Theoretical Relationship between the Confining Pressure and Poisson’s Ratio of Intact Rock

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Abstract
The Poisson's ratio of the intact rock material is one of the basic material constants for rock engineering calculations in and on rock environments. Recently, several relationships were developed to calculate the Poisson's ratio of the intact rock from different mechanical parameters (e.g., rigidity index, cohesion, internal friction angle). This value can be measured under standardized laboratory conditions using traditional uniaxial compressive test at zero confining pressure. Analyzing the laboratory tests carried out at different confining pressure, it was found that this material constant should be increasing as a function of confining pressure. This paper aims to present a theoretical relationship between Poisson's ratio of intact rock and the confining pressure, using the Hoek-Brown failure criteria. It was assumed that the Poisson's ratio is increasing linearly, and at the brittle-ductile transaction point, it is 0.5. The proposed relationship can be extended to rock mass, using the Geological Strength Index (GSI).

Keywords
rock mechanics, brittle-ductile transaction, Poisson's ratio, Hoek-Brown failure criteria

1 Introduction
Poisson's ratio (\(\nu\)) is defined as the negative of ratio of the radial strain (\(\varepsilon_r\)) and the corresponding axial strain (\(\varepsilon_a\)) caused by uniformly distributed axial stress. According to the definition, the Poisson's ratio of isotropic, linear elastic material is between -1 (lower limit) and + 0.5 (upper limit) because of the requirement for Young's modulus, the shear modulus and bulk modulus to have positive values. The vast majority of intact rock materials is between 0.1–0.4 [1], and using cylindrical rock samples can be calculated from the measured axial and lateral strains using traditional uniaxial compressive tests [2]. The Poisson's ratio depends on several mechanical/pertophysical parameters, i.e., porosity, water content, confining pressure, among the others.

Although the Poisson's ratio is an important mechanical parameter for rock engineering design, it is less investigated because it is difficult to measure accurately [1]. They summarized and published the typical ranges of values for Poisson's ratio of the most important rock types – see Fig. 1.

Unfortunately, in many cases it is not possible to determine the Poisson's ratio of the intact rock. It can be assumed that the Poisson's ratio depends on the mechanical...
behavior, namely the rigidity of the intact rock – increasing the brittleness of the rock material the Poisson's ratio should be decreasing. These equations are based on Mohr-Coulomb theory.

Zhang et al. [3] summarized the most important relationships between the different Mohr-Coulomb failure parameters (such as internal friction angle ($\phi$), cohesion ($c$)) and the Poisson's ratio ($\nu$) of the intact solid material.

One can find the equations describing the relationship between the Poisson's ratio and the internal friction angle without cohesion:

$$\nu = \frac{1}{2} \left(1 - \sin \phi \right), \quad (1)$$

$$\nu = \frac{\cos^2 \phi}{1 + \cos^2 \phi}, \quad (2)$$

$$\nu = \frac{\arctan \left[ \cos \phi - (1 - \sin \phi) \tan \phi \right]}{90^\circ}, \quad (3)$$

$$\nu = \frac{1 - \sin \phi}{2 + \sin \phi}, \quad (4)$$

$$\nu = \frac{\tan \left(\frac{45^\circ - \phi}{2}\right)}{1 + \tan \left(\frac{45^\circ - \phi}{2}\right)}, \quad (5)$$

• The Poisson's ratio can be calculated by the use of other mechanical parameters, such as cohesion ($c$) and uniaxial compressive strength ($\sigma_c$):

$$\nu = \frac{c}{\sigma_c}, \quad (6)$$

$$\nu = \frac{2c}{\sigma_c + 2c}. \quad (7)$$

• In rock mechanics the uniaxial compressive strength ($\sigma_c$) and the tensile strength ($\sigma_t$) play fundamental role. Their proportion ($R$) is also used as a key parameter and called rigidity, where $R = \sigma_c/|\sigma_t|$. The Poisson's ratio can also be calculated as a function of compressive and tensile strength, i.e., rigidity is the key parameter. In [3–5] one can find the most commonly used relations that are listed below:

