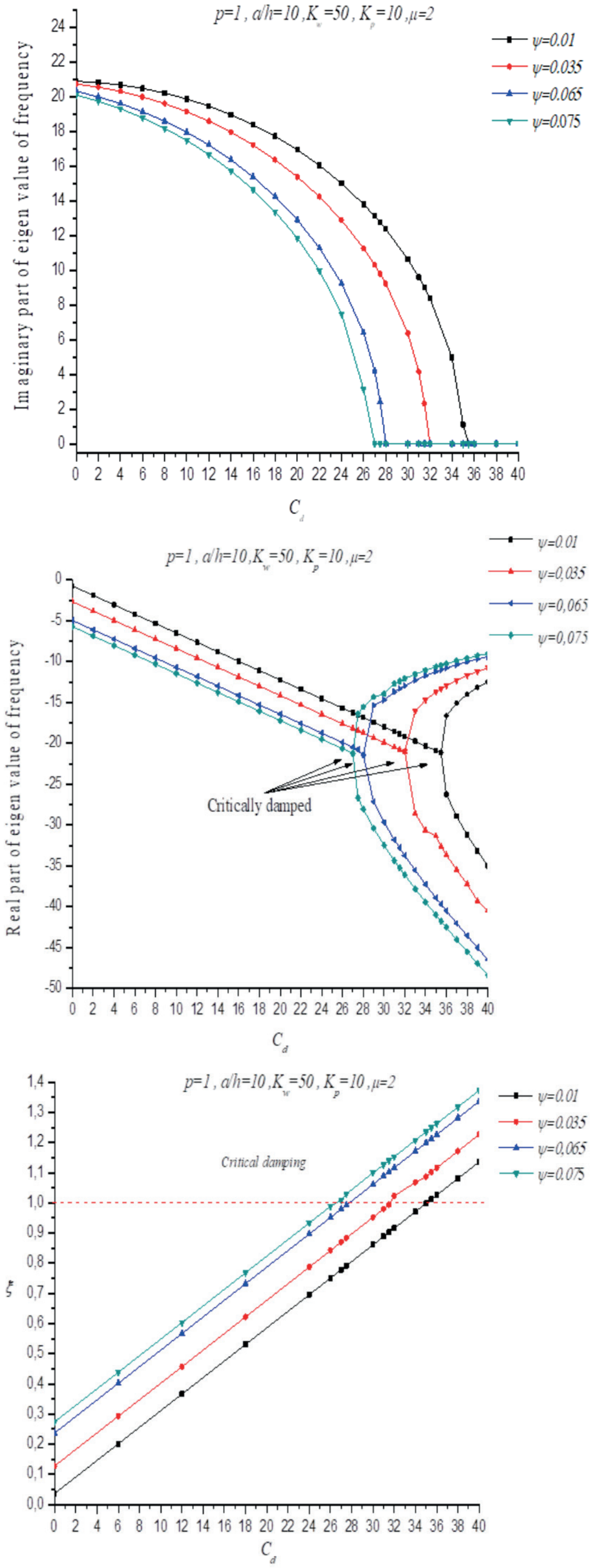

Fig. 11 Effect of the damping coefficient C_d of the foundation on the free vibration of simply supported FGM nanoplates *SSSS* for different values of the viscoelastic parameter Ψ of the material that constitutes the plate, (a) Imaginary part, (b) Real part and (c) Damping ratio

