

Symmetry Measure of Truss Structures with Disturbed Higher-order Symmetries

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Abstract

This paper covers aspects of quantifying the symmetry of two- and three-dimensional elastic bar-and-joint structures. The concept of symmetry as a quantitative property instead of a binary question of 'yes' or 'no' is widely accepted and thoroughly investigated, for example, in molecular physics but also in engineering sciences, mainly in chemical engineering. Similarly to most of the articles written on this topic, our method is also based on the comparison of specific metrics of the analyzed structure and a reference one, i.e., which possesses the desired (perfect) symmetry. The deviation of the analyzed (imperfect) structure from the reference structure is quantified by one scalar. The novelty in our approach is that we consider not just the relative position of the nodes but also the normal stiffness of the truss members, even for structures with higher-order, i.e., polyhedral symmetries. For both geometric and material properties to be accounted for, the eigenvalues of the stiffness matrix were chosen as metrics. The difficulty lies in finding the reference structure which will be carried out based on energy principles.

Keywords

measure of symmetry, imperfect symmetry, symmetric truss structures, near-symmetric structures

1 Introduction

1.1 Motivation

The state of symmetry in structural engineering is still mostly considered binary; a structure is either symmetric or not symmetric. In other fields of science, however, several methods exist that allow us to consider symmetry as a quantitative property. This paper aims to give an easy-to-use method to measure the degree of symmetry of truss structures with *imperfect* or *disturbed* higher-order symmetries, considering both the geometry and the stiffness properties.

In the field of civil engineering, structures with higher-order symmetries also exist; it is sufficient to mention *geodesic domes*, which are sensitive to geometric imperfections: even minor disturbances in their geometry may cause considerable changes in their structural behavior. Quantifying the 'health' of such structures is possible using an adequate symmetry measure.

Our further motivation is to understand the shape evolution of objects (e.g., biological tissues) that can also be modeled as truss structures. With an adequate measure of

symmetry, the direction of a process of shape evolution starting from a nearly symmetric state could be understood, namely, if it approaches a more or less symmetric state.

1.2 State-of-the-art

This subsection briefly introduces the already existing approaches to the quantification of symmetry. A good starting point could be mentioning the concept of *continuous symmetry measure (CSM)* for measuring the degree of symmetry of molecules introduced by David Avnir and coauthors [1–6].

Among all the papers dedicated to this topic, the most influential to our work was the paper dealing with the regular (Platonic and Archimedean) polyhedra of Pinsky and Avnir [4]. In its Appendix A, a detailed algorithm can be found for the determination of the value of *CSM*, which will be shortly summarized here.

Let us suppose that we have a disturbed set of points with the center of mass at the origin, which had, in its

undisturbed configuration, the symmetries of one of the regular polyhedra. The goal is to find a perfectly symmetric set of points best fitted to the disturbed one, which means here that the sum of the squares of the distance between each point in the original configuration and its pair in the symmetric one is minimal. In order to find this set, we depart from a perfectly symmetric set whose points are on the unit sphere. This set is scaled and rotated till a minimum deviation of the squares of the distances between the corner points (in terms of the scaling factor and the angles of rotation, e.g., the Euler angles) is obtained. If the pairing of the points from the two sets is not trivial, all possible pairings must be checked to determine whether a better fit can be achieved. Once the reference set is obtained, the *CSM* value is the sum of the squares of the distances of the pairs normalized by the sum of the squares of the norms of the vectors pointing from the origin to the points of the reference set. Note that, because of the normalization, the value of *CSM* is between 0 and 1 (in the above-referenced papers, it is given as a percentage); 0 stands for the perfectly symmetric state, so it is a measure of asymmetry instead.

Several authors updated or made somehow more efficient the original algorithm of Avnir and coauthors; for example, Alikhanidi and Kuzmin [7] introduced the *MCSM*, the 'modified continuous symmetry measure' which can handle sets of points in which the points have different mass or Alon and Tuvi-Arad [8] reached better running times with their update of the original algorithm.

A way different but interesting method uses *Voronoi entropy* [9, 10]. A set of points (seeds) in a plane defines Voronoi cells (sets of closest points to each seed), forming a *Voronoi tessellation*. The Voronoi entropy is defined as follows: $S_{vor} = -\sum_i P_i \ln P_i$, where $i=3, 4, \dots$, and P_i is the number of n -sided polygons over the total number of polygons in the Voronoi tessellation (the simplest polygon, i.e., having the lowest number of sides, is the triangle). The Voronoi entropy gives information on the *orderliness* of the set of points. Bormashenko et al. [11] compared the *CSM* and the Voronoi entropy specifically for a Penrose tiling and pointed out that these symmetry measures are not objective.

