

GEOID COMPUTATION BY ANALYTICAL AND DISCRETE SPECTRUM

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Abstract

For the computation of geoid undulation applying gravity anomalies, FFT technique is used. The effects of the analytical and the discrete spectra of kernel functions and that of zero-padding in FFT are studied. The conclusion obtained from the practical computation is that the discrete spectrum seems to be better than the analytical one. Also, to eliminate the effect of circular convolution, 100 % zero-padding should be used. The computed geoidal undulations were compared with 45 GPS/Levelling stations. Results show that the differences between obtained geoidal undulations and GPS/Levelling have a standard deviation at 35 cm.

Keywords: geoid, zero-padding, circular convolution.

1. Introduction

Gravity is caused primarily by the gravitational attraction of masses inside the earth and the centrifugal acceleration resulting from the rotation of the earth about its spin axis. It is a vectorial quantity; that is, it has a magnitude and a direction.

Mathematically, gravity is the gradient of the scalar gravity potential. A particular equipotential surface (that is, a surface on which the potential is everywhere equal to the same numerical value) of the gravity potential, which closely approximates the mean surface of the oceans, is the so-called geoid. This geoid serves as the reference surface for topographic heights, for example, as they are shown on maps. The geoid is the surface of zero height.

The determination of the geoid has been one of the main interests of geodesists in the last few decades. In recent years, FFT has been widely used in the field of physical geodetic applications, the computation of geoid undulations and terrain corrections, because it can efficiently handle huge amounts of gridded data and gives good results on all grid points simultaneously.

But in FFT processing we have some problems, such as aliasing, leakage, the singularity of kernel functions at the origin, the proper handling of

mean and point data, phase shifting, circular convolution and, sometimes, planar approximation that affect the accuracy of the results that are common to all methods.

In fact, phase shifting, circular convolution and planar approximation are unique to spectral methods. Phase shifting can easily be corrected by using the time/space shifting property of the Fourier Transform. The effect of planar approximation is not significant and is truly negligible in most cases. The only problem we have to pay attention to is the circular convolution (SIDERIS, 1993).

To handle the problem of the singularity of the kernel functions at the computation point, SIDERIS (1985, 1993) proposed the use of a modified kernel function and suggested to use analytical spectra. To prevent the effect of circular convolution, zero-padding must be used (BRIGHAM, 1988). To treat the case where the input data are mean values, SIDERIS and TZI-AVOS (1988) suggested the use of 2-D sinc function in the frequency domain.

In this study, it was applied a set of new formulae which is proposed by Sideris for computing geoid undulations by using FFT when the input is mean values, and give some numerical results obtained to test the effect of analytical and discrete spectra on the computation of geoid undulations.

2. FFT Evaluation with Point and Mean Data

2.1 Stokes' Integral Formula

Stokes' boundary value problem is the gravimetric determination of the geoid, which is the equipotential surface of the Earth's gravity field corresponding to the mean sea level. Stokes' problem deals with the determination of a potential, harmonic outside the masses, from gravity anomalies given everywhere on the geoid surface.

The geoid undulations N are obtained from anomalous potential at every point on the geoid by Bruns' formula.

$$N = \frac{T}{\gamma} = \frac{R}{4\pi\gamma} \iint_{\sigma} \Delta g S(\psi) d\sigma . \quad (1)$$

The use of Eq. (1) requires gravity anomalies all over the Earth for the computation of a single geoid undulation. Obviously, this is not practical to say at least and thus, in practice, some modifications of the equations are necessary.

The contribution of gravity anomalies can be computed in a variety of ways. Here the planar approximation of Stokes' integral is briefly explained.

