PERIODICA POLYTECHNICA SER. CIVIL ENG. VOL. 39, NO. 1, PP. 37–75 (1995)

TEST INTERPOLATION OF DEFLECTION OF THE VERTICAL IN HUNGARY BASED ON GRAVITY GRADIENTS

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Received: January 6, 1995

Abstract

A program package for computers was developed to test empirical methods of interpolation of deflection of the vertical which can be used to determine deflections of the vertical by any method of interpolation either along triangulation chains or in networks covering arbitrary large areas. In the course of our test computations in Hungary first we compared different empirical methods of interpolation then we tried to get an answer to the question whether the reliability of interpolated data can be increased by introducing appropriate weighting. Another important object of our investigations was to determine optimal geometrical arrangement for interpolation networks.

Keywords: deflection of the vertical, torsion balance measurement, gravity gradients.

1. Computer Program Package for Interpolation

The computer program package developed by us is able to determine deflections of the vertical based on torsion balance measurements either along triangulation chains or in networks covering arbitrary areas using any of the interpolation methods fully described in (VÖLGYESI, 1993). It can plot the interpolation network and vector diagram of interpolated deflections of the vertical, calculate geoid heights by astronomic levelling and also plot either perspective or isoline map of geoid for the area.

The operation of the program package developed for personal computers is visualized in *Fig. 1*.

A *catalogue file* has to be created as a first step of the computing process which contains all known data of torsion balance measurements within the area to be processed (codes of measurement points, co-ordinates, second derivatives W_Δ, W_{xy}), and the catalogue should contain the known values of deflection of the vertical for astrogeodetic points available.

Besides the catalogue file another input file should be made for the program package which contains *data of the interpolation network* (the point codes of point pairs forming sides of triangles in a triangulation net-