Fang et al. [12] introduced the *degree of symmetry (DoS)* and simultaneously the *degree of asymmetry (DoAS)* in the field of quantum physics. Suppose we have a quantum system at a certain energy level (characterized by its Hamiltonian). In that case, an operation that leaves the system at the same energy level is considered a symmetry operation. If the energy level changes, then the

degree of asymmetry can be determined from the deviation from the original energy level. Similar formulations of this method are available in [13, 14]. Note that 'degree of symmetry' in the present paper is used as a general term to describe any measure of symmetry, including the one introduced in the forthcoming section.

We mention some examples of symmetry measures in biology, too. Esposito et al. [15] examined the symmetry of the connectivity diagram of the human brain, setting up the adjacency matrix of the neurons, which is quite similar to the stiffness matrix considered later in this paper. Gandhi et al. [16] used the concept of entropy to process and measure the symmetry of biological patterns (note that Avnir and coauthors also dealt with incomplete symmetries appearing in images; see [17]). Jaśkowski and Komosinski [18, 19] quantified the reflection symmetry of '3D stick creatures', which are similar to the truss structures analyzed in this paper, and the authors called attention to the problem of the different weights of the sticks.

Finally, let us cite some papers close to our topic and related to the field of structural engineering. Kaveh and Khazaee [20] dealt with originally symmetric structures with minor disturbance in their symmetry and showed examples in which the calculation of eigenvalues of the stiffness matrix (providing information about the eigenmodes) of the closest symmetric structure instead of the actual one causes negligible error. Once the error is negligible (which, from the point of view of this paper, means that the symmetry measure of the current state shows closeness to the perfect one) and a symmetric reference state can be found, then several methods exist to make use of the symmetry in order to simplify the eigenvalue problem, see for example the paper of Kangwai et al. [21] or the textbook of Kaveh [22].

Chen and Büyüköztürk [23] dealt with health monitoring and damage to structures. Damages usually imply certain asymmetry (symmetry breaking) in originally symmetric structures, and this asymmetry shows up in the mode shapes of the structures. Hence, applying *CSM* to the mode shapes gives information about the health of a structure.

2 The method

This section presents the steps for finding a reference bar-and-joint structure in the geometric neighborhood of the analyzed structure. Further, it introduces our measure of symmetry as the normalization of the sum of the squares of the deviation between the eigenvalues of the analyzed and the reference structure by the sum of the squares of the eigenvalues of the latter one.

It is meaningful to mention that one can read more about the utilization of symmetry by (i.e., a more effective way of) determining the eigenvalues of perfectly symmetric structures with a high degree of freedom, among others, in [24–26].

First, the following assumptions are adopted: the structure consists of linearly elastic straight bars with no initial strains or stresses; bars are connected by frictionless ball joints; external supports and loads along bars are not considered. We also assume that deformations are small (this condition might not be met in cases where the initial imperfect structure is far from being symmetric, but such cases are not dealt with here).

2.1 Finding the reference structure

As has been mentioned above, our method is also one of those that first determines a reference structure, considered to have a perfect symmetry in contrast to the imperfect symmetry of the analyzed, 'near-symmetric' structure. In the present case, the symmetry of the reference structure is required from both the aspects of geometry and stiffness. The reference structure is identified based on energy principles as follows. Consider a structure with imperfect symmetry: For simple arguments of geometry, it is always possible to deform this structure into a perfect one by applying an appropriate set of nodal forces. The perfect structure is defined by some *primary nodes* and the *orbit* (set of images of a primary node under all symmetry operations of the given symmetry) of each. During this deformation process strain energy accumulates in the structure: we will consider the structure to be the closest one, i.e., that defines the geometry of the reference structure, for which the strain energy $U(\mathbf{r}_{10}, \dots, \mathbf{r}_{p0}, \dots, \mathbf{r}_{q0}, \dots, \mathbf{r}_{n0}) = U(\mathbf{R}_0)$, as a function of initially *unknown position vectors* of primary nodes Fig. 1, has a minimum (\mathbf{R}_0 contains the column vectors of the primary nodes).

$$U(\mathbf{R}_0) = \sum_{i,j=0}^{m-1} \sum_{p,q=1}^n \chi_{ij,pq} \frac{EA_{ij,pq}}{2l_{0,ij,pq}} \left(|\mathbf{T}_j \mathbf{r}_{q0} - \mathbf{T}_i \mathbf{r}_{p0}| - l_{0,ij,pq} \right)^2. \quad (1)$$

