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Probabilistic Modelling of Bending Strength of Timber Beams with the Help of Weak Zones Model

Ireneusz Czmoch

[1] Faculty of Civil Engineering, Warsaw University of Technology, al. Armii Ludowej 16, Warsaw, Poland
[*] Corresponding author, e-mail: i.czmoch@il.pw.edu.pl

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Abstract

The variability of longitudinal bending strength of timber beams due to the presence of knots and other defects is analyzed. The weak zones model of timber beam bending strength used in the analysis consists of short weak zones (knots or group of them) and strong sections of clear wood. The load bearing capacity of timber beams is defined as an extreme (minimum) value problem or as a first downcrossing problem. Assuming a marked Poisson random field with zero correlation between all random variables, cumulative probability distribution functions of the load carrying capacity of timber beams were determined by analytical methods for typical load cases: pure bending, midspan point load and one-third point load. These results, as well as the marked Poisson random field as a model of longitudinal variability of the bending strength of timber beams, can be applied in the reliability analysis of timber structures. Furthermore, the analytical formulae for the cumulative distribution functions of load carrying capacity of timber beams can provide a good reference for numerical analysis conducted with Monte Carlo simulation to determined statistics for specific timber members.

Keywords

structural timber, knots, weak zones model, bending strength, probabilistic modelling

1 Introduction

Structural timber, which is a building material of natural origin, is characterized by many positive features: high strength relative to weight, ease of processing with machines and tools, nice appearance of timber structures. All trees have branches that are necessary to form a crown with a leaf system where solar energy can be used to produce cellulose and other organic matter. However, broken branches and intertwining of branches with the trunk results in dead and live knots. Because of the specific structure of the trunk and branches, structural timber can be characterized as a composite of clear wood and growth defects. Clear wood is anisotropic material and its properties does not change significantly along the grain. On the other hand, growth defects such as knots, cross grains, wavy grains and interlocked knots are the main sources of longitudinal variation in bending strength of timber beams. Clear wood usually fails in the compression zone due to grain crushing, while a piece of structural timber containing defects (knots) often fails in the tension zone near one of the defects where the grains are distorted and stresses perpendicular to the grain can initiate crack growth. Growth defects are more or less randomly distributed along structural timber. Therefore, the bending strength of structural timber depends on the loading configuration and the length effect what is an important issue in codes of practice dealing with the determination of characteristic values for structural timber strength.

In design practice, a homogeneous model of timber beam has been used for many years, together with a specified test procedure for determining the characteristic value of bending strength of structural timber: "the defect that determines the class should be located in the shear-free zone formed by 1/3 point loads; the tensile edge should be randomly selected". This conservative approach cannot lead to cost-effective designs. Developments in reliability methods over the past forty years enable more accurate modelling and safety analysis of timber structures. However, the application of probabilistic analysis requires statistical data on the variability of structural timber properties. Statistical data on bending strength variability are very sparse.

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The development of more accurate computational methods for glulam construction has stimulated experimental studies aimed at obtaining statistical data on the longitudinal distribution of knots and other defects as well as the longitudinal variation in bending or tensile strength.

Colling and Dinort [1] studied the incidence of knots in softwood used in glulam using the Knot-Area-Ratio (KAR), which depends on the predicted area of knots (within a segment equal to the beam height) relative to a reference area that is equal to the total cross-sectional area (in tension) or half of the cross-sectional area (in bending). KAR is used in visual grading as a measure of strength reduction due to knots. Using 456 boards taken from three different growing areas (Germany, Austria, Nordic countries), the average distance of 0.5 meters between major knots or groups of knots was determined.

Riberholt and Madsen [2] fitted an exponential distribution to experimental data on the distance between knots or groups of knots in Danish and Swedish spruce timber and found that the mean distance depended on the species and was 0.3–0.5 meters. Lam and Varoglu [3] investigated the variation of tensile strength within a grain-parallel element in spruce-pine-fir timber. The study showed a significant strength correlation between cross sections 0.5–1.0 meters apart. The strength correlation decreases to zero for distances greater than 1.8 meters.

According to [4], the probability distribution of bending strength of structural timber is dependent on the tensile and compressive strengths. Therefore, the coefficient of variation for bending strength is usually higher than those coefficients for tensile and compressive strengths. Isaksson [5] investigated the variation of timber bending strength within a member on the basis of experiments for 133 boards of Norway spruce with 4–7 weak zones within a 5 m long board. Weibull and Normal models were fitted to the experimental data for the bending strength of the weak zones. The high correlation between bending strength and localized modulus of elasticity is the basis of machine stress classification procedures. On the other hand, the correlation between strengths determined for sections 0.5 meters apart is close to zero. Baño et al. [6] used a finite element model to simulate structural timber beams with defects and predict their maximum bending load. Assuming the elastoplastic constitutive law of wood, the prediction of the failure load gives information about the failure mechanisms of the timber, in particular with respect to the influence of knots and their local deviation from the grain. Pereira and Machado [7] investigated a probabilistic method to evaluate the bending strength of maritime pine beams in a probabilistic framework based on Monte Carlo simulations, in which the reference properties of the beams were randomly assigned based on their probability distributions, considering that weak zones are associated with reduced bending strength that is caused by the presence of knots or grain deviation. Köhler and Faber [8] presented a probabilistic model code for timber that accounts for the influence of a hierarchical model of spatial variation along with cross-sectional properties by taking a timber beam as a longitudinal sequence of weak sections.