The Stokes' integral, in its planar form, can be transformed to the frequency domain using gridded gravity anomalies over a grid element (SIDERIS, 1993; TZIAVOS, 1993; SIDERIS and TZIAVOS, 1988). These gravity anomalies are treated as either mean or point approximation of Stokes' integral. In planar approximation over an area E and using rectangular coordinates, the Stokes' integral takes the form

$$N(x_p, y_p) = \frac{1}{2\pi\gamma} \iint_E \Delta g(x, y) \frac{1}{\sqrt{(x_p - x)^2 + (y_p - y)^2}} dx dy, \quad (2)$$

where γ is the mean gravity value and Δg are the point gravity anomalies. Note that $[(x_p - x)^2 + (y_p - y)^2]^{-1/2}$ is an approximation to only the term of the spherical Stokes kernel function. We can express (2) as:

$$N(x_p, y_p) = \frac{1}{2\pi\gamma} \Delta g(x_p, y_p)^* l_N(x_p, y_p), \quad (3)$$

where $*$ denotes the 2D convolution of two functions and $l_N(x, y) = (x^2 + y^2)^{-1/2}$ is the integral kernel function.

2.1.1 Gridded Point Gravity Anomalies

Using $M \times N$ gridded point gravity anomalies with grid spacing Δx , Δy in the x and y directions, respectively, the geoid undulation at point (x, y) can be evaluated by the following discrete convolution (SIDERIS, 1994):

$$N(x_k, y_l) = \frac{1}{2\pi\gamma} \sum_{k=1}^{M-1} \sum_{l=0}^{N-1} \Delta g(x_i, y_j) l_N(x_k - x_i, y_l - y_j) \Delta x \Delta y, \quad (4)$$

where

$$l_N(x_k - x_i, y_l - y_j) = \frac{1}{\sqrt{(x_k - x_i)^2 + (y_l - y_j)^2}}, \quad x_p \neq x \text{ and } y_p \neq y. \quad (5)$$

To account for the singularity of l_N , the kernel is set to zero at the origin and the contribution to geoid undulation of the gravity anomaly at the computation point can be evaluated separately. Approximately, this contribution is (HEISKANEN and MORITZ, 1967 and SCHWARZ et al., 1990):

$$\delta N(x_k, y_l) \approx \frac{\sqrt{\Delta x \Delta y}}{\sqrt{\pi\gamma}} \Delta g(x_k, y_l). \quad (6)$$

Making use of the spectra of Δg and l_N , i.e., $\Delta G(u, v)$ and $L(u, v)$, geoid undulations can be evaluated then by FFT as follows:

$$\begin{aligned} N(x_k, y_l) &= \frac{1}{2\pi\gamma} F^{-1} \{ \Delta G(u_m, v_n) L_N(u_m, v_n) \} \\ &= \frac{1}{2\pi\gamma} F^{-1} \left\{ \Delta G(u_m, v_n) \frac{1}{q} \right\}, \end{aligned} \quad (7)$$

where F and F^{-1} denote the direct and inverse Fourier transform operators, respectively, and u_m, v_n are the frequencies corresponding to x_k, y_l . ΔG has to be computed by the Discrete Fourier transform (DFT).

$$\begin{aligned} \Delta G(u_m, v_n) &= F \{ \Delta g(x_k, y_l) \} \\ &= \Delta x \Delta y \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \Delta g(x_k, y_l) e^{-2\pi i \left(\frac{mk}{M} + \frac{nl}{N} \right)}. \end{aligned} \quad (8)$$

$L(u, v)$ can be evaluated either by the DFT

$$\begin{aligned} L_N(u_m, v_n) &= F \{ l_N(x_k, y_l) \} \\ &= \Delta x \Delta y \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} l_N(x_k, y_l) e^{-2\pi i \left(\frac{mk}{M} + \frac{nl}{N} \right)} \end{aligned} \quad (9)$$

or by the Continuous Fourier Transform (CFT)

$$\begin{aligned} L_N(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l_N(x_k, y_l) e^{-2\pi i (ux + vy)} dx dy \\ &= \frac{1}{(u^2 + v^2)^{1/2}} = \frac{1}{q}. \end{aligned} \quad (10)$$

where q is the radial frequency, and $L_N(u, v)$ defined by Eq. (10) is called the analytically-defined spectrum of Stokes' kernel.