The interpretation of the variables is as follows: \mathbf{r}_{p0} denotes the position vector of the p th out of n primary nodes; its orbit can therefore be expressed as $\mathbf{r}_{pi} = \mathbf{T}_i \mathbf{r}_{p0}$ where \mathbf{T}_i , $i=0, \dots, m-1$ stand for matrices of symmetry operations, and m is the order of the respective symmetry group. The internal sum in (1) is governed by the connectivity parameter $\chi_{ij,pq}$ set to 1 if there is a directed bar with starting node \mathbf{r}_{pi} and final node \mathbf{r}_{qj} and zero otherwise. Observe that both $p=q$ and $i=j$ are

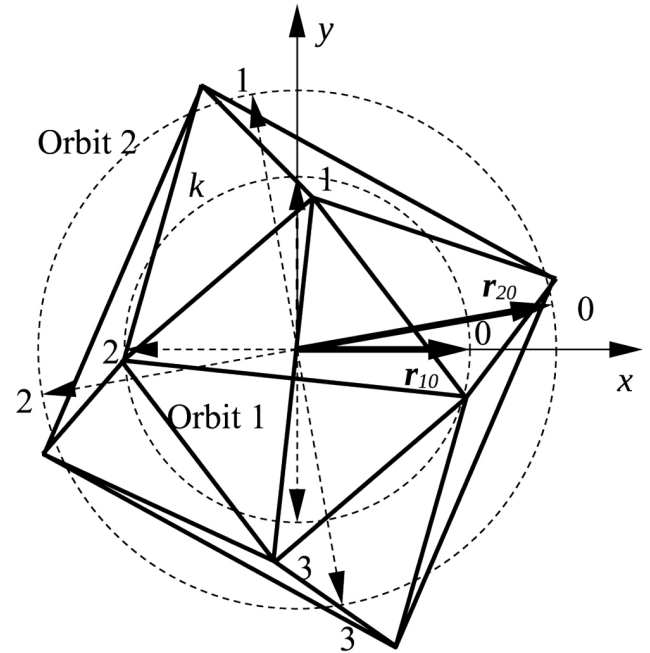


Fig. 1 Interpretation of variables of the strain energy: the number of orbits of nodes, the order of symmetry, and the primary nodes. The reference structure (drawn only by its nodes and position vectors in dashed lines) with C_4 symmetry is generated by two primary nodes (\mathbf{r}_{10} and \mathbf{r}_{20}) and their orbit, but for the sake of simplicity, y_{10} can be set to zero. The initial structure is drawn in solid lines.

possible but not simultaneously. It should also be noted that the number of scalar unknowns in U does not equal $2n$ (in 2D) or $3n$ (in 3D) since, on the one hand, the orientation of the position vector of the first primary node is free up to a rigid-body rotation of the structure; on the other hand, there exist reference structures with some nodes lying on symmetry elements (i.e., mirror planes or axes of rotation) that remove some of the geometric freedoms of node positions. An example structure with $n=2$ and C_4 symmetry (hence $m=4$) can be seen in Fig. 1, where some orbits of bars contain 4, but others only have 2 bar members.

Setting all first derivatives of (1) to zero can be expressed in a compact form as follows:

$$\text{grad}U(\mathbf{R}_0) = \mathbf{0}. \quad (2)$$

Note that a simple stationarity condition is sufficient to be checked because a minimum always exists, and (2) results in a linear system of equations. Note also that in more complex cases, e.g., when there are more primary nodes, (2) can typically be solved for vectors \mathbf{r}_{p0} numerically; however, since rigid body motions do not affect the strain energy, we can simplify the problem departing from some position vectors, e.g., parallel to one of the coordinate axes. Analytical solutions to examples with one single orbit will be shown in the next section.

Once the coordinates of the nodes of the reference structure have been determined, the rest length of the bars of the reference structure can also be obtained.

As soon as the geometry of the reference structure is known, the stiffness values are to be determined. Here, the normal stiffness of the bars of different lengths can differ from each other without disturbing the symmetry; the only requirement is that the bars of the same orbit must have the same normal stiffness. These unknown stiffness values are determined from the following condition: the analyzed and the reference structure have to store the same amount of strain energy if the same normal strain is applied to all bars of both structures (one can imagine that the structure is proportionally 'swollen'). In order to write this strain energy in a proper form, it is convenient to consider a symmetric structure as d sets of bars where each set contains n_d symmetry images (as an orbit) of a particular bar element, including that element itself (for instance, a diagonally braced near-square plane truss shown later in Fig. 4 implies a square reference structure with $d=2$, hence $n_1=4$ for the contour and $n_2=2$ for the diagonals). Let us apply a normal strain ε to the bars in both structures and equate the strain energies by orbits of bars (each orbit t in the perfect structure should have the same normal stiffness EA_{pt} , but they may be different for each orbit):