To summarize a brief review of research on the dependence of bending strength on knots and other defects, there is a systematic shift away from the homogeneous timber beam model to models that take into account a well-known characteristic of timber that has a major impact on the safety of timber structures: structural timber consists of longer sections of clear wood and short sections associated with defects. Since the bending strength of a cross section with a knot is usually much lower than that of clear wood, failure of a timber beam usually occurs or is initiated near a knot or group of knots. This paper presents the results of analyses performed for the weak zones model proposed by Riberholt and Madsen [2]. The reliability analysis, based on the weak zones model, presented in [9] confirms the advantages of using the weak zones model instead of the homogeneous timber beam model. This paper presents analytically derived cumulative probability distribution functions for the load carrying capacity of a timber beam subjected to standard load cases: pure bending and 1/3 point loading.

2 Weak zones model of the bending strength variation in structural timber

The bending strength of a cross-section with a knot depends on many factors, such as:

- knot size and shape
- knot position within the cross-section (in tensile or in compression zone)
- inclination of grains in the vicinity of a knot
- intersection of grains around a knot by the beam surface (especially in the tensile zone)

This paper presents application of a simple mechanical model of a timber beam under bending, as shown in Fig. 1, which is called the weak zones model of a timber beam and is based on the following assumptions:

- timber beam is modelled as a composition of short weak zones connected by longer sections of clear wood

- weak zones correspond to knots or group of knots and are randomly distributed
- failure can only occur within weak zones
- bending strength (modulus of rupture, MOR) of a weak zone is a random variable

The distribution of bending strength should rather be described with the help of a continuous function, which allows smooth transition between strong clear wood and weak zones. However, the inhomogeneous random field model seems to be the most general model which could be employed in the description of the lengthwise variability in bending strength. The weak zones model reflects in a simplified way a very important feature of structural timber, i.e., the variability of the bending strength depends on knots or group of knots.

Two versions of this model can be considered

- both the distance between weak zones as well as the length of weak zones are random variables; the random strength of a weak zone depends on the random variable and some deterministic weight function, which describes the variation of strength within a weak zone (e.g., rectangular, parabolic or triangular distribution).

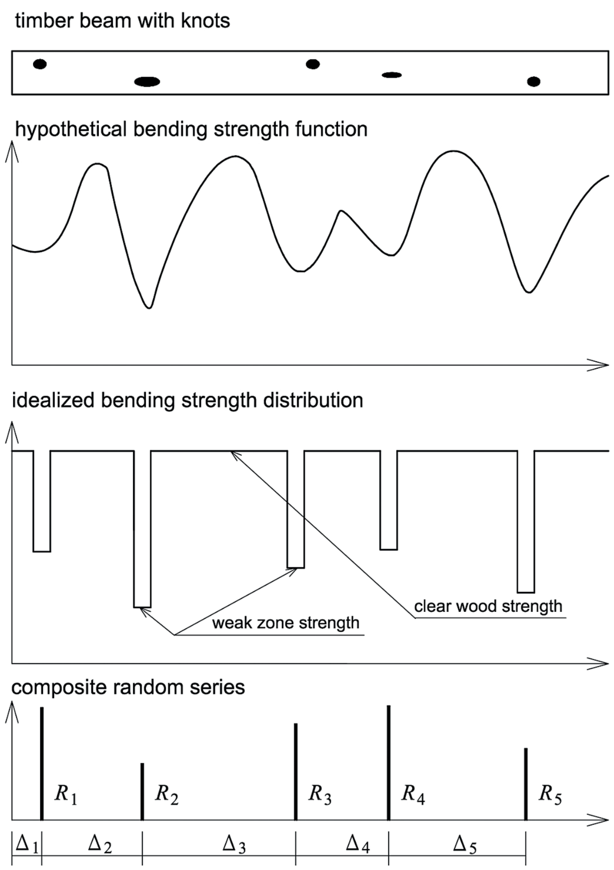

Fig. 1 Modelling the lengthwise variation bending strength in timber

- the length of a weak zone is equal to zero, thus a weak zones becomes a weak cross-section.

This paper deals with the application of the second model mostly, since both analytical solutions as well as parametric studies employing simulation could be carried out for the 'spike' version of the weak zones model. The first model is applicable only in specific cases, e.g., the analysis of a simply supported beam under pure bending. It can be shown that under certain conditions the probability distribution function of the load carrying capacity is identical for the two versions of the weak zones model.

3 Two probabilistic models of variability of strength of timber beams

The weak cross-section model or weak zones model is composed of two independent random point series:

- distances between consecutive weak zones: $\{\Delta_i\}$
- strengths of weak zones: $\{R_i\}$

In this paper both random point series are treated as homogeneous series, i.e., the statistics of random series are invariant to the translation of the origin, in particular, the nth order probability density function has the property for any distance dx.

$$f_{s\ldots s}\left(s_1\left(x_1\right),\ldots,s_n\left(x_n\right)\right) \\ = f_{s\ldots s}\left(s_1\left(x_1+dx\right),\ldots,s_n\left(x_n+dx\right)\right) \tag{1}$$

Another concept used in bending strength modelling is the filtered random field, which is defined by the linear transformation

$$Z\left(x\right)\rightarrow h\left(x\right)\rightarrow Y\left(x\right), \tag{2}$$

where $h(x)$ is a homogeneous impulse response function. In particular $h(x)$ can be a deterministic response function.

Furthermore, when the impulse response function is restricted to a point pulse at a particular position x_o, then

$$h\left(x\right)=\delta\left(x-x_o\right), \tag{3}$$

where $\delta(x)$ is the Dirac's delta function and the filtered random field becomes "the marked random field", see [10].