2.1.2 Mean Gravity Anomalies

If the input data are $M \times N$ gridded mean gravity anomalies $\overline{\Delta g}$, the planar Stokes formula can be formulated as

$$N(x_k, y_l) = \frac{1}{2\pi\gamma} \sum_{k=1}^{M-1} \sum_{l=0}^{N-1} \overline{\Delta g}(x_i, y_j) \overline{I}_N(x_k - x_i, y_l - y_j), \quad (11)$$

$$\overline{I}_N(x_k, y_l) = \int_{x_k - \Delta x/2}^{x_k + \Delta x/2} \int_{y_l - \Delta y/2}^{y_l + \Delta y/2} \frac{1}{\sqrt{x^2 + y^2}} dx dy \quad (12)$$

$$\begin{aligned} &= x \ln(y + \sqrt{x^2 + y^2}) + \\ &+ y \ln(x + \sqrt{x^2 + y^2}) \Big|_{x_k - \Delta x/2}^{x_k + \Delta x/2} \Big|_{y_l - \Delta y/2}^{y_l + \Delta y/2}. \end{aligned} \quad (13)$$

Eq. (11) can also be efficiently evaluated via FFT, i.e.

$$\begin{aligned} N(x_k, y_l) &= \frac{1}{2\pi\gamma} \{F^{-1}\{F\{\overline{\Delta g}(x_k, y_l)\}F\{\overline{I}_N(x_k, y_l)\}\}\} \\ &= \frac{1}{2\pi\gamma} F^{-1}\{\overline{\Delta G}(u_m, v_n) \overline{L}_N(u_m, v_n)\}, \end{aligned} \quad (14)$$

where

$$\overline{\Delta G}(u_m, v_n) = F\{\overline{\Delta g}(i, j)\}, \quad (15)$$

$$\overline{L}_N(u_m, v_n) = F\{\overline{I}_N(i, j)\}. \quad (16)$$

To distinguish between the spectra defined by Eqs. (5) and (13), we call \overline{L}_N the mean Stokes' kernel spectrum.

It is worth mentioning here that a simple 2D sinc function can also be used to relate the spectrum of data considered as either representing point values at the nodes of a grid or mean values in the area of a grid element (SIDERIS and TZIAVOS, 1988).

By using this technique, if the input gravity anomalies are mean values, the geoid undulations can be expressed as

$$N(x_k, y_l) = \frac{1}{2\pi\gamma} F^{-1}\left\{\sin c\left(\frac{m}{M}\right) \sin c\left(\frac{n}{M}\right) \overline{\Delta G}(u_m, v_n) L_n(u_m, v_n)\right\}, \quad (17)$$

where $\overline{\Delta G}$ is the spectrum of mean gravity anomalies as in Eq. (15), L_N is the spectrum of the kernel function as expressed in Eq. (5) and the sinc function is defined as

$$\sin c(t) = \frac{\sin(\pi t)}{\pi t}. \quad (18)$$

By comparing Eq. (14) with (17), we see that

$$\overline{L_N}(u_m, v_n) = \sin c\left(\frac{m}{M}\right) \sin c\left(\frac{n}{N}\right) L_N(u_m, v_n). \quad (19)$$

Eq. (19) indicates that the Fourier Transform of mean kernel function can be obtained by multiplying the Fourier Transform of ‘point’ kernel function, obtained either analytically or by the discrete transform, by a 2D sinc function.

2.2 Analytical versus Discrete Spectrum

Although the analytically-defined spectrum has some advantages compared with the discrete one, such as no DFT required for its evaluation and no effect of leakage and aliasing, it is not suitable for the computation of discrete convolution. If we use the analytically-defined spectrum, for example, in the computation of geoid undulations, Eq. (6) becomes (SIDERIS, 1994)

$$N(x, y) = \frac{1}{2\pi\gamma} F^{-1}\{\Delta G(m, n) L_N(m, n)\}, \quad (20)$$

where

$$\Delta G(m, n) = F\{\Delta g(k, l) = \Delta x \Delta y \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \Delta g(k, l) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})}\}, \quad (21)$$

$$L_N(m, n) = F\{l_N(k, l)\} = \Delta x \Delta y \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} l_N e^{-2\pi i(\frac{mk}{M} + \frac{nl}{N})}. \quad (22)$$

The spectrum L_N can be evaluated either by DFT or by the Continuous Fourier Transform (CFT):

$$\begin{aligned} L_N(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l_N(x, y) \cdot e^{2\pi j(ux+uv)} dx dy \\ &= \frac{1}{(u^2 + v^2)^{-1/2}} = \frac{1}{q}. \end{aligned} \quad (23)$$

L_N defined by Eq. (23) is the analytically-defined spectrum.