$$\sum_{k=1}^{n_d} \frac{EA_{kt}}{2} l_{0,kt} \varepsilon^2 = n_d \frac{EA_{pt}}{2} l_{pt} \varepsilon^2, \tag{3}$$

where subscript p stands for *perfect*, $t=1, \dots, d$. We can express the t th perfect normal stiffness from (3) as follows:

$$EA_{pt} = \frac{\sum_{k=1}^{n_d} EA_{kt} l_{0,kt}}{n_d l_{pt}}. \tag{4}$$

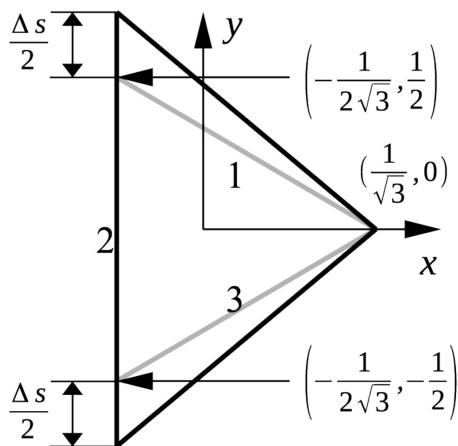


Fig. 2 Geometry and numbering of bars of the triangle in Example 1. Coordinates are to be understood in meters.

2.2 The measure of symmetry

Once the geometry and the normal stiffness(es) of the reference structure are known, the quantification of symmetry will be done based on the comparison of the eigenvalues of the stiffness matrices of the analyzed and the reference structure. The stiffness matrix for skeletal structures is as follows [27]:

$$\mathbf{K} = \mathbf{G}\mathbf{F}^{-1}\mathbf{G}^T, \tag{5}$$

where \mathbf{G} is the equilibrium matrix, and \mathbf{F} is the flexibility matrix containing values l_i/EA_i in its main diagonal. The stiffness matrix seems to be an adequate object to quantify symmetry because it contains both geometric data (\mathbf{G}^T is also known as compatibility matrix) and element flexibilities.

It is essential to mention that in this paper we do not deal with the supports or external loads of the structures (clearly, supports in an asymmetric arrangement would also break the symmetry of the structure). Thus, zero eigenvalues will also appear in structures with no internal (inextensional) flexibility: their number is 3 and 6 for two and three-dimensional structures, respectively, in accordance with the number of independent rigid-body displacements of the whole assembly. Sorting the eigenvalues is also of key importance since each eigenvalue is to be compared to its 'pair' obtained from the imperfect structure.

The measure of the symmetry is introduced as the sum of squares of the deviations of the eigenvalues normalized by the square of the length of the vector containing the eigenvalues of the reference structure as follows:

$$S = \frac{\sum_{i=1}^N (\lambda_i - \lambda_{p,i})^2}{\sum_{i=1}^N \lambda_{p,i}^2}, \tag{6}$$

where λ_i is the i_{th} eigenvalue of the stiffness matrix of the analyzed structure, while $\lambda_{p,i}$ is the i_{th} eigenvalue of the stiffness matrix of the reference structure, and N is the number of degrees of freedom of the structure. Note that $S=0$ holds for a perfectly symmetric structure; values close to 0 mean almost perfect symmetry, and moving away from 0 means increasing asymmetry.

Several remarks are to be given regarding this method. Firstly, just like other mentioned measures of symmetry, our method is not an objective one, e.g., the reference structure could have been determined differently, for example, by measuring directly the deviation between the *Frobenius norm* of the stiffness matrix of the analyzed structure and that of the stiffness matrix of a (possible) perfectly symmetric structure which would depend on rotation angles and a

scaling factor. This approach (which would have resulted in the generalization of the CSM [28]) was discarded because of the lack of a clear physical interpretation.

Secondly, handling the geometry and normal stiffness together in one measure might cause the measure to be more sensitive to disturbances in geometry than in the stiffness values or vice versa. This behavior will be illustrated through some examples in the next section.

3 Examples

This section presents examples of determining the reference structure and computing the measure of symmetry.

Example 1 (triangle). First, we show how our method is differently sensible to disturbances of geometry and normal stiffness, respectively, analyzing a simple triangle (Fig. 2). In two different cases, we apply two types of disturbances to an initially regular triangle written in a circle (the normal stiffness values are initially set to some basic unit, e.g., $EA=1$ kN, and the side lengths are $s=1$ m). In the first case, we displace vertically the two ends of bar #2 with the same distance $\Delta s/2$ (Δs is positive for elongation and negative or shortening). Fig. 3 (a) shows the effect of Δs on the measure of symmetry for the cases $\Delta s=-0.2, \dots, 0.2$ m.