3.1 Bending strength modelled by the filtered and marked Poisson random fields

Assuming the distances between consecutive weak zones: $\{\Delta_i\}$ as homogeneous Poisson random field what means that the random number $N(x)$ of weak zones within interval $\langle 0, x\rangle$ follows the Poisson distribution, the bending strength can be modelled by the filtered Poisson random field.

$$R(x)=\sum_{i=1}^{N(x)} w(x,x_i,R_i)\,, \quad (4)$$

where $\{x_i\}$ - the random positions of weak zones generated by the Poisson random field; $w(x, x_i, R_i)$ - a deterministic response function with different shapes: e.g., rectangular, triangular or parabolic; $\{R_i\}$ - independent random variables assigned to weak zones.

The random variable R_i denotes a reference strength within a weak zone, usually the minimum value and can have any type probability distribution function (*pdf*), $F_d(r)$, but usually is assumed of the same type for all weak zones. In general the series $\{R_i\}$ can be inhomogeneous, i.e., it may depend on the position of weak zones.

If the weak zone lengths are considered to be much smaller than the distances between weak zones, then the marked Poisson random field is used to model the bending strength of structural timber.

$$R(x)=\sum_{i=1}^{N(x)} R_i\,\delta(x-x_i) \quad (5)$$

In case of the filtered and marked Poisson random field the important parameter is the intensity of weak zones (weak cross-section) per unit length, $v(x)$, which can be a function of the space coordinate for a non-homogeneous Poisson random field. The application of the homogenous Poisson random fields defined by Eqs. (1), and (2) makes analytical solutions possible.

3.2 Bending strengths and weak zone intervals as a translation random series

When both random point series $\{R_i\}$, $\{\Delta_i\}$ are assumed as translation random series, then the mean value, the correlation function and the first-order probability distribution functions F_d, F_Δ are assumed to be known. This model is more general than the Poisson model. Translation random series allow any type of distribution functions F_d, F_Δ and correlation between pairs of random variables R_i, R_j and pairs of random variables Δ_i, Δ_j. In general, both series can be inhomogeneous. Although this model allows a more general description of the variation of bending strength, Monte Carlo simulations are required to solve practical problems.

4 Load carrying capacity of timber beams as the extreme value problem and the first downcrossing problem

The load carrying capacity generally denotes the minimum external load which causes the failure of a structure. Assuming that timber is a brittle material, the failure of a timber beam occurs when the bending strength (the modulus of rupture, MOR) is attained at any cross-section. The minimum stress at which the i^{th} weak zone fails is equal to

$$Z_i=\frac{R_i}{m(x_i)}\,, \quad (6)$$

where R_i, x_i the bending strength and the position of the i^{th} weak zone and $m(x) = M(x)/M_{max}$ is the normalized bending moment function. Since both R_i, x_i are random variables, the stress Z_i is a random variable, and for a beam of length L the number of weak zones is a random variable, $N_L = N(L)$. The probability distribution function of the load carrying capacity depends on random variables $\{Z_1,\ldots, Z_N\}$ at all weak zones within the beam span L and can be determined by solving one of two equivalent problems, described in the following.

4.1 The extreme-value problem

The probability distribution function of the load carrying capacity Z is expressed as the extreme-value distribution.

$$F_Z(z)=1-P[Z>z]=1-P\left[\min\left(Z_1,\ldots,Z_{N_L}\right)>z\right] \quad (7)$$

In general the random variables Z_i Eq. (6) are correlated. However, when the function $m(x)$ is constant and random variables R_i are uncorrelated than the random variables Z_i are also uncorrelated. The extreme-value definition can be applied together with the Monte Carlo simulation and the transition series to study the influence of correlation in the point series $\{R_i\}$, $\{\Delta_i\}$ on the statistics of the load carrying capacity Z of timber beams.

4.2 The first downcrossing problem

Failure occurs when at least one strength variable R_i down-crosses the maximum bending stress function, i.e., $S = Z\, m(x)$. The load carrying capacity is greater than z, if the first downcrossing occurs at a distance longer than the beam span L.

Thus, the cumulative probability distribution function of Z is derived with the help of the probability of the survival.

$$1-F_Z(z)=P\left[R_1>zm(x_1)\cap\ldots\cap R_{N_L}>zm\left(x_{N_L}\right)\right] \quad (8)$$

The first downcrossing definition has been employed together with the Poisson random field to derive the analytical expressions for the distribution functions of the load carrying capacity of simply supported timber beams.

4.3 CDF of load carrying capacity of a simply supported timber beam subjected to pure bending

If a timber beam is subjected to a constant bending moment $m(x) = 1$, then the filtered and marked Poisson random field could be employed to model the lengthwise variability of bending strength.

4.3.1 The extreme-value distribution for independent random variables

Assuming the marked Poisson random field the independent random variables $\{R_1,\ldots, R_N\}$ are assigned to N_L weak zones. For a fixed number of weak zones N_L the distribution function of the load carrying capacity of a timber beam subjected to pure bending can be determined as the minimum value distribution of N_L independent random variables $\{R_1,\ldots, R_N\}$.

$$F_Z(z) = 1 - \left[1 - F_d(z)\right]^{N_L}, \tag{9}$$

where $F_d(z)$ is the cumulative probability distribution function of strength for each weak zone.

At a low stress level the distribution function is close to zero, $F_d(z) << 1$, so the distribution can be approximated by

$$F_Z(z) \approx 1 - exp\left(-N_L F_d(z)\right). \tag{10}$$

4.3.2 The mixed model of bending strength of a timber beam

The more general solution can be obtained for the mixed model of bending strength presented in Fig. 2, where the bending strength of a timber beam depends both on the filtered Poisson random field of bending strength of weak zones and the homogeneous random field of bending strength of clear wood. The following assumptions establish the model:

- the distance between weak zones (defects) follows the Poisson law with the intensity $v(x)$; the mean number of weak zones N_L within beam span L is equal to

$$N_L = \int_0^L v(x)\,dx. \tag{11}$$

- the length of a weak zone δ_i is a random variable with common distribution for all weak zones and the mean value denoted as $E[\delta_i]$.
- the bending strength R_d is constant within each weak zone and is a random variable with common distribution function F_d for all weak zones
- the bending strength of clear wood R_c is a homogeneous random field with the first order distribution function F_c.