Considering the symmetry of the kernel function $l_N(x, y)$, we can rewrite Eq. (22) as follows:

$$\begin{aligned} N(x, y) &= \frac{1}{2\pi\gamma} F^{-1}\{\Delta G(m, n) L_N^1(u, v)\} \\ &\quad + \frac{1}{2\pi\gamma} F^{-1}\{\Delta G(m, n) L_N^2(u, v)\}, \end{aligned} \quad (24)$$

where

$$L_N^1(u, v) = \int_{-T_x/2}^{T_x/2} \int_{-T_y/2}^{T_y/2} l_N(x, y) e^{-2\pi j(ux+vy)} dx dy ,$$

$$L_N^2(u, v) = 2 \int_{T_x/2}^{\infty} \int_{T_y/2}^{\infty} l_N(x, y) e^{-2\pi j(ux+vy)} dx dy ,$$

and T_x, T_y are the dimensions of the area E . If the grid interval is small enough and the effect of aliasing is negligible when discretizing L_N and Δg , the first term of the right-hand side in Eq. (24) is equal to the discrete convolution, i.e., Eqs. (7), (9), and the second term of the right-hand side are the errors due to the analytically defined spectra used.

2.3 The Proper Use of Zero-Padding

There are some problems that affect the accuracy of the results and are usually believed to be unique to FFT-based methods. Actually, many of these problems, such as aliasing, leakage, the singularity of kernel functions at the origin, and the proper handling of mean and point data, are common to all methods using the same data. The problems that are indeed unique to spectral methods only include phase shifting, edge effect or circular convolution and, sometimes, planar approximation (SIDERIS, 1994). In fact, phase shifting can be very easily corrected by using the time/space shifting property of the Fourier transform. The effect of planar approximation is not significant in most applications. Nevertheless, FFT can also be used on the sphere.

The use of windowing and the rejection of results in the edges of the grid are regularly applied to minimize this problem. It is possible, however, to eliminate this problem completely by applying (at least 100 %) zero-padding to the two data sets before evaluating the convolution.

3. Practical Computation

3.1 Used Data

A number of data includes information belonging to various wavelength signals of the total spectrum of the gravity field. Therefore the appropriate combination of the data is very important for the precise resolution of the whole gravity spectrum. The geopotential model as a reference sur-

face, gravity data and Digital Terrain Model can be selected as a suitable combination of the data for geoid computation.

1. Geopotential Model

One of the most important kinds of information for gravity field approximation today is a good reference field supplied by a geopotential model (ZHAO S., 1990). Geopotential models are generally used as reference surface for the local geoid computation. Removing the contribution of a high resolution geopotential model from gravity field in a local area makes possible to determine precisely the medium and the short wavelength signals to the geoid undulation. For the available models, the medium and short wavelength coefficients have large errors and are not as reliable as the coefficients of the long wavelength (SIDERIS and SCHWARZ, 1986; SCHWARZ et al. 1987). Although errors of the long wavelength coefficients cause the biased and tilted geoid solutions, it seems possible to eliminate those effects using additional space data or a transformation method (FORSBERG and MADSON, 1990). To determine the optimum reference field, from several tests OSU91A model up to degree and order 360 fits better for test area (H.S. YUN and J. ADAM, 1994).