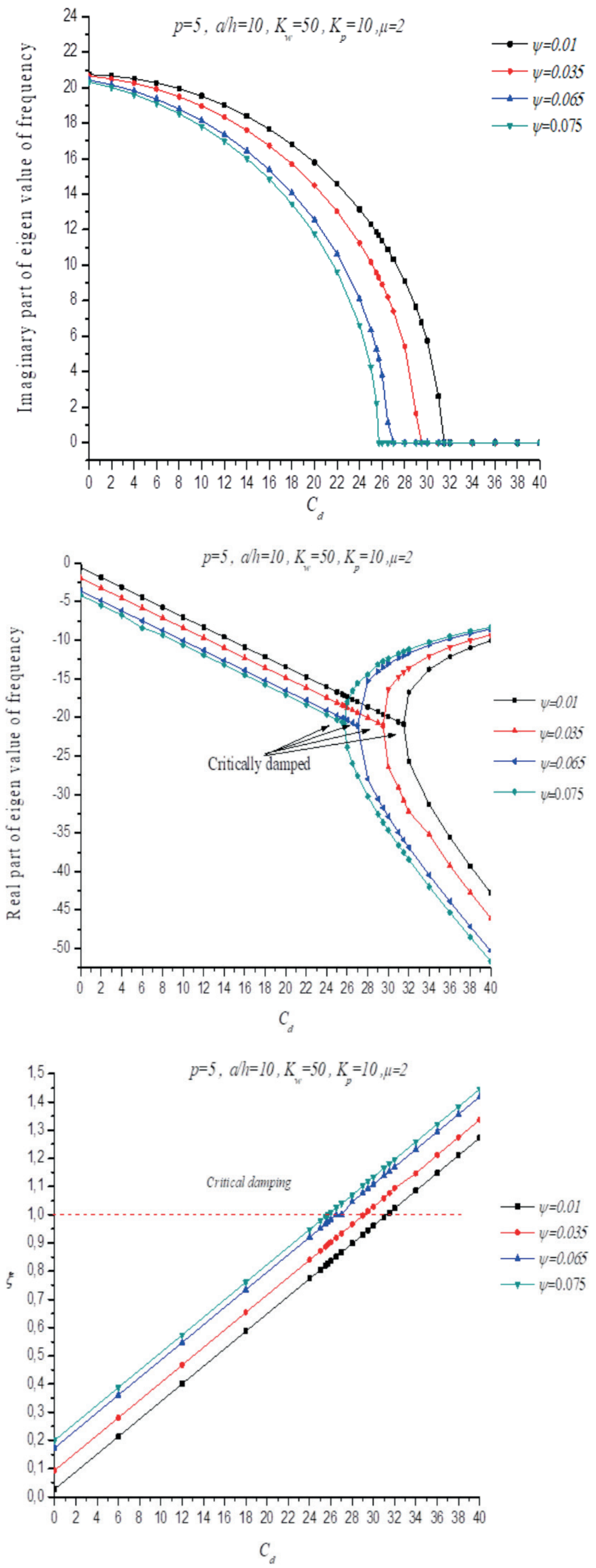

Fig. 12 Effect of the damping coefficient C_d of the foundation on the free vibration of the simply supported FGM nanoplates *SSSS*, for various values of the viscoelastic parameter Ψ; (a) Imaginary part, (b) Real part and (c) Damping ratio

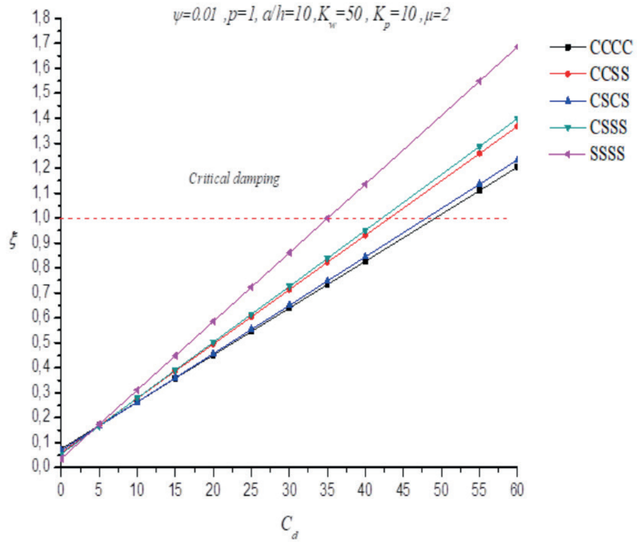

Fig. 13 Effect of the viscoelastic parameter C_d of the foundation on the free vibration of FGM nanoplates under different boundary conditions

8 Conclusions

The present study aims at investigating the free vibration of an FGM viscoelastic nanoplate, lying on a viscoelastic foundation, using the Kelvin-Voigt model. A new model, which takes into account the higher-order transverse deformation theory, was utilized to develop nonlocal equilibrium equations that are based on Hamilton's principle, under various boundary conditions. The results obtained are presented for the purpose of showing the impact of different parameters on the free vibration of the FGM nanoplate.