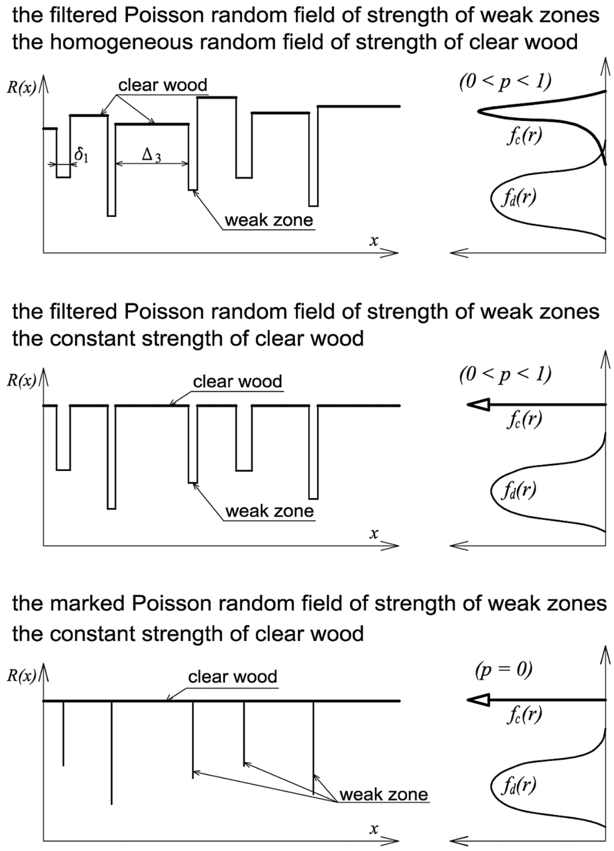

Fig. 2 Mixed models of bending strength of structural timber

The probability density function and the probability distribution function of bending strength are defined as follows

$$\begin{aligned} f_R(r) &= (1-p) f_c(r) + p f_d(r), \\ F_R(r) &= (1-p) F_c(r) + p F_d(r), \end{aligned} \tag{12}$$

where p is equal to the fraction of the mean length of a weak zone with respect to the mean distance between weak zones.

$$p = \frac{E\left[\delta_i\right]}{E\left[\Delta_i\right]} \tag{13}$$

If $p \to 0$, then the marked (spike) Poisson random field of weak zones is valid. If $p \to 1$ then the filtered Poisson random field becomes the Poisson rectangular-wave random field, i.e., the strength of the whole beam is represented by the weak zones random field.

The weak zones model in Fig. 1 and the marked Poisson model in Fig. 2 depend on the constant strength of clear wood. The variations in bending strength of clear wood are considered unimportant in relation to the variations in bending strength of weak zones. In such a case the functions given by Eq. (12) have got simpler forms,

$$f_R(r) = (1-p)\delta(R_c) + pf_d(r),$$
$$F_R(r) = (1-p)H(R_c) + pF_d(r), \tag{14}$$

where R_c is the constant strength of clear wood, $\delta(R_c)$ is the Dirac's delta function and $H(R_c)$ is the Heaviside's function.

The load carrying capacity of a timber beam subjected to pure bending is equal to the minimum value of the strength along the beam span

$$F_Z(z) = P\left[\min R(x) \le z; x \in [0, L]\right],$$
$$F_Z(z) = 1 - P\left[\min R(x) > z; x \in [0, L]\right]. \tag{15}$$

The Poisson random field is defined for the left-end open interval $x \in [0,L]$, i.e., the Poisson distribution is based on the assumption $N(0) = 0$. Thus, $R(x)$ over the whole beam span depends both on the distribution at $x = 0$ (i.e., the distribution at arbitrary point on axis) and on the minimum of the filtered Poisson random field,

$$1 - F_Z = P\left[min\left\{R(0), R(x); x \in (0, L]\right\} > z\right], \tag{16}$$

$$1 - F_Z = P_1 P_2 . \tag{17}$$

The first probability in Eq. (17) is equal to

$$P_1 = P\left[R(0) > z\right] = 1 - F_R(z). \tag{18}$$

The second probability in Eq. (17) shall be computed with help of the total probability theorem (19)

$$P_2 = P\left[minR(x); x \in (0, L] > z\right] =$$
$$= \sum_{n=0}^{\infty} P\langle min\left\{R_1, \ldots, R_{N(L)}\right\} > z | N(L) = n\rangle P\left[N(L) = n\right].$$

Since it has been assumed that the strengths at weak zones are independent random variables and are assigned to weak zones distributed according to the Poisson law

$$P\left[N(L) = n\right] = \frac{(\nu L)^n}{n!} e^{-\nu L}, \tag{20}$$

the probability distribution function of the load carrying capacity of a beam under pure bending is equal to

$$F_Z(z, p) = 1 - \left[1 - (1-p)F_c(z) - pF_d(z)\right] e^{-\nu L F_d(z)}. \tag{21}$$

There are two limit cases as follows:

- $p = 0$: the marked (spikes) Poisson random field of weak zone.

$$F_Z(z, 0) = 1 - \left[1 - F_c(z)\right] e^{-\nu L F_d(z)} \tag{22}$$

- $p = 1$: the rectangular-wave Poisson random field of weak zones, i.e., the timber is composed of weak zones only.

$$F_Z(z, 1) = 1 - \left[1 - F_d(z)\right] e^{-\nu L F_d(z)} \tag{23}$$

It can be shown that the distribution function by Eq. (21) is also valid in case of weak zones with variable strength.