$$\nu = \frac{1}{R + 1}, \quad (8)$$

$$\nu = \frac{4R}{R^2 + 6R + 1}. \quad (9)$$

$$\nu = \arcsin \left[ \frac{1}{(R + 1) \sqrt{\frac{R + 1}{R} - \frac{R}{(R + 1)^2}}} \right], \quad (10)$$

$$\nu = \frac{2}{3R + 1}, \quad (11)$$

$$\nu = \frac{1}{\sqrt{R + 1}}, \quad (12)$$

$$\nu = \frac{1}{2\sqrt{R}}. \quad (13)$$

Lógó and Vásárhelyi [4, 5] investigated the formulation in Eqs. (1)–(13), and validated by parametric experiences. Their computed values for the Poisson's ratio obtained by the use of Eqs. (1)–(13) were compared with the available internationally published data sets. They concluded:

• the Poisson's ratio highly depends on the ratio of uniaxial compressive strength and tensile strength (i.e., brittleness) of the intact rock,
• increasing the brittleness of the rock, the Poisson's ratio decreasing.

The usual range of the ordinary Poisson's ratios of different intact rock types can be seen in Fig. 1, as it was indicated earlier. We elaborated parametric studies [4] to investigate the applicability of the Eqs. (8)–(13). This study is based on a simple data analysis operation where the upper and the lowers bounds of the available ranges of the Poisson's ratios are plotted from the papers by Gercek [1] and AASHTO. Among these equations Eq. (12) was the best fitted one – however one can see the difference between the measured/published and the theoretically calculated values. This difference is significant even with the best approximation (see Fig. 2).
Accepting that the rigidity of the intact rock \( (R = \sigma_c/|\sigma_t|) \) is equal to the Hoek-Brown material constant \( (m_i) \), thus \( R \approx m_i \), \([6]\), one can use the following form \([5]\):

\[
v = \frac{1}{\sqrt{m_i} + 1}.
\]  

(14)

Hereafter this form will be used.

In \([7]\) Carniero and Puga examined various materials by dynamic mechanical analysis. This method is able to monitor the instant values of load and displacement to determine the instant specimen stiffness and the Poisson’s ratio and complex modulus. Carried out several tests they found: increasing the temperature, the Poisson’s ratio is also increases (see Fig. 3).

As shown in Fig. 4, if the Poisson's ratio is considered constant, the behavior of the model does not correspond to the actual measurement results. In Fig. 5 one can see the variation of the Poisson's ratio and the corresponding stresses and strains in case of tensile strength test.

The structure of the rest of this paper is as follow: Section 2 contains an overview of selected papers dealing with the influence of the confining pressure on the Poisson’s ratio. Section 3 contains the main goal of this paper, namely a recommendation is suggested to give a theoretical relationship between the confining pressure and the Poisson’s ratio of intact rock material. In the last section (Section 4) one can find the conclusions and applicability of the results.

2 An overview on the influence of the confining pressure on the Poisson’s ratio

The above-presented equations were developed for zero confining pressure; however, the environmental stress should have influence on this value. It is well known that increasing the confining pressure both the ultrasound velocities (both \( v_p \) and \( v_s \)) and the deformation modulus increase. It can be assumed that the Poisson's ratio of the intact rock increases, as well.

Xu et al. \([8]\) investigated the influence of the increasing lateral stress \( (\sigma_l) \) for the Poisson’s ratio of the intact rock (they investigated cryptocrystalline amygdaloidal basalt samples). According to their results, the Poisson's ratio \( (v) \) linearly increases with increasing lateral stress \( (\sigma_l) \), and the lateral stress loading on the rock samples has some softening effects (see Fig. 6).

Dong et al. \([9]\) investigate experimentally the variation of the Poisson’s ration for intact rocks and its variation as deformation develops. They conclude, by analyzing the deformation processes of a wide variety of rocks under uniaxial compression, that the Poisson's ratio and Young's modulus behave quite differently: there is no plateau in Poisson's ratio because the secant and average Poisson's ratios both increase monotonously with increasing stress.
This behavior could be related to the irreversibility of compaction in the direction of compression and micro-crack propagation in the principal stress direction.