In the second case, however, the perfect initial symmetry of the geometry of the structure is not changed, unlike the normal stiffness values: only bar #2 is affected; its normal stiffness is modified as $EA_2=EA+\Delta EA$. In Fig. 3 (b), the effect of the changes in the normal stiffness of bar #2 on the measure of symmetry can be observed from the values $\Delta EA=-0.2, \dots, 0.2$ kN, while the contour plot in Fig. 3 (c) illustrates the change in S for both disturbances simultaneously.

Fig. 3 allows two different conclusions to be drawn. Firstly, if pure disturbances are only considered, the method seems to be more sensitive to changes in geometry. Secondly, there exists a particular combination of disturbances in geometry and stiffness that has little impact on the symmetry measure. The 'valley' in Fig. 3 (c) is found to have a characteristic oblique direction, which can be explained by the fact that a proportional increase in both EA and l leaves the flexibility values unchanged. Of course, the constant growth in l_2 induces changes in both the slope and flexibility of the inclined members, which makes the symmetry measure gradually deviate from zero even in this direction.

Example 2 (square). Consider disturbed square shapes with only one, e.g., horizontal, diagonal bracing of length $2s$ (Fig. 4), allowing modifications only in the geometry

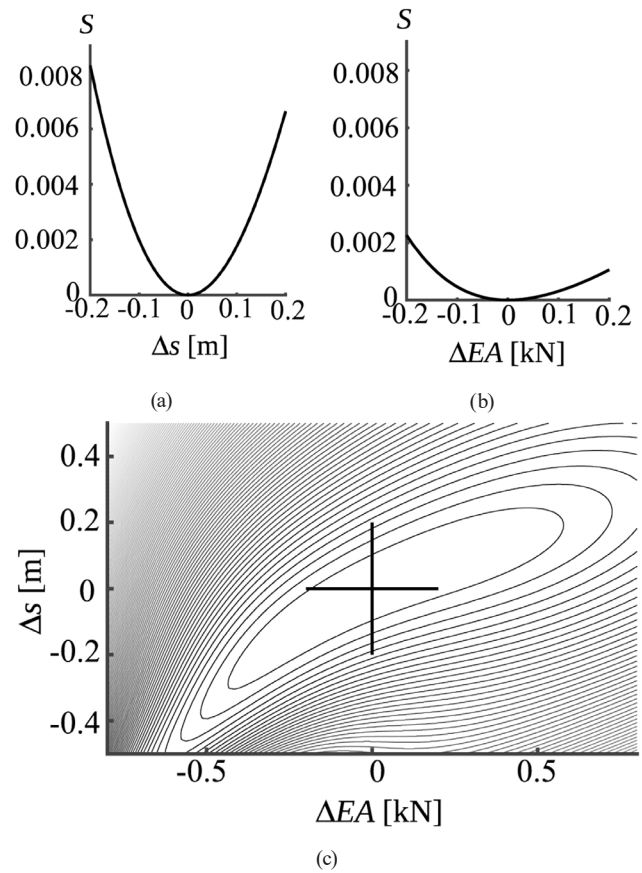


Fig. 3 Plots of measure of symmetry in Example 1: (a) Effect of the change of geometry; (b) Effect of the change of stiffness; (c) Contour plot of the measure of symmetry as a function of disturbance in length and stiffness (vertical and horizontal lines in the center correspond to sections displayed in panels (a) and (b), respectively).

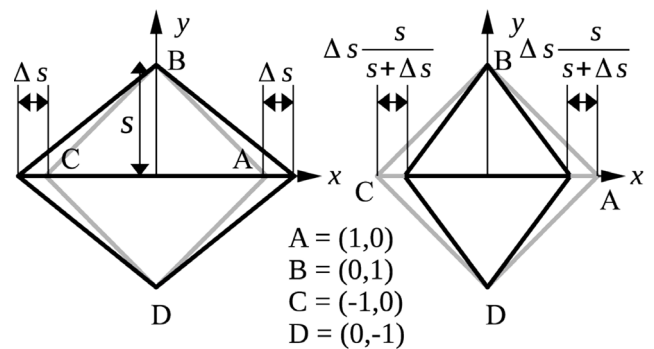


Fig. 4 Examples of disturbed square-shaped structures. Coordinates are to be understood in meters.

but keeping the normal stiffness EA to be 1 kN for all bars. The disturbance is carried out by pulling away or pushing towards each other two opposite corners A and C of the square by Δs , such that two geometrically similar assemblies could be obtained (of course, rotated by 90 degrees). We give the solutions to the particular case of $\Delta s=0.2$ m summarized in Table 1.