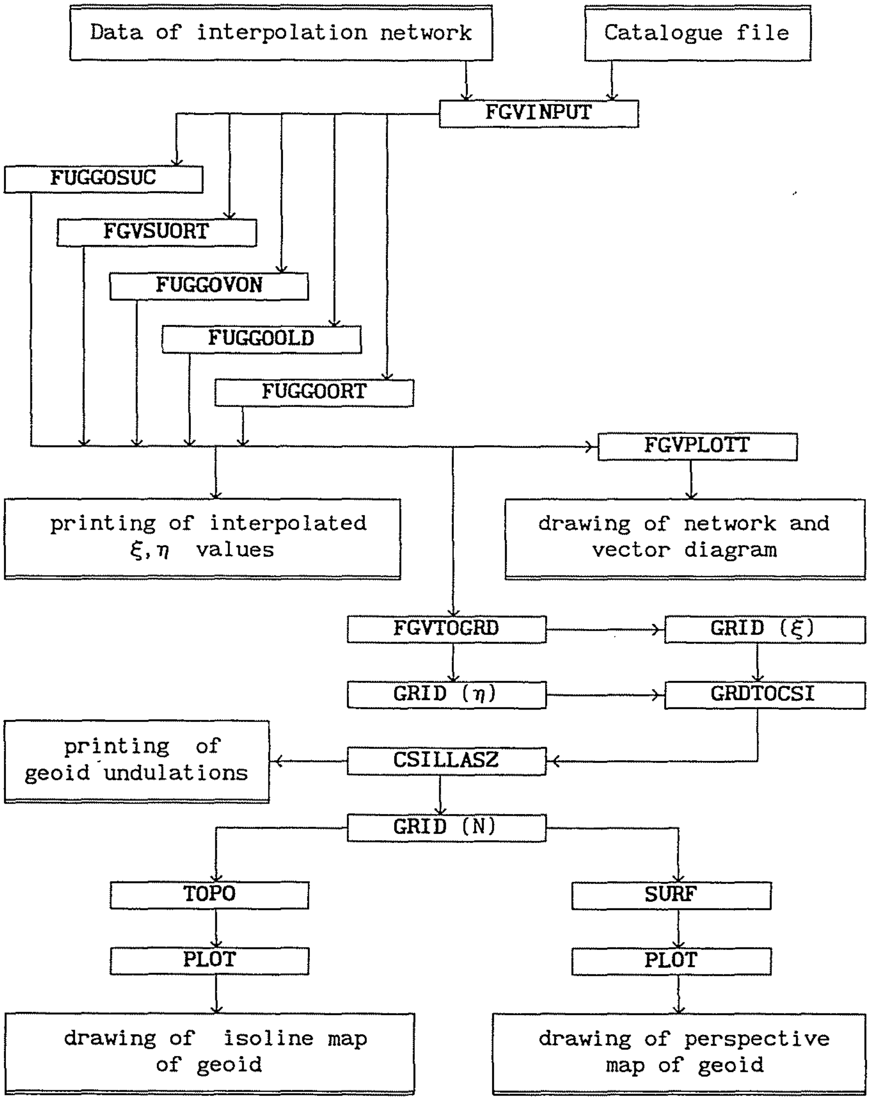


Fig. 1.

work should be given in pairs and those points should be noted where components of deflection of the vertical are known).

According to the process visualized in *Fig. 1* the first program of the package, named FGVINPUT, selects from catalogue file the data needed for process, using file containing data of the interpolation network, and depending on the interpolation method to be used for further computation it produces input data set with appropriate format for certain (FUGGOSUC, FGVSUORT, FUGGOVON, FUGGOOLD, FUGGOORT) programs. Here we can choose for example whether to perform interpolation with unreduced or topographically reduced torsion balance measurements; it can be given which point to choose as origin of the local co-ordinate frame and here it should be fixed which interpolation method will be used for further computations. According to *Fig. 1* we can choose from the following five possibilities:

Program FUGGOSUC determines along arbitrary interpolation chains between two astrogeodetic points from torsion balance measurements direct values of components of deflection of the vertical in points of the *chain* by the successive elimination method discussed in (VÖLGYESI, 1993). Input data for program are: co-ordinates of the points of interpolation network; at each point either directly measured torsion balance W_Δ and W_{xy} second derivatives or ΔW_Δ and ΔW_{xy} values that are computed using these derivatives through the formula $\Delta W = W - U$; as well as known ξ_1, η_1 and ξ_n, η_n deflections of the vertical at starting and closing points of chain. Using those data the program computes n_{ij} length and α_{ij} azimuth of each side of the chain; by using formula

$$T_{ij} = \frac{n_{ij}}{4g}[(\Delta W_{\Delta_i} + \Delta W_{\Delta_j}) \sin 2\alpha_{ij} + (\Delta W_{xy_i} + \Delta W_{xy_j}) 2 \cos 2\alpha_{ij}] \quad (1)$$

calculates T_{ij} values for each triangle side, its variances and covariances; and for these sides

$$\Delta\xi_{ji} \sin \alpha_{ij} - \Delta\eta_{ji} \cos \alpha_{ij} = T_{ij} \quad (2)$$

expression is utilized to get the $\Delta\xi_{ij}$, $\Delta\eta_{ij}$ component differences; and so using these values and the input data unknown ξ, η deflections of the vertical and their variances are computed at points of the interpolation network. Finally output files are generated by the program either to print out results or for an optional post processing.

By the program FGVSUORT the $\Delta\xi$, $\Delta\eta$ component differences of deflections of the vertical between points of an interpolation *chain* between two astrogeodetic points can be determined using given torsion balance measurements by the method of matrix orthogonalization. In input data of the program are completely identical with input data of the program FUGGOSUC and is similar to the FUGGOSUC program in that it can only be used to compute interpolation chains for which deflections of the vertical are given at its starting and closing points.

Program FUGGOVON can be applied to compute not only simple interpolation chains but networks of arbitrary shape. Direct values of ξ, η

components of deflections of the vertical at points of the network are determined by the program based on given torsion balance measurements by giving consideration to fixed ξ, η values at astrogeodetic points of arbitrary number and distribution. Input data for program are: co-ordinates of points of the interpolation network, W_Δ and W_{xy} second derivatives which are measured directly by torsion balance or ΔW_Δ and ΔW_{xy} values computed from them; known ξ_0 and η_0 deflections of the vertical at points of constraint of arbitrary number and finally sides of interpolation network orderly (given by pairs of point numbers). By using these data it can be computed first by the program the length and azimuth of each side then the coefficient matrix and vector of constant terms of observation equations are set up, unknown ξ, η values and their standard deviations are determined and finally output files are produced both for printing out the results and for an optional post processing.