One can see the variation of the secant Poisson's ratios of several marble specimens in Fig. 6(e) [9]. The evolution of the secant Poisson's ratio can be divided into two parts. The first part is linear and here the increment of stress is 20–30% of the peak strength. The values are in the normal range of values for Poisson's ratio [1]. When the stress increases to 70–80% of the peak strength, the secant Poisson's ratio grows in a rapid and nonlinear manner, and it exceeds the normal range of values for Poisson's ratio [1] prior to failure. The transitions into the nonlinear rapid growth stage roughly corresponds to the failure stress. A similar behavior can be seen in Fig. 6(f) [5] where the authors of this paper presented as the results of series of parametric experiments that the value of the Poisson's ratio is not only dependent on the magnitude of the environmental pressure but is significantly influenced by the fragmentation of the rock. This numerical experiment is based on Eq. (3). One can see quasi–linear relations between the Poisson’ ratio and the confining pressure.

Fig. 6 According to the measured data, the influence of the lateral stress on the Poisson's ratio of intact rock with given s1 [8]: a) Relationship among \( \sigma_3 \), \( \sigma_3 \), and \( \varepsilon_1 \); b) Relationship among \( \sigma_1 \) and \( \sigma_3 \); c) Relationship among \( \sigma_3 \) and \( \varepsilon_1 \); d) Relationship among \( \nu \) and \( \sigma_3 \); e) Variation of the secant Poisson's ratios of several marble specimens (according to Dong et al. [9]); f) Poisson's ratio in the function of GSI and confining pressure [5]
Theoretical considerations

It is well known, that by increasing the confining pressure, the rigid rock material becomes plastic. First, Kármán observed this mechanical behavior of the rigid rock samples [10–12], namely Carrara marble and Mutenberg sandstone. They documented the deformation of the samples, both at zero and high confining pressure (see Fig. 7) [13–14]. As it is known, the plastic behavior can be described by yield functions. In these functions the yield of the structural element depends on the deviatoric part of the stress tensor. In the following it is assumed, that $\sigma_2 = \sigma_3$ and $\sigma_1$ and $\sigma_3$ are used in our investigation.

In this paper, the transition point from brittle to ductile failure is calculated using $\sigma_{tr}$ (transitional stress) as referred to Mogi’s widely used brittle-ductile transition limit [15]:

$$\sigma_1 - \sigma_3 = 3.4 \cdot \sigma_3 .$$

Here $\sigma_1$ is the axial stress. Hence, the axial stress can be considered as:

$$\sigma_1 = 4.4 \cdot \sigma_3 .$$

The Hoek–Brown failure criterion is widely used in rock mechanics and rock engineering practice. For determining the transition point of the intact rock the Hoek-Brown failure criterion can be applied. This semi-empirical failure criterion was introduced by Hoek and Brown [16] and the following form was suggested for intact rock (see also [17]):

$$\sigma_1 = \sigma_3 + \sigma_c \left( m_i \frac{\sigma_3}{\sigma_c} + 1 \right)^{0.5} ,$$

where $\sigma_1$ and $\sigma_3$ are major and minor principal stresses at failure, respectively, $m_i$ is the Hoek-Brown material constant and $\sigma_c$ is the uniaxial compressive strength of intact rock. In case of the experiments above the major and minor principal stresses are equivalent to the axial and lateral stresses, respectively. In the following axial and lateral stresses are used instead of the expression of major and minor principal stresses.

According to Eq. (17), two independent parameters are necessary, namely:

- $\sigma_c$: Uniaxial compressive strength of intact rock,
- $m_i$: Hoek–Brown material constant of intact rock.

Note, the Hoek-Brown material constant ($m_i$) is equal to the ratio of the uniaxial compressive strength ($\sigma_c$) and the tensile strength ($\sigma_t$) of the intact rock [18]. This ratio can be used as the rigidity of the rock ($R$).