Table 1 The results obtained for squares with $\Delta s=0.2$ m.

	Case 1	Case 2
Initial side lengths	$\sqrt{61}/5$ m	$\sqrt{61}/6$ m
Initial (horizontal) diagonal length	12/5 m	5/3 m
Obtained primary node r_{10}	$\begin{bmatrix} 1.128\text{m} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.8960\text{m} \\ 0 \end{bmatrix}$
Perfect side lengths	1.595 m	1.267 m
Perfect contour normal stiffness ($EA_{p,s}$)	0.9792 kN	1.027 kN
Perfect diagonal normal stiffness ($EA_{p,d}$)	0.5319 kN	0.4650 kN
Measure of symmetry (S)	0.02689	0.01486

Since we have only one orbit of joints, we now have one unknown vector $r_{10}=[x_{10},y_{10}]^T$ ($p=1$, q is not used). We have C_4 symmetry (so $m=4$, $i,j=0,1,2,3$), and the matrices representing the symmetry operations are as follows (subscripts 0, 1, 2, and 3 correspond to nodes A, B, C and D , respectively):

$$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, T_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The $\chi_{ij,11}$ values are as follows: $\chi_{13,11}$ is zero, and the others are units since all of the nodes are connected except for the nodes B and D . When the diagonal AC is extended (Case 1), the initial length for the contours and the horizontal diagonal are $\sqrt{61}/5$ m and 12/5 m, respectively. However, with a shortened diagonal AC (Case 2), the same lengths are $\sqrt{61}/6$ m and 5/3 m, respectively, to maintain similarity. As was mentioned before, setting the unknown vector r_{10} parallel to one of the coordinate axes is a reasonable simplification since the rigid body rotation does not affect the strain energy. Assuming that y_{10} is zero, only the first component of the gradient vector is to be dealt with, and the solution in both cases can be obtained analytically from (2) as follows:

$$\text{Case 1: } \left(\frac{5}{3} + \frac{40}{61}\sqrt{61}\right)x_{10} - 4\sqrt{2} - 2 = 0, x_{10} = 1.128\text{m}.$$

$$\text{Case 2: } \left(\frac{12}{5} + \frac{48}{61}\sqrt{61}\right)x_{10} - 4\sqrt{2} - 2 = 0, x_{10} = 0.8960\text{m}.$$

The joints of the reference structure are now determined by the orbit of the position vector $r_{10}=[x_{10},0]^T$, i.e., $T_0r_0, T_1r_0, T_2r_0, T_3r_0$, and restoring both diagonal bars according to the requirements of a fourfold symmetry we get the geometry of the reference structure. The perfect stiffness values are to be calculated from (4) separately for $d=2$ orbits, one for $n_1=4$ bars at the contour and another for $n_2=2$ bars in diagonal position. Numeric results for both the geometry and stiffness are summarized in Table 1.

Example 3 (tetrahedron). Let us consider spatial examples starting with a regular tetrahedron with nodal coordinates $(1, -1, -1), (-1, 1, -1), (-1, -1, 1)$, and $(1, 1, 1)$ [m]. All normal stiffness values are initially of 1 kN. The following cases will be considered (these cases can also exemplify the different effects on the measure of symmetry of the changes in geometry and normal stiffness):

(a) The normal stiffness of three edges sharing the same node is reduced; meanwhile, the geometry remains unchanged. This way, from a mechanical point of view, we obtain a structure with one corner that can be displaced more easily.

(b) One of the corners is pushed radially towards the center of mass of the tetrahedron, while the other corners and the normal stiffness values remain unchanged.

(c) The cases (a) and (b) together.

We give the results for disturbances of 10 % (i.e., the reduced normal stiffness values in Case (a) are 0.9 kN, and the distance of the modified node to the origin in Case (b) is $0.9\sqrt{3}$ m).

In these examples, consider the initial unknown vector $r_{10}=[x_{10},x_{10},x_{10}]^T$ and rotate it about the axis $[-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]^T$ by 120 and 240 degrees and about axis z by 180 degrees. Note that the choice of the applied symmetry operations in this case is not unique since the nodes all lie on some symmetry elements (axes of threefold rotation). We give a possible equation for x_{10} (the gradient of U , which is a simple differentiation now):

$$\text{Case (a): } \frac{57\sqrt{2}}{5}x_{10} - \frac{57\sqrt{2}}{5} = 0, x_{10} = 1\text{m}.$$

$$\text{Case (b): } \left(6\sqrt{2} + \frac{80\sqrt{723}}{241}\right)x_{10} - 12\sqrt{2} = 0, x_{10} = 0.9747\text{m}.$$

$$\text{Case (c): } \left(6\sqrt{2} + \frac{72\sqrt{723}}{241}\right)x_{10} - \frac{57\sqrt{2}}{5} = 0, x_{10} = 0.9760\text{m}.$$

The results are summarized in Table 2. The conclusion is the same as in Example 1., i.e., modification in the geometry causes one magnitude larger measure of asymmetry.