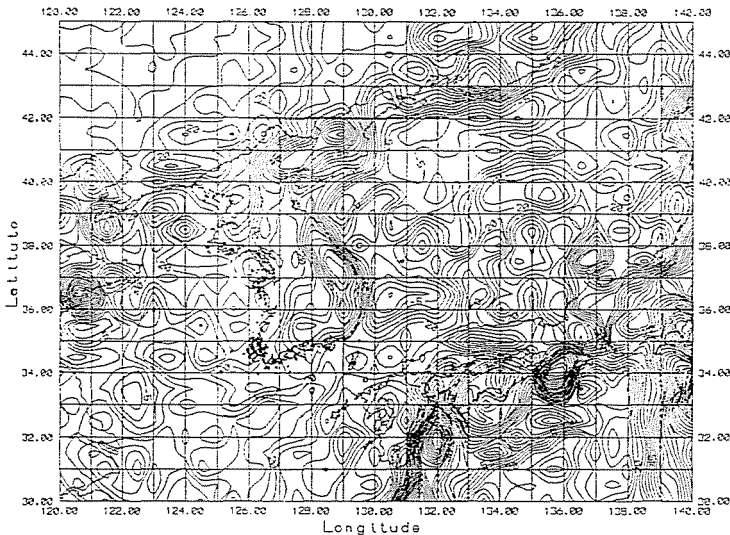


Fig. 1. The reference anomaly map from geopotential model OSU91A up to 360 degrees (Contour interval : 5 mGal)

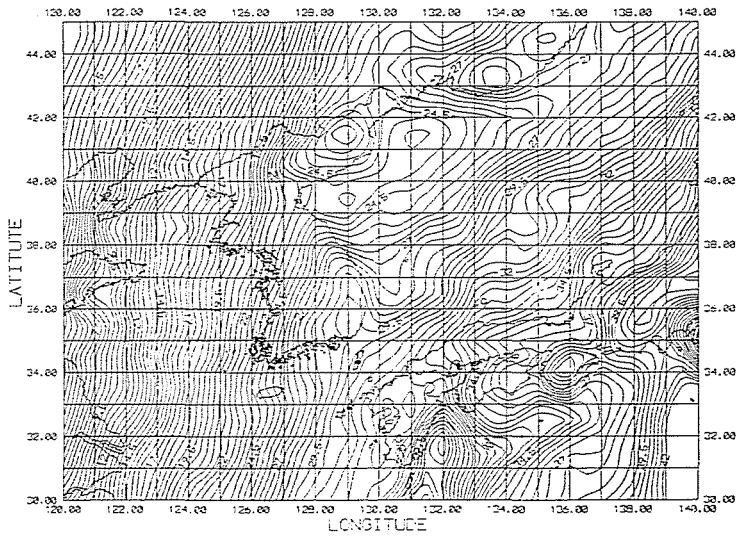


Fig. 2. The geoid undulation map from geopotential model OSU91A up to 360 degrees (Contour interval : 0.5 m)

Fig. 1 is the reference anomaly map for the Korean Peninsula area with harmonic expansions of up to 360 degrees. Fig. 2 is the geoid undulation map computed from the geopotential model up to 360 degrees.

2. Gravity Data

The prepared gravity data consists of 7259 point gravity values within the following boundaries [$33^\circ < \varphi < 39^\circ$, $125^\circ < \lambda < 130^\circ$] as shown in Fig. 3. All these gravity data were used to predict a $5' \times 5'$ point free-air anomaly data (4320 gravity values, 60×72 grid). The gridding was done by minimum curvature method and using the commercial software SUFER. A contour map showing the free-air anomalies covering part of the study area is shown in Fig. 4. The reference frame of these gravity data is the GRS80. For the checking of possible errors in the data, a precheck is performed on contour map with a visual inspection. There was no possible error in the data. Some statistical values of the measured data and free-air anomalies are given in Table 1.

Table 1
Statistics of raw gravity anomalies and free-air anomalies

Data	Max.	Min.	Mean	S.D.
Raw gravity data (mGal)	176.79	-25.21	22.72	23.26
Free-air anomaly (mGal)	131.80	-11.56	26.22	14.61