All distribution functions (Eqs. (21)–(23)) are exact functions under the given assumption valid for the corresponding models. Assuming $z \ll E[R_c]$, then $F_c(z) \approx 0$ and the approximate distribution, valid at low stress level, can be obtained from Eq. (21).

$$F_Z(z, p) \approx 1 - \left[1 - pF_d(z)\right] e^{-\nu L F_d(z)} \tag{24}$$

If $p \ll 1$, i.e., the weak zones model is valid, then $pF_d(z) \approx 0$ and the approximate distribution function is obtained.

$$F_Z(z, p) \approx 1 - e^{-\nu L F_d(z)} \tag{25}$$

The above function depends on the mean number of weak zones within a beam span, but it is similar to the function Eq. (10), which has been derived for a fixed number of weak zones.

For a higher stress level the distribution function given by Eq. (25) tends towards

$$\lim_{z \to \infty} F_Z(z) = \lim_{z \to \infty} \left[1 - e^{-\nu L F_d(z)}\right] = 1 - e^{-\nu L}. \tag{26}$$

4.3.3 Riberholt-Madsen model

Riberholt and Madsen [2] have derived the approximate distribution function given by Eq. (25), assuming the weak zones model and the marked Poisson probabilistic model, which is only valid at low stress level, i.e., the load carrying capacity is governed by weak zones with defects. Furthermore, the distribution function of the load carrying capacity for all stress levels has been derived under the assumption that a timber beam fails only at weak zones, i.e., a beam without defects can survive any load. The total probability theorem applied to the probability of survival (conditional on the number of weak zones) results in the following formula for the probability that the load carrying capacity is greater than z, since it has been assumed that the failure can only occur at weak zone, i.e.,

$$P\langle Z > z | N_L = 0\rangle = 1,$$
$$P\left[Z > z\right] = P\left[N_L = 0\right] + P\langle Z > z | N_L \ge 1\rangle P\left[N_L \ge 1\right]. \tag{27}$$

It is assumed that the weak zones constitute a Poisson random field with intensity $\nu(x)$. Those weak zones at which random variables $\{R_i\}$ down cross the stress function $S(x) = Z$ constitute a new Poisson random field with the intensity $v_z(x) = v(x)F_d(z)$. Thus, the following probabilities can be derived

$$P[Z > z] = e^{-N_L F_d(z)}, \tag{28}$$

$$P[N_L = 0] = e^{-N_L} \quad P[N_L \geq 1] = 1 - e^{-N_L}, \tag{29}$$

and the distribution function of the load carrying capacity $F_z(z)$ for the marked Poisson random field of weak zones and constant strength of clear wood is as follows:

$$F_Z^{RM}(z) = \frac{1 - e^{-N_L F_d(z)}}{1 - e^{-N_L}} \tag{30}$$

This function has been derived by Riberholt and Madsen [2] and is valid at all stress levels. The distribution function Eq. (30) can be represented as the conditional probability.

$$F_Z^{RM}(z) = P\langle Z \leq z | N_L \geq 1\rangle = \frac{P[Z \leq z \cap N_L \geq 1]}{P[N_L \geq 1]} \tag{31}$$

In the following, the distribution functions similar to the conditional probability, Eq. (31), are called distribution function according to Riberholt-Madsen model, whereas a distribution function defined by Eq. (25) is referred to as the distribution function according to the weak zones model.

The approximate functions defined by Eqs. (25), Eq. (30) are compared in Fig. 3, for different mean number of weak zones. Both distributions are close to each other at low level of stresses. The difference in the upper part tail diminishes with growing number of weak zones.

Two distribution function: according to the rectangular-wave Poisson random field, Eq. (23), and the Riberholt-Madsen function, Eq. (30), which are valid at all stress levels, are compared in Fig. 4. Both functions are similar to each other. The difference between them decreases slower than between the functions shown in Fig. 3.

4.4 Bending strength of timber modelled by the inhomogeneous marked Poisson random field

Let's assume that the bending strength of a timber beam is modelled by the random variables $\{R_i\}$ assigned to weak zones with the lengthwise distribution governed by the Poisson random field with the intensity $v(x)$. Definition of the load carrying capacity as the first downcrossing problem is employed in this case.

To derive the probability distribution function $F_z(z)$ it is important to notice that those weak zones at which random variables $\{R_i\}$ downcross the stress function $S(x) = Zm(x)$ constitute Poisson random field with the intensity

$$v_Z(x) = v(x)F_d(S(x)) = v(x)F_d(Z\,m(x)), \tag{32}$$

where $F_d(r)$ is the first-order distribution function of random variables $\{R_i\}$.

Hence, the distribution function of the load carrying capacity $F_z(z)$ can be computed with help of three equivalent definitions:

- by using the probability of survival, i.e., the beam survives the stress level z, if the strength function $R(x)$, defined by Eq. (4) or Eq. (5), is above the stress function $S(x) = Zm(x)$ within the beam span L,

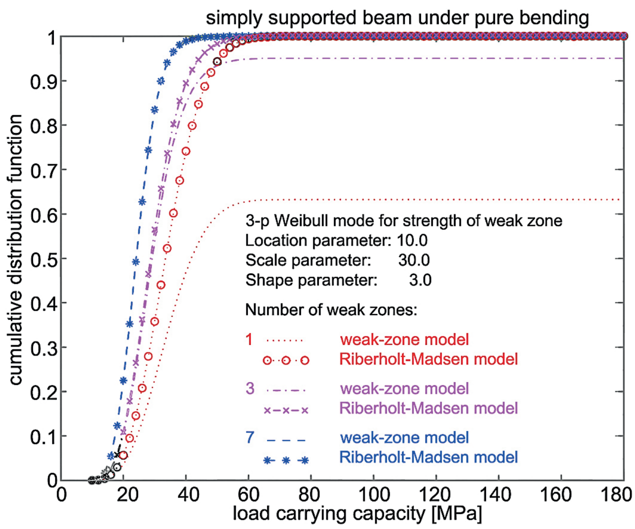

Fig. 3 Comparison between two distribution functions of load carrying capacity of timber beams subjected to pure bending