Example 4 (octahedron). Our next example is a randomly disturbed octahedron, both from the aspect of geometry and stiffness (Fig. 5). Both disturbances have been carried out with the help of randomly generated numbers. The octahedron was initially written in the unit sphere with one of the coordinates being ± 1 m, and the bars were all of normal stiffness $EA=1$ kN. To modify the stiffness values, we generated twelve integers (for the twelve edges) in the closed interval $[20,20]$ with uniform distribution, which were all multiplied by 0.01 kN

Table 2 The results obtained for differently disturbed tetrahedra.

	Case (a)	Case (b)	Case (c)
Initial edge lengths	$2\sqrt{2}m$	$\frac{\sqrt{723}}{10}m, 2\sqrt{2}m$	$\frac{\sqrt{723}}{10}m, 2\sqrt{2}m$
Obtained primary node r_{10}	$\begin{bmatrix} 1m \\ 1m \\ 1m \end{bmatrix}$	$\begin{bmatrix} 0.9747m \\ 0.9747m \\ 0.9747m \end{bmatrix}$	$\begin{bmatrix} 0.9760m \\ 0.9760m \\ 0.9760m \end{bmatrix}$
Perfect edge length	2.828 m	2.757 m	2.761 m
Perfect normal stiffness (EA_p)	0.9500 kN	1.001 kN	0.9506 kN
Measure of symmetry (S)	1.214×10^{-5}	5.950×10^{-4}	5.972×10^{-4}

Table 3 Coordinates of the nodes of the disturbed octahedron.

	x [m]	y [m]	z [m]
Node 1	1.14	0.02	0.08
Node 2	0.06	1.08	0.12
Node 3	-1.00	0.10	0.04
Node 4	0.18	-0.92	0.02
Node 5	0.10	0.14	-0.84
Node 6	0.02	0.08	1.10

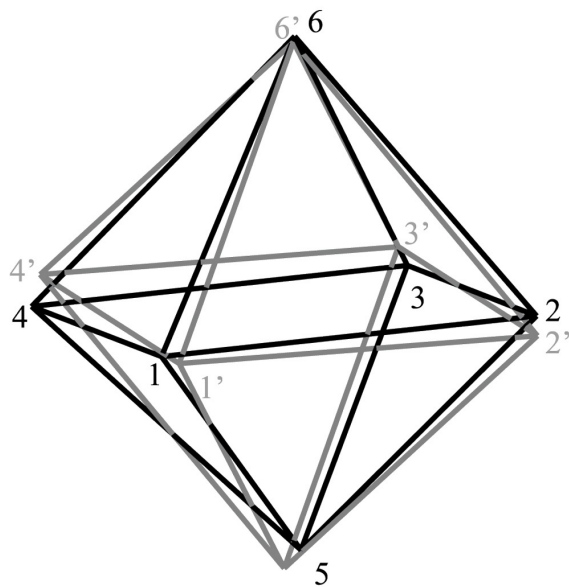


Fig. 5 The disturbed octahedron (in black, 1–6) and the reference structure (in gray, 1'–6').

and added to the initial value of the normal stiffness. Likewise, in the case of the geometry, the closed interval in which 18 integers were randomly generated with uniform distribution was [0,9], and the integers were multiplied by 0.02 m and added to the original coordinate values. In Fig. 5., both the analyzed and the reference structure are shown, while Tables 3 and 4 give the initial data of the disturbed structure. Without getting into the details, we obtain $l=1.434$ m for the perfect edge length, $EA_p=0.9806$ kN for the perfect normal stiffness, and the symmetry measure evaluates to $S=0.002852$.

Example 5 (spatial truss structure with fivefold rotational symmetry). As a last and more practical example, let us quantify the symmetry of a structural unit of a triangulated shell shown in Fig. 6. Initially, the lower nodes were the nodes of a regular pentagon written in the unit circle; the upper nodes, however, were written in a circle of radius 0.5 m at the height of 1 m, the original axis of rotational symmetry is the vertical axis z . The bars were all of normal stiffness $EA=1$ kN. The disturbances were carried out the same way as shown at the octahedron. For the sake of reproducibility, we provide the initial data of the disturbed structure in Tables 5 and 6. Fig. 6. shows the reference structure as well.