It should be noted that the Hoek-Brown criterion is proposed to deal with shear failure in rocks. Therefore, the Hoek-Brown criterion is only applicable for confining stresses within the range defined by $\sigma_3 = 0$ and the transition from shear to ductile failure, as shown in Fig. 2. Research by [19, 20] indicated that the range of confining stress $\sigma_3$ can have a significant influence on the calculation of $m_i$.

Also, triaxial test data of Indiana limestone by Schwartz [21] in Fig. 8. shows that the applicability of the Hoek-Brown criterion is determined by the transition from shear to ductile failure at approximately $\sigma_1 = 4.0 \sigma_3$ [22].

Substituting Eq. (16) into Eq. (17) one can get the following equations:

$$4.4\sigma_3 = \sigma_3 + \sigma_c \left( m_i \frac{\sigma_3}{\sigma_c} + 1 \right)^{0.5} .$$

![Fig. 7 Marble samples: from unconfined (left) to increasing confining pressure (to right) (according to Kármán [13–14])](image)

![Fig. 8 Limit of applicability of the Hoek-Brown criterion [22]](image)
Making the necessary calculation, the final form is:

\[ 3.4\sigma_3 = \sigma_c \left( m + \frac{\sigma_3}{\sigma_c} + 1 \right)^{0.5}. \]  

(19)

It means that \( \sigma_3 \) can be derived from the following equation:

\[ 3.4^2 \sigma_3^2 - m_2 \sigma_3 \sigma_c - \sigma_c^2 = 0. \]

(20)

Without taking into account the negative value of \( \sigma_3 \), the transitional stress \( \sigma_3 = \sigma_{tr} \) point can be calculated from Eq. (20):

\[ \sigma_{tr} = \sigma_c \sqrt{\frac{m_2}{23.12}}. \]

(21)

where, \( \sigma_c \) and \( \sigma_{tr} \) are the uniaxial compressive strength and brittle-ductile transient stress, respectively, \( m_2 \) is the Hoek-Brown material constant. This equation was analyzed for a different types of rock samples by Davarpanah et al. [23].

The goal of this research is to suggest a theoretical relationship between the confining pressure and the Poisson's ratio \( (\nu) \) of intact rock material. The following assumptions were used for determining the influence of the confining pressure to Poisson’s ratio:

1. The intact rock is the linearly elastic homogeneous material.
2. In case of zero lateral stress \( (\sigma_y = 0) \), the Poisson's ratio of the intact rock is constant: \( (\nu = \nu_{im} = \text{const.}) \)
3. According to the definition, the maximum value of the Poisson's ratio is 0.5 \( (\nu = \nu_{im} = 0.5) \)– reaching this value, the rock has plastic behavior.
4. There is a linear relationship between the confining pressure and the Poisson's ratio (accepting the results of Xu et al. [8]) and the results of the parametric study of Lógó and Vásárhelyi [5]).

According to the theoretical assumptions the Poisson's ratio of the linearly elastic intact rock is increasing linearly from \( \nu = \nu_{im} \) (intact rock without confining pressure) to \( \nu = \nu_{tr} = 0.5 \) (confining pressure reaches the brittle-ductile transient stress, \( \sigma_{tr} \) – see Fig. 9.

If the relationship between the Poisson's ratio and the axial stress is assumed to be linear, the relationship can be obtained using the equation of a line between the points P and Q, where the coordinates are: \( P(\sigma_0, \nu_1) \) and \( Q(\sigma_{tr}, 0.5) \).