The new model was validated based on studies previously carried out by Aghababaei and Reddy et al. [25] and Sobhy [38]. The findings in this article turned out to be in good agreement with those reported by the aforementioned authors.

The dimensionless frequency increased when the side-to-thickness ratio (a/h) of the FG nanoplate went up. In addition, the effect of the transverse deformation was clearly noticed for thick nanoplates.

The vibration eigenfrequencies of simply supported FG nanoplates (*SSSS*) were smaller than those of the embedded ones (*CCCC*). It is interesting to note that for plates with intermediate boundary conditions (*CSSS*, *CCSS* and *CSCS*), the eigenfrequencies exhibited intermediate values.

The functionally graded (FG) nanoplates were better damped than the local FG plates ($\mu = 0$); this is certainly due to the fact that the nonlocal parameter reduces the structure stiffness significantly.

In the presence of an elastic foundation, the stiffness of the FG nanoplates rises, which leads to a significant increase in the vibration eigenfrequency.

The findings of the study indicate that the predominant real part of the vibration frequency of the FGM structures is significantly influenced by the viscoelastic parameter of the material and by the damping coefficient of the foundation as well. In addition, it was found that the real part of the frequency decreases as the structural and external damping values of the foundation go down.

The free vibration eigenfrequencies of FG nanoplates are generally lower than those of homogeneous ceramic nanoplates. Moreover, a decrease in the power law index value (p = 0, pure ceramic) leads to a reduction in the damping factor, which indicates a lower capacity to dissipate energy.

The damping effects in viscoelastic material will reduce the peak values of the dynamic response, and therefore can be applied to the design to reduce the system vibration.

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Appendix A

$$
\begin{aligned}
L_{11}=\int_0^a\int_0^b &-\frac{\partial X(x)}{\partial x}Y(y)\Bigg(-I_1\lambda^2\frac{\partial X(x)}{\partial x}Y(y)+I_1\mu\lambda^2\frac{\partial X(x)}{\partial x}\frac{\partial^2 Y(y)}{\partial y^2}\\
&+I_1\mu\lambda^2\frac{\partial^3 X(x)}{\partial x^3}Y(y)+A_{66}\frac{\partial X(x)}{\partial x}\frac{\partial^2 Y(y)}{\partial y^2}+A_{66}\eta\lambda\frac{\partial X(x)}{\partial x}\frac{\partial^2 Y(y)}{\partial y^2}\\
&+A_{11}\frac{\partial^3 X(x)}{\partial x^3}Y(y)+A_{11}\eta\lambda\frac{\partial^3 X(x)}{\partial x^3}Y(y)\Bigg)dxdy
\end{aligned}
$$

$$
L_{12}=\int_0^a\int_0^b -\left(\frac{\partial X(x)}{\partial x}\right)^2 Y(y)\frac{\partial^2 Y(y)}{\partial y^2}\left(A_{12}+A_{66}\right)\left(1+\eta\lambda\right)dxdy\frac{\partial X(x)}{\partial x}Y(y)
$$

$$
\begin{aligned}
L_{13}=\int_0^a\int_0^b &\Bigg(\Bigg(\Bigg(\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial X(x)}{\partial x}+\frac{\partial^3 X(x)}{\partial x^3}Y(y)\Bigg)\mu-\frac{\partial X(x)}{\partial x}Y(y)\Bigg)\mathrm{I}_2\lambda^2\\
&+\left(2\mathrm{B}_{66}+\mathrm{B}_{12}\right)\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial X(x)}{\partial x}+\mathrm{B}_{11}\frac{\partial^3 X(x)}{\partial x^3}Y(y)+\\
&\left(\left(2\mathrm{B}_{66}+\mathrm{B}_{12}\right)\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial X(x)}{\partial x}+\mathrm{B}_{11}\frac{\partial^3 X(x)}{\partial x^3}Y(y)\right)\eta\lambda\Bigg)dxdy
\end{aligned}
$$