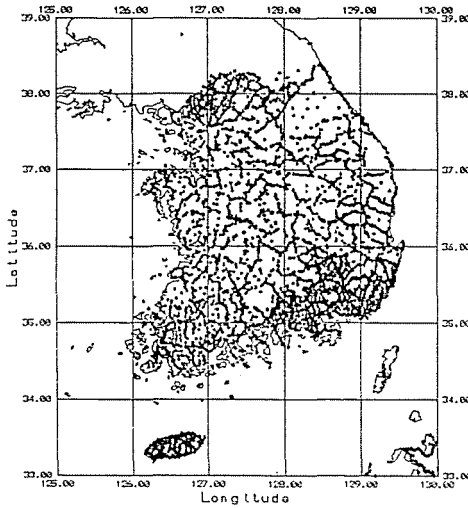


Fig. 3. The distribution of gravity data

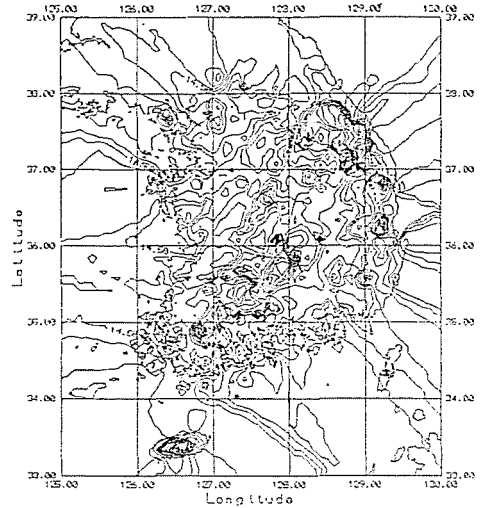


Fig. 4. Free-air anomaly map based on $5' \times 5'$ grid data (CI=5 mGal)

In *Fig. 5*, the residual gravity anomalies are depicted and in *Table 2* the statistics of the residual gravity anomalies is given.

Table 2
Statistics of the residual gravity anomalies

Data (Δg)	Max.	Min.	Mean	S. D.
Δg -OSU91A	131.84	-11.56	3.61	18.21

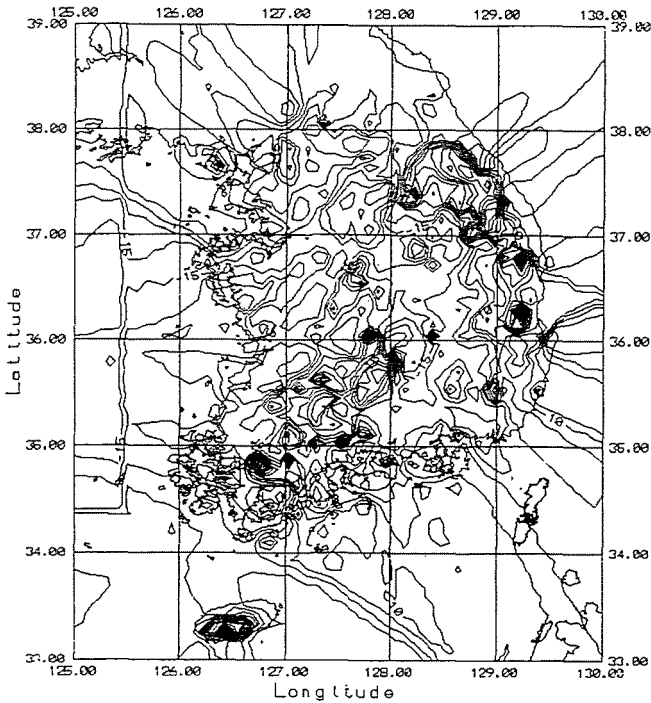


Fig. 5. Residual gravity anomalies

3.2 The Computation of Residual Geoid Undulations by FFT

For the computation of residual geoid undulations, analytical and discrete defined spectra were applied. The only problem that in FFT one has to be pay attention to is the circular convolution. To prevent the effect of circular convolution, the zero-padding applied by appending 100 % zeros (50 % to each side) around the gravity anomalies. The kernel function is evaluated by discrete Fourier transform (DFT) and analytical Fourier transform (AFT). Table 3 shows that when the comparisons are made in the test area, the residual geoid undulation computed with the discrete spectra of the kernel functions is superior to the analytically-defined ones and the discrete spectra with zero-padding give optimal results in the computation of the residual geoidal undulations.

Figs. 6 and 7 show the residual geoid undulations computed by analytical spectra of kernel function and the discrete one.