Program FUGGOOLD is a modified version of program FUGGOVON. Difference between the two programs lies in the fact that in the input data of FUGGOOLD sides of the interpolation network are not included because these are generated automatically by the program FUGGOOLD. Instead of the sides a maximum distance should be given very carefully; and neighbouring points (torsion balance measurement sites) are searched for by the program within this maximum distance where network sides can be formed. This program can very advantageously be applied to such 'homogenic' areas where torsion balance measurement stations are lying from each other at an approximately equal distance and sides of nearly the same length will be resulted.

FUGGOORT is considerably a more capable program than the preceding four ones and it can handle very large matrices and besides this it is the most fast and accurate among interpolation programs we programmed. FUGGOORT can be applied to compute interpolation networks of arbitrary shape. Direct values of ξ, η components of deflections of the vertical can be determined through matrix orthogonalization adjustment process from given torsion balance measurements for network points by considering fixed ξ, η values at astrogeodetic points of arbitrary number and distribution. Input data of program are identical with that of FUGGOVON. FUGGOORT can handle very large matrices using a programming idea, hence interpolation networks containing several hundred points can also be computed and adjusted simply by its use. Moreover, this algorithm may also be used very well to solve any adjustment task of surveying, the coefficient matrix of which is large and sparse (containing many zero elements), because the coefficient matrix of observation equations can be stored up for the matrix orthogonalization adjustment process by saving much storage space.

The preceding five interpolation programs create data files of two types: the first displays computation results in the form of a table which can easily be viewed while the second is for purpose of post processing. During post processing – according to the sketch in *Fig. 1* – we have the opportunity either to plot the interpolation network and interpolated deflections of the vertical or to plot detailed geoid map of the area by using computed deflections of the vertical.

The interpolation network and vector diagram of interpolated deflections of the vertical are displayed on the screen by program FGVPLOTT where the plot on screen can be directed from optionally to a printer or plotter if it is required. It can be selected from the menu system of FGVPLOTT, among others, whether to draw on plot point numbers of network points, to connect network points forming sides and whether to plot (and on what scale) the vector diagram of interpolated deflections of the vertical. (It should be noted, however, that by this program without change one is able to plot any kind of a geodetic network).

If a detailed geoid map of an area from interpolated deflections of the vertical is required, the way indicated in *Fig. 1* should be followed. Detailed geoid map of the area of the interpolation network is computed by program CSILLASZ by astronomical levelling. Input data for program CSILLASZ are computed by utility program FGVTOGRD, GRID and GRDTOCSI stepwise.

2. Data of Test Computations

The area surrounding *Cegléd*, as can be seen in *Fig. 2*, extended over some 1200 km^2 and well-measured by torsion balance was chosen for the purpose of our test computation.

Distances between astrogeodetic points and density of the torsion balance stations among them in our test area correspond to average flatland conditions in Hungary as can be seen in *Fig. 2*; however, in the upper area of the figure near *Pilis* and *Albertirsa* it is apparent that torsion balance stations were located more densely as it was usual in the Hungarian Plain. This can be found along the southern extension area of *Gödöllő Hills* where the change of gradients and second derivatives, that can be measured by torsion balance, is greater.