$$
\begin{aligned}
L_{14}=\int_0^a\int_0^b &-\frac{\partial X(x)}{\partial x}Y(y)\Bigg(\Bigg(\Bigg(\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial X(x)}{\partial x}+\frac{\partial^3 X(x)}{\partial x^3}Y(y)\Bigg)\mu-\frac{\partial X(x)}{\partial x}Y(y)\Bigg)\mathrm{I}_4\lambda^2\\
&\left(2\mathrm{B}^{\mathrm{f}}_{66}+\mathrm{B}^{\mathrm{f}}_{12}\right)\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial X(x)}{\partial x}+\mathrm{B}^{\mathrm{f}}_{11}\frac{\partial^3 X(x)}{\partial x^3}Y(y)+\begin{pmatrix}\left(2\mathrm{B}^{\mathrm{f}}_{66}+\mathrm{B}^{\mathrm{f}}_{12}\right)\dfrac{\partial^2 Y(y)}{\partial y^2}\dfrac{\partial X(x)}{\partial x}\\ +\mathrm{B}^{\mathrm{f}}_{11}\dfrac{\partial^3 X(x)}{\partial x^3}Y(y)\end{pmatrix}\eta\lambda\Bigg)dxdy
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
L_{22}=\int_0^a\int_0^b &-X(x)\frac{\partial Y(y)}{\partial y}\Bigg(\Bigg(\Bigg(\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\Bigg)\mu-X(x)\frac{\partial Y(y)}{\partial y}\Bigg)\mathrm{I}_1\lambda^2\\
&+\mathrm{A}_{22}\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\mathrm{A}_{66}\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\\
&+\left(\mathrm{A}_{22}\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\mathrm{A}_{66}\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\right)\eta\lambda\Bigg)dxdy
\end{aligned}
$$

$$
\begin{aligned}
L_{23}=\int_0^a\int_0^b &X(x)\frac{\partial Y(y)}{\partial y}\Bigg(\Bigg(\Bigg(\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\Bigg)\mu-X(x)\frac{\partial Y(y)}{\partial y}\Bigg)\mathrm{I}_2\lambda^2\\
&+\left(\mathrm{B}_{12}+2\mathrm{B}_{66}\right)\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}+\mathrm{B}_{22}\frac{\partial^3 Y(y)}{\partial y^3}X(x)\\
&+\left(\mathrm{B}_{22}\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\left(\mathrm{B}_{12}+2\mathrm{B}_{66}\right)\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\right)\eta\lambda\Bigg)dxdy
\end{aligned}
$$

$$
\begin{aligned}
L_{24}=\int_0^a\int_0^b &-X(x)\frac{\partial Y(y)}{\partial y}\Bigg(\Bigg(\Bigg(\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\Bigg)\mu-X(x)\frac{\partial Y(y)}{\partial y}\Bigg)\mathrm{I}_4\lambda^2\\
&+\left(\mathrm{B}^{\mathrm{f}}_{12}+2\mathrm{B}^{\mathrm{f}}_{66}\right)\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}+\mathrm{B}^{\mathrm{f}}_{22}\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\\
&\left(\mathrm{B}^{\mathrm{f}}_{22}\frac{\partial^3 Y(y)}{\partial y^3}X(x)+\left(\mathrm{B}^{\mathrm{f}}_{12}+2\mathrm{B}^{\mathrm{f}}_{66}\right)\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial Y(y)}{\partial y}\right)\eta\lambda\Bigg)dxdy
\end{aligned}
$$