Table 3
Statistics of the residual geoid undulation

	Spectrum	Max.	Min.	Mean	S.D.
No Applied	Analytical Spectrum	1.010	-2.328	-0.903	0.732
Zero-Padding	Discrete Spectrum	0.646	-2.223	-0.918	0.701
100 % Applied	Analytical Spectrum	0.746	-1.332	-0.530	0.410
Zero-Padding	Discrete Spectrum	0.476	-1.258	-0.510	0.390

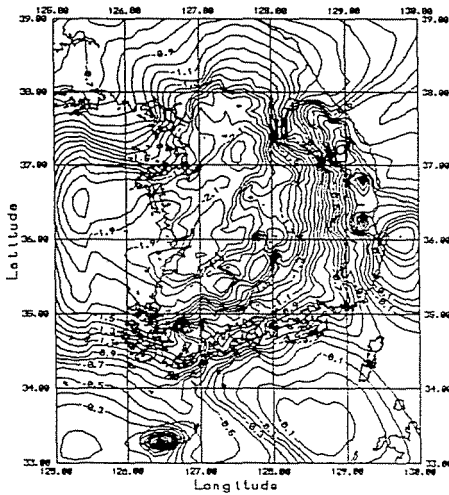


Fig. 6. The residual geoid undulation computed by analytical spectrum without zero-padding (CI=0.1 m)

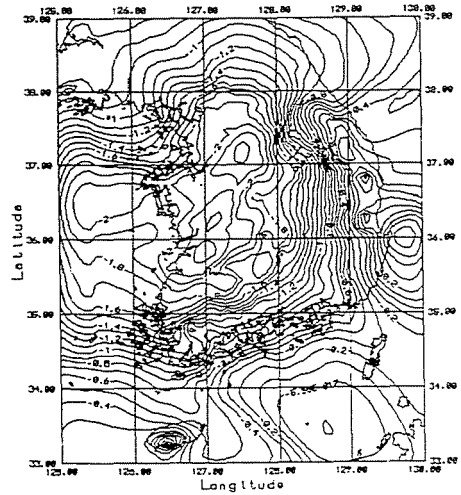


Fig. 7. The residual geoid undulation computed by discrete spectrum without zero-padding (CI=0.1 m)

4. Discussion and Conclusions

The main objective of this study was the comparison of the discrete spectrum between analytical ones in computing the residual geoid undulation. The main advantage of spectral methods is that they can efficiently handle large amounts of gridded data and give results on all grid points simultaneously, which has made them a standard and indispensable tool for geoid computations.

The numerical test shows that the discrete spectra of the kernel function should be used instead of the analytically-defined one in the plane in

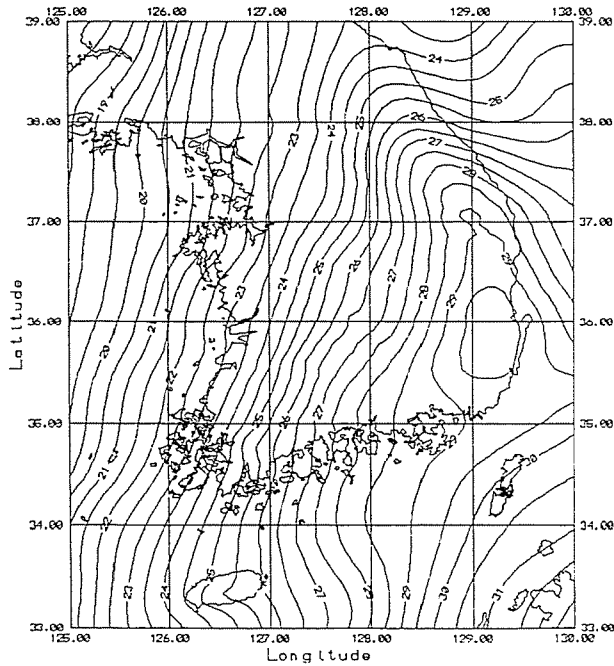


Fig. 10. Complete gravimetric geoid solution

the S. D. of the differences between GPS/Levelling derived heights and corresponding heights derived from DFT with 100 % zero-padding reached a value of 35 cm.

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