Again, for the lower initial node (denoted by r_{10}), we get $[1.055,0,0]^T$, and for the upper one (r_{20}) is obtained as $[0.6282,-0.04342,1.034]^T$. Note that a relative rotation about axis z of the two orbits is permitted without disturbing the fivefold rotational symmetry. To determine perfect stiffnesses, we distinguished four sets of bars: Set 1 contains the edges of the lower pentagon ($EA_{p,1}$), Set 2 is composed of the edges of the upper pentagon ($EA_{p,2}$), in Set 3 contains all bars which were initially coplanar with z (e.g., bar 1–6; $EA_{p,3}$) and finally, the diagonals of the trapezoidal faces belong to Set 4 ($EA_{p,4}$). The perfect stiffnesses are as follows: $EA_{p,1}=1.033$ kN, $EA_{p,2}=0.9582$ kN, $EA_{p,3}=1.035$ kN, and $EA_{p,4}=1.048$ kN, while the symmetry measure evaluates to $S=0.004664$.

4 Conclusions

In this paper, a possible measure of symmetry of bar-and-joint structures has been presented. The method consists of two main parts: first, a reference structure that is perfectly symmetric both from the point of view of

Table 4 Normal stiffness of the bars of the disturbed octahedron in kN.

Bar	1–2	1–4	1–5	1–6	2–3	2–5	2–6	3–4	3–5	3–6	4–5	4–6
EA	0.93	0.86	1.12	1.12	0.89	1.00	1.03	1.04	0.85	0.81	1.10	0.99

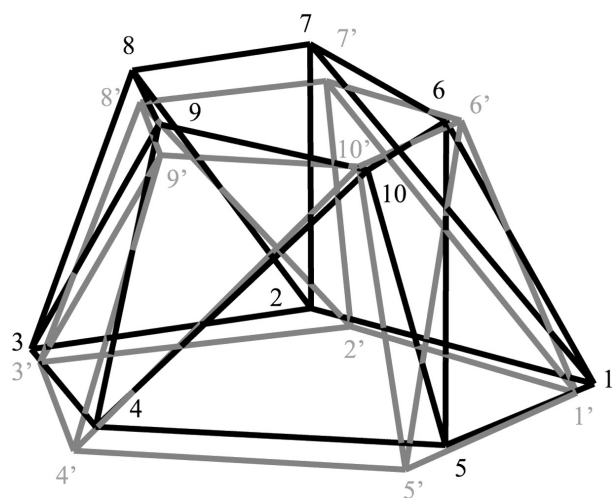


Fig. 6 The disturbed spatial truss structure with fivefold rotational symmetry (in black, 1–10) and the reference structure (in gray, 1'–10'). Nodes 1–5 belong to the lower orbit, and 6–10 belong to the upper orbit.

Table 5 Coordinates of the nodes of the disturbed structure.

	x [m]	y [m]	z [m]
Node 1	1.050	0.1479	0.06000
Node 2	0.1338	0.9910	0.1000
Node 3	-0.8705	0.5317	0.1000
Node 4	-0.7766	-0.7505	0.08000
Node 5	0.3884	-1.051	0.02000
Node 6	0.6567	0.06589	1.160
Node 7	0.1609	0.6194	1.160
Node 8	-0.5490	0.3663	1.160
Node 9	-0.4329	-0.3861	1.000
Node 10	0.2227	-0.5355	1.020

Table 6 Normal stiffness of the bars of the disturbed structure in kN.

Bar	1–2	2–3	3–4	4–5	1–5	6–7	7–8	8–9	9–10	6–10
EA	0.84	0.97	1.11	1.08	1.14	0.93	0.89	0.95	0.87	1.14
Bar	1–6	2–7	3–8	4–9	5–10	1–7	2–8	3–9	4–10	5–6
EA	0.92	1.18	1.06	0.89	1.12	0.83	1.04	1.18	1.09	1.08

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geometry and stiffness is found, and second, the measure of symmetry is determined by comparing the eigenvalues of the stiffness matrix of the reference and the analyzed structure. The main achievement of the method is a novel procedure of finding the reference structure: it is done based on energy principles, and the eigenvalues of the stiffness matrix as metrics to be compared have been chosen because they possess information regarding both the geometry and the stiffness properties. We emphasize that our symmetry measure, like all other measures of symmetry, is not objective; there might be other ways to determine the reference structure and different metrics can also be chosen for comparison.

The method seems to be applicable in a further investigation of the shape evolution of (nearly) symmetric objects that can be modeled as truss structures under different types of loading. A symmetry measure of general scope is fundamental in understanding the shape and structural evolution processes more deeply from the point of view of symmetry, namely if a structure with disturbed symmetry tends to reach a more or even less symmetric state under given conditions.

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