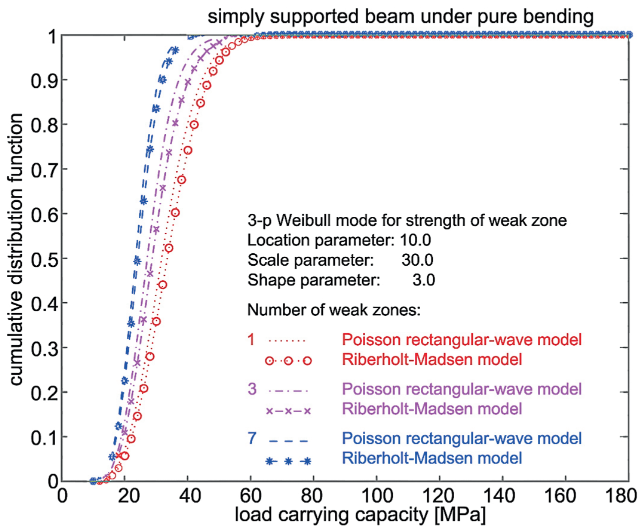

Fig. 4 Comparison between two distribution functions of load carrying capacity of timber beams subjected to pure bending

$$F_Z(z) = 1 - P\left[R(x) > S(x) = zm(x)\right],\quad \text{for}\left[0 \le x \le L\right], \tag{33}$$

- by using the probability of survival, i.e., the beam survives the stress level z, if the strength function $R(x)$, defined by Eq. (4) or Eq. (5), down crosses the stress function $S(x) = Zm(x)$ first at the distance $x_{(1)}$ which is longer than the beam span L.

$$F_Z(z) = 1 - P\left[R\left(x_{(1)}\right) < zm\left(x_{(1)}\right); x_{(1)} > L\right] \tag{34}$$

- by using the probability that the inhomogeneous Poisson random field with the intensity $v_z(x)$, defined by Eq. (32), has no occurrence within the beam span L, i.e., the first downcrossing does not occur within beam span.

$$F_Z(z) = 1 - P\left[N(L) = 0; v_Z(x)\right] \tag{35}$$

Hence

$$F_Z(z) = 1 - exp\left\{-\int_0^L v_Z(x)\,dx\right\} = 1 - e^{-I(z)}$$
$$\text{where } I(z) = \int_0^L v(x) F_d\left(zm(x)\right)dx. \tag{36}$$

The distribution function by Eq. (35) is an extension of the relation by Eq. (25) for the case of the variable moment function $m(x)$ and inhomogeneous random series $\{R_i\}$.

The integral defined by Eq. (36) depends on the intensity function $v(x)$ defining the distribution of weak zones, on the probability distribution function $F_d(r)$ for the bending strength of weak zones as well as on the normalized bending moment function $m(x)$. In most cases the integral by Eq. (36) must be evaluated by means of numerical methods for specific data. However, if the variability of the strength of weak zones is modelled with the Weibull model, then an analytical expressions for the distribution function $F_z(z)$ can be obtained.

4.4.1 Timber beam with inhomogeneous distribution of weak zones

Application of a homogeneous Poisson random field for the distribution of weak zones means that distances between weak zones have an Exponential distribution function and are not correlated as well as the shorter distances between weak zones are assumed to be more probable than longer.

In order to study another arrangement of weak zones a particular type of a inhomogeneous Poisson random field has been applied, with the intensity function

$$v(x) = \sum_{i=1}^{N(L)} \delta\left(x - x_i\right), \tag{37}$$

where $N(L)$ is the number of weak zones within a beam of length L and x_i is the position of the i^{th} weak zone defined by

$$x_i = x_1 + \Delta(i-1) \quad for\ i = 1,\ldots,N(L). \tag{38}$$

In general, both the distance to the first weak zone, x_1, and the distance between weak zones, Δ, are random variables.

In particular case assuming that the distance to the first weak zone, x_1, is random, but the distance between weak zones, Δ, is deterministic and assuming the weak zones intensity function according to Eqs. (37) and (38) the integral Eq. (36) can be determined analytically and the function $I(z)$ is equal to

$$I(z) = \sum_{i=1}^{N(L)} F_d\left[zm(x_i)\right]. \tag{39}$$

Thus, the conditional cumulative distribution function (CDF) for the load carrying capacity of the weak zones model is

$$F_Z\left\langle z|x_1\right\rangle = 1 - exp\left\{-\sum_{i=1}^{N(L)} F_d\left[zm(x_i)\right]\right\}, \tag{40}$$

and for the given probability density function $f_{x_1}(x_1)$ of the distance to the first weak zone, the unconditional CDF for the load carrying capacity of the weak zones model is obtained.

$$F_Z(z) = \int_{x_1} F_d\left\langle z|x_1\right\rangle f_{x_1}(x_1)\,dx_1 \tag{41}$$

The above formula is valid for any functions F_d and f_{x_1}.

Moreover, in the case of a beam under pure bending does not depend on f_{x_1}, i.e., the probability distribution function for the distance to the first weak zone.