Writing the equation of the line crossing on these points, one can get the following relation:

\[ \nu(\sigma) = \sigma \times \frac{0.5 - \nu_1}{\sigma_{tr}} + \nu_1. \]

(22)

Based on Eq. (22), the steepness of the line can be determined:

\[ \tan \alpha = \frac{0.5 - \nu_1}{\sigma_{tr}}. \]

Thus, the slope of the line is:

\[ \tan \alpha = \frac{1}{\sigma_c} \frac{11.56 - 23.12 \nu_1}{m_2 + \sqrt{m_2^2 + 46.24}}. \]

It means the Poisson's ratio in the function of confining pressure \( (\nu_{(c)}) \) can be calculated using the following equation:

\[ \nu_{(c)} = \sigma \times \frac{1}{\sigma_c} \frac{11.56 - 23.12 \nu_1}{m_2 + \sqrt{m_2^2 + 46.24}} + \nu_1. \]

In Eq. (25) one can see the closed form of the linearly approximated variation of the Poisson's ratio \( \nu_{(c)} \) in function of the uniaxial compressive stress \( \sigma \).

4 Conclusions

The goal of this theoretical research is to determine a simple relationship between the confining stress and the Poisson's ratio value of intact rock. This simple linear relation can be created easily and does not need any other information than the Poisson's ratio of the intact rock without confining pressure \( (\nu = \nu_1) \) and the transient stress value.

It has to be mentioned, that the Poisson's ratio can be differently handled. As it was indicated in Section 2, it can be more accurately described as an elastic deformation parameter that monotonously increases with stress during the compressive processes. According to the experimental results of Dong et al. [9], different rocks exhibit different behaviors in their Poisson's ratios. In hard rocks, the secant and average Poisson's ratios grow approximately linearly with stress in the main stages of the loading process. However, in soft rocks, these ratios quickly increase...
beyond the theoretical maximum of 0.5 in the initial loading stage. In that case a bilinear expression can be created, but the determination of the intermediate point is critical in that approximation. The intermediate point is the case where the axial stress reaches 20–30% of the transition stress. If this intermediate point is known a Lagrange or a Hermitian approximation can be applied, as well. In these last cases non-linear approximations are used instead of linear ones.

The influence of the confining pressure to the Poisson's ratio value, using Eq. (25), is plotted in Fig. 10: the uniaxial compressive strength ($\sigma_c$) = 100 MPa and the Hoek-Brown constant ($m_i$) = 10 in this example. The Poisson's ratios were plotted in the function of confining pressure for $v_i = 0.1, 0.2, 0.3$ and 0.4 values of the intact rock at zero confining pressure.

It should be emphasized that this relationship applies to intact rock. In rock engineering practice it is important to know the mechanical parameters of the rock mass (Poisson's ratio, as well). The Poisson's ratio value depends on the rock mass quality. Increasing the quality of the rock mass, the Poisson's ratio is decreasing, according to [24, 25]. The Poisson's ratio also depends on the water content [26]. It should be also used for determining the different elastic moduli for rock engineering design – see Davarpanah et al. [27].

Using the above presented theory it is possible to calculate of the Poisson's ratio for rock mass in different ambient stress conditions. According to the suggestion of Vásárhelyi [24], the Poisson's ratio of the rock mass ($v_{rm}$) can be calculated from the Geological Strength Index (GSI) [22]:

$$v_{rm} = -0.002\text{GSI} + v + 0.2.$$  \hspace{1cm} (26)

It is assumed that Eq. (25) and Eq. (26) can be used together in case of increasing confining pressure around the rock mass.

Lógó and Vásárhelyi [4] suggested a calculation method of the Poisson’s ratio from the Hoek-Brown material constant ($m_i$) of the intact rock. They compared the different theories with published results and received that the Poisson's ratio is decreasing in case of increasing rigidity (i.e., $m_i$ value [18]). The following equation was suggested for calculating the Poisson's ratio ($v_i$) of intact rock using the Hoek-Brown material constant ($m_i$):

$$v_i = \frac{1}{\sqrt{m_i + 1}}.$$  \hspace{1cm} (27)

Using the above presented theory it is possible to calculate of the Poisson's ratio for rock mass in different ambient stress conditions.

Using parametric analysis, Lógó and Vásárhelyi [5] plotted the Poisson's ratio in function of both Geological Strength Index (GSI) and confining pressure ($v_3$) (Fig. 6(f)).

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