$$
\begin{aligned}
L_{33} = \int_0^a\int_0^b -X(x)Y(y)\Bigg(&\left(\left(\left(\left(\frac{\partial^2 Y(y)}{\partial y^2}X(x)+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)I_1 - \left(2\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial^2 Y(y)}{\partial y^2}+Y(y)\frac{\partial^4 X(x)}{\partial x^4}+X(x)\frac{\partial^4 Y(y)}{\partial y^4}\right)I_3\right)\mu - \right.\right.\right.\\
&X(x)Y(y)I_1+\left(X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)I_3\Bigg)\lambda^2+\left(\left(X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mu\text{-}X(x)Y(y)\right)\lambda \mathrm{c_d}+\\
&\left((\text{-}2\mathrm{D}_{12}\text{-}4\mathrm{D}_{66})\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial^2 X(x)}{\partial x^2}\text{-}\mathrm{D}_{11}Y(y)\frac{\partial^4 X(x)}{\partial x^4}\text{-}\mathrm{D}_{22}X(x)\frac{\partial^4 Y(y)}{\partial y^4}\right)\lambda\eta\text{-}\\
&\mathrm{D}_{11}Y(y)\frac{\partial^4 X(x)}{\partial x^4}\text{-}\mathrm{D}_{22}X(x)\frac{\partial^4 Y(y)}{\partial y^4}+\left(\left(-2\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial^2 Y(y)}{\partial y^2}-Y(y)\frac{\partial^4 X(x)}{\partial x^4}-X(x)\frac{\partial^4 Y(y)}{\partial y^4}\right)\mu +X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mathrm{k_p}+\\
&\left(\left(\frac{\partial^2 Y(y)}{\partial y^2}X(x)+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mu\text{-}X(x)Y(y)\right)\mathrm{k_w}+(\text{-}2\mathrm{D}_{12}\text{-}4\mathrm{D}_{66})\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial^2 X(x)}{\partial x^2}\Bigg)dxdy
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
L_{34} = \int_0^a\int_0^b -X(x)Y(y)\Bigg(&\left(\left(\left(\left(\frac{\partial^2 Y(y)}{\partial y^2}X(x)+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)I_1 - \left(2\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial^2 Y(y)}{\partial y^2}+Y(y)\frac{\partial^4 X(x)}{\partial x^4}+X(x)\frac{\partial^4 Y(y)}{\partial y^4}\right)I_5\right)\mu - \right.\right.\right.\\
&X(x)Y(y)I_1-\left(X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)I_5\Bigg)\lambda^2+\left(\left(X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mu\text{-}X(x)Y(y)\right)\lambda \mathrm{c_d}+\\
&\left(\left(2\mathrm{D}^{\mathrm{f}}_{12}+4\mathrm{D}^{\mathrm{f}}_{66}\right)\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial^2 X(x)}{\partial x^2}\text{-}\mathrm{D}^{\mathrm{f}}_{11}Y(y)\frac{\partial^4 X(x)}{\partial x^4}\text{-}\mathrm{D}^{\mathrm{f}}_{22}X(x)\frac{\partial^4 Y(y)}{\partial y^4}\right)\lambda\eta+\mathrm{D}^{\mathrm{f}}_{11}Y(y)\frac{\partial^4 X(x)}{\partial x^4}\\