4.4.2. Timber beam with homogeneous distribution of weak zones and Weibull distribution of bending strength of weak zones

The function $I(z)$ defined by Eq. (36) may be derived analytically, for the following cases:

- a homogeneous Poisson random field is assumed for the distribution of weak zones, with constant $v(x) = v$.
- Weibull distribution is assumed as the probability model of bending strength of weak zones.

$$F_d(r) = 1 - exp\left\{-\left(\frac{r - \mu_d}{\sigma_d}\right)^{\lambda_d}\right\}, \tag{42}$$

where μ_d is the location parameter, σ_d is the scale parameter and λ_d is the shape parameter.

Assuming the Weibull model of weak zones strength and constant intensity v of weak zones distribution, the integral by Eq. (36) has been evaluated for the normalized bending moment functions and the cumulative probability distribution functions of load carrying capacity have been obtained for three typical load cases, considered in the design of simply supported timber beams: a) the midspan point load F_Z^a, b) the one-third point loads F_Z^b, c) the pure bending F_Z^c.

$$F_Z^a(z) = 1 - exp\left[-I_1(z)\right],$$
$$F_Z^b(z) = 1 - exp\left[-\frac{2}{3}I_1(z) - \frac{1}{3}I_2(z)\right], \quad (43)$$
$$F_Z^c(z) = 1 - exp\left[-I_2(z)\right],$$

where

$$I_2(x) = vL\left\{1 - exp\left[-\left(\frac{z-\mu_d}{\sigma_d}\right)^{\lambda_d}\right]\right\}, \quad (44)$$

and the function is

$$I_1(x) = vL\left\{\left(1 - \frac{\mu_d}{\sigma_d}\right) - \frac{\sigma_d}{z\lambda_d}\gamma\left(\frac{1}{\lambda_d}\left(\frac{z-\mu_d}{\sigma_d}\right)^{\lambda_d}\right)\right\}, \quad (45)$$

and is the incomplete gamma function

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \,. \quad (46)$$

It is worth noticing that in the case of the one-third point load the function $I(z)$ is the combination of two functions $I_1(x)$, $I_2(x)$ with coefficients equal to the proportion of beam subjected to the linear moment function and constant moment function, respectively. It can be shown that the same CDF as for a midspan point load, Eq. (43), is obtained for: a simply supported beam under concentrated force applied at any point, a cantilever subjected to point load at free end, a fixed-fixed beam under a midspan point load. All three distribution functions, given by Eq. (43) are valid at the low stress levels.

In order to obtain the cumulative probability distribution functions valid at all stress levels, i.e., according to Riberholt-Madsen model, functions by Eq. (43) should be divided by the probability that there is at least one weak zone within the beam span.

$$P\left[N_L \geq 1\right] = 1 - e^{-N_L} \quad (47)$$

For the pure bending load case, see Fig. 3, the difference between the two types of distribution depends on the mean number of weak zones within a beam span. Fig. 5 shows the comparison for the midspan point load and Fig. 6 for the one-third point loads. For a typical mean number of weak zones observed in timber, i.e., $N_L \geq 5$, the difference is small, particularly at the lower tail of distributions.

In design practice the most important parameter is the characteristic value of the load carrying capacity Z_k defined as the p^{th} quantile of the distribution function F_z, where $p = 0.05$ is usually assumed for the resistance variables.

$$F_Z(Z_k) = P\left[Z \leq Z_k\right] = p \quad (48)$$

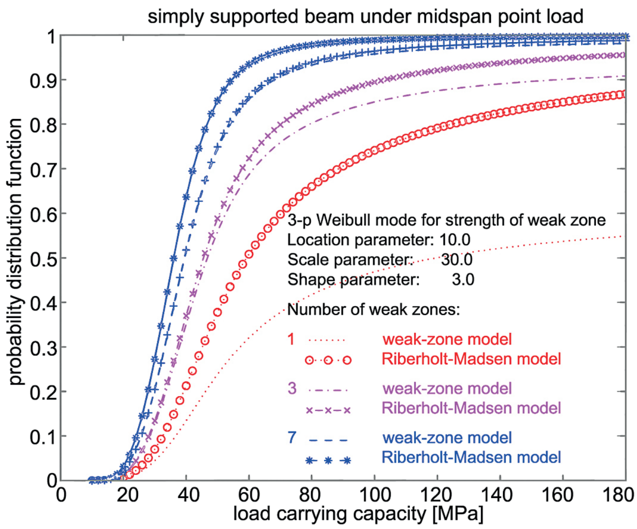

Fig. 5 Comparison between two probability distribution functions of load carrying capacity of a timber beam under midspan point load

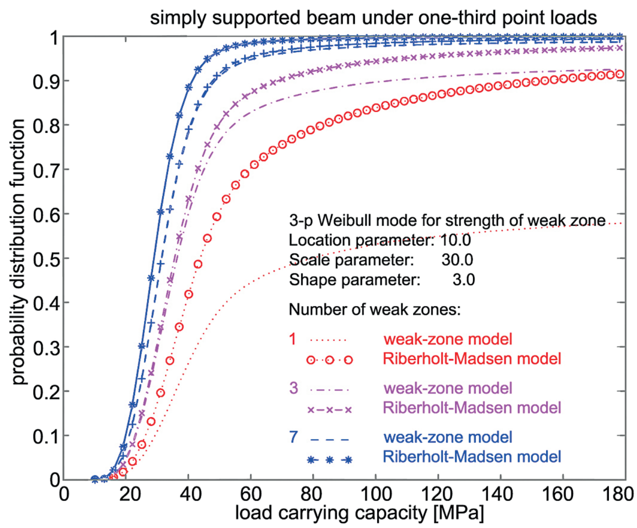

Fig. 6 Comparison between two probability distribution functions of load carrying capacity of timber beams under one-third point loads

The characteristic values of load carrying capacity can be computed by solving the following equations

a) the middle point load

$$I_1\left(Z_k\right) = -\ln\left(1-p\right) \tag{49}$$

b) the one-third point loads

$$\frac{2}{3}I_1\left(z\right)+\frac{1}{3}I_2\left(z\right) = -\ln\left(1-p\right) \tag{50}$$

In case of the Riberholt-Madsen model the right sides of Eqs. (49) and (50) must be changed, i.e.,

a) the middle point load

$$I_1\left(Z_k\right) = -\ln\left(1-p\left(1-e^{-N_L}\right)\right) \tag{51}$$

b) the one-third point loads

$$\frac{2}{3}I_1\left(z\right)+\frac{1}{3}I_2\left(z\right) = -\ln\left(1-p\left(1-e^{-N_L}\right)\right) \tag{52}$$

If the mean number of weak zones N_L is greater than 5, the factor $e^{-N_L} \leq 0.00674$ is quite small. Thus, in a typical case the solutions of Eqs. (49) and (50) are sufficiently accurate. It is worth noticing that the characteristic values computed according to the weak zones model are always greater than corresponding values according to the Riberholt-Madsen model. On the other hand distribution functions derived according to the Riberholt-Madsen model cannot be considered as accurate solutions. An exact solution is unknown and depends both on the strength of weak zones and clear wood.

For a timber beam under pure bending the characteristic value is given by the formula

$$Z_k = \mu_d + \sigma_d\left[-\ln\left(1+\frac{1}{\nu L}\ln\left(1-p\right)\right)\right]^{\frac{1}{\lambda_d}}, \tag{53}$$

which can be simplified with help of the linear expansion of the logarithm function

$$Z_k = \mu_d + \sigma_d\left[-\frac{\ln\left(1-p\right)}{\nu L}\right]^{\frac{1}{\lambda_d}} \tag{54}$$

Dependence of the bending strength on the beam length and load configuration has been observed in many experimental tests, see [6], [11]. Taking into account formulae Eqs. (53) and (54) the relationships between the characteristic values Z_{k1}, Z_{k2} corresponding to the beams spans L_1, L_2 can be obtained. In case of a beam under pure bending, the relationship is similar to a formula presented in [11] for the length effect on the basis of the Weibull weakest link theory,

$$\frac{Z_{k1}-\mu_d}{Z_{k2}-\mu_d} = \left(\frac{L_2}{L_1}\right)^{\frac{1}{\lambda_d}} \tag{55}$$

Thus, the shape parameter λ_d in the Weibull distribution assigned to the bending strength of weak zones determines the degree of the length effect. This means that experimental data concerning the relation between span of beams and the characteristic value of load carrying capacity of timber beams under pure bending could be used in determination of the parameters μ_d and λ_d in the Weibull distribution related to the bending strength of weak zones. The length effect described by Eq. (55) does not depend on the intensity of weak zones ν, since the distribution of weak zones $\{x_i\}$ is modelled by the homogeneous Poisson random field with $\nu(x) = \nu$.

5 Conclusions

A homogeneous model of a timber beam has been used in design practice for many years. It is assumed that both the modulus of elasticity and the modulus of rupture (load bearing capacity) are constant along the beam axis. On the other hand, the brittleness of structural timber due to the presence of knots or other defects is well known. Therefore, in order to guarantee the required margin of safety, the test procedure for determining the characteristic value of the bending strength of structural timber requires that the class defining defect should be placed in the shear free zone produced by a one-third point loads, i.e., the part of a beam with a significant defect should be subjected to pure bending. This approach leads to safe and conservative design. The development of reliability methods over the last forty years enables more accurate modelling and safety analysis of timber structures. However, the application of probabilistic analysis requires statistical data on the variability of structural timber properties.

This paper presents an analysis of timber bending strength variability based on the weak zones model proposed in [2]. It is a simple mechanical model, but takes into account the basic failure mode of timber beams, which in bending usually fail at knots or groups of knots. The growth defects are typical for structural timber produced from conifer trunks. The weak zones model is based on the assumption that a structural beam can only break in a weak zone or in a weak cross section. The presented analysis is independent of the analysis described in [2]. It is shown that both solutions are equivalent in special cases.

The load bearing capacity of timber beams was determined by solving the first downcrossing problem assuming that the distances between weak zones are a Poisson random field, with both the strength of the weak zones and the distances between them being independent. Two types of homogeneous random series (one dimension random field) were considered: a filtered random series and a marked random series. Analytical formulae for the cumulative probability distribution function of the load carrying capacity were obtained for the cases of: (1) a timber beam subjected to pure bending with a filtered and marked Poisson random field to model the strength of the timber, (2) a timber beam with a particular type of inhomogeneous Poisson random field modelling the distribution of weak zones, (3) a timber beam with a homogeneous distribution of weak zones and a Weibull distribution of the bending strength of weak zones. Example calculations were performed for three typical load cases: pure bending, mid-span point load and one-third point loads, which are typical in the grading process of structural timber. Unlike structural materials produced under controlled conditions (e.g., steel or concrete), each timber beam must be graded. Therefore, the relationship between two cumulative probability distribution functions is important: the load carrying capacity of timber beam and the strength of weak zones. The analysis performed confirms the load configuration effect and size effect on the characteristic values of structural timber strength.

From the example calculations determined by the analytical formula, it can be concluded that the probability distribution of the load carrying capacity of a timber beam depends more on the number of weak zones than on the type of analysis. As the number of weak zones increases, the probability distributions obtained in this paper and in [2] are close to each other. Thus, the filtered Poisson random field applied to the weak zones model appears to be a suitable probabilistic model of bending strength and is recommended for practical applications, especially in reliability analysis of timber structures.

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