&+\mathrm{D}^{\mathrm{f}}_{22}X(x)\frac{\partial^4 Y(y)}{\partial y^4}+\left(\left(-2\frac{\partial^2 X(x)}{\partial x^2}\frac{\partial^2 Y(y)}{\partial y^2}-Y(y)\frac{\partial^4 X(x)}{\partial x^4}-X(x)\frac{\partial^4 Y(y)}{\partial y^4}\right)\mu+ X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mathrm{k_p}+\\
&\left(\left(\frac{\partial^2 Y(y)}{\partial y^2}X(x)+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mu\text{-}X(x)Y(y)\right)\mathrm{k_w}+\left(2\mathrm{D}^{\mathrm{f}}_{12}+4\mathrm{D}^{\mathrm{f}}_{66}\right)\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial^2 X(x)}{\partial x^2}\Bigg)dxdy
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
L_{44} = \int_0^a\!\!\int_0^b -X(x)Y(y)\Bigg(\Bigg(\Bigg(&-\left(\left(\frac{\partial^2 Y(y)}{\partial y^2}X(x)+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)I_1+\left(\begin{array}{l}2\dfrac{\partial^2 X(x)}{\partial x^2}\dfrac{\partial^2 Y(y)}{\partial y^2}+\\ Y(y)\dfrac{\partial^4 X(x)}{\partial x^4}\\ +X(x)\dfrac{\partial^4 Y(y)}{\partial y^4}\end{array}\right)I_6\right)\mu+\\
&X(x)Y(y)I_1-\left(X(x)\frac{\partial^2 Y(y)}{\partial y^2}+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)I_6\Bigg)\lambda^2+\left(\begin{array}{l}-\left(\begin{array}{l}X(x)\dfrac{\partial^2 Y(y)}{\partial y^2}+\\ \dfrac{\partial^2 X(x)}{\partial x^2}Y(y)\end{array}\right)\mu\\ +X(x)Y(y)\end{array}\right)\lambda \mathrm{c_d}+\\
&\left(\begin{array}{l}\left(2\mathrm{F}_{12}^{\mathrm{f}}+4\mathrm{F}_{66}^{\mathrm{f}}\right)\dfrac{\partial^2 Y(y)}{\partial y^2}\dfrac{\partial^2 X(x)}{\partial x^2}+\mathrm{F}_{11}^{\mathrm{f}}Y(y)\dfrac{\partial^4 X(x)}{\partial x^4}+\mathrm{F}_{22}^{\mathrm{f}}X(x)\dfrac{\partial^4 Y(y)}{\partial y^4}-\\ A_{44}^{f}X(x)\dfrac{\partial^2 Y(y)}{\partial y^2}-A_{55}^{f}\dfrac{\partial^2 X(x)}{\partial x^2}Y(y)\end{array}\right)\lambda\eta+\\
&\mathrm{D}_{11}^{\mathrm{f}}Y(y)\frac{\partial^4 X(x)}{\partial x^4}+\mathrm{D}_{22}^{\mathrm{f}}X(x)\frac{\partial^4 Y(y)}{\partial y^4}+\mathrm{F}_{11}^{\mathrm{f}}Y(y)\frac{\partial^4 X(x)}{\partial x^4}+\mathrm{F}_{22}^{\mathrm{f}}X(x)\frac{\partial^4 Y(y)}{\partial y^4}-\\
&A_{44}^{f}X(x)\frac{\partial^2 Y(y)}{\partial y^2}-A_{55}^{f}\frac{\partial^2 X(x)}{\partial x^2}Y(y)\left(\begin{array}{l}\left(\begin{array}{l}2\dfrac{\partial^2 X(x)}{\partial x^2}\dfrac{\partial^2 Y(y)}{\partial y^2}+Y(y)\dfrac{\partial^4 X(x)}{\partial x^4}+\\ X(x)\dfrac{\partial^4 Y(y)}{\partial y^4}\end{array}\right)\mu\text{-}\\ X(x)\dfrac{\partial^2 Y(y)}{\partial y^2}\text{-}\dfrac{\partial^2 X(x)}{\partial x^2}Y(y)\end{array}\right)\mathrm{k_p}+\\
&\left(X(x)Y(y)-\left(\frac{\partial^2 Y(y)}{\partial y^2}X(x)+\frac{\partial^2 X(x)}{\partial x^2}Y(y)\right)\mu\right)\mathrm{k_w}+\left(2\mathrm{F}_{12}^{\mathrm{f}}+4\mathrm{F}_{66}^{\mathrm{f}}\right)\frac{\partial^2 Y(y)}{\partial y^2}\frac{\partial^2 X(x)}{\partial x^2}\Bigg)dxdy
\end{aligned}
\tag{42}
$$