2.1 Co-ordinates of Interpolation Points

For our test computations in Hungary co-ordinates of torsion balance measurement stations in Budapest Stereographic System were available with the reliability of the order of m. This was completely enough for us be-

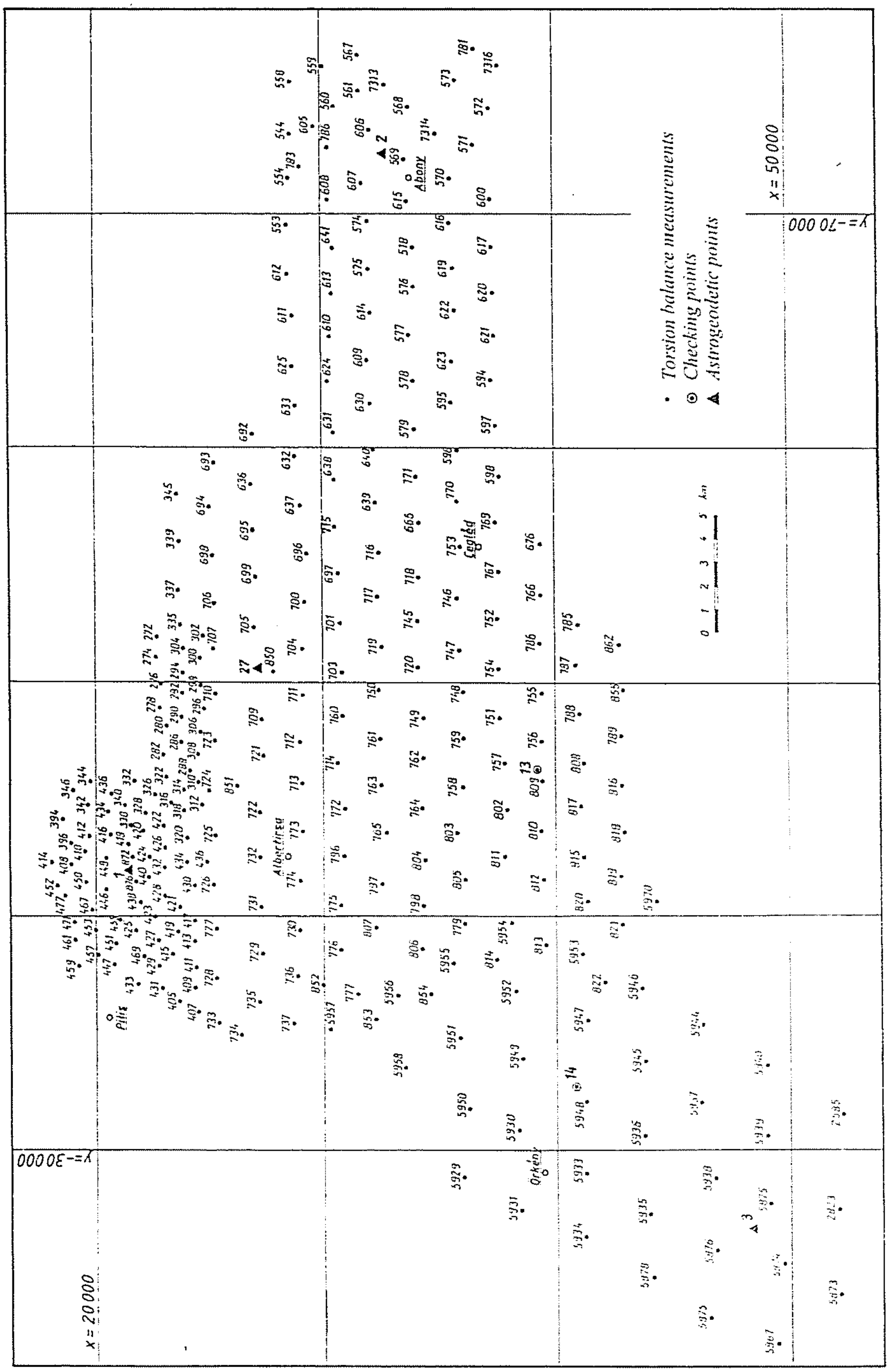

Torsion balance measurements
Checking points
Astrogeodetic points
x = 20 000
x = 50 000
y = −30 000
y = −70 000
0 1 2 3 4 5 km
Albertirsa
Cegléd
Abony
Pilis
Örkény

Fig. 2.

cause ±50 m error in co-ordinates used for vertical interpolation of deflection causes an error in the components of vertical deflections of only one hundredth of a second of arc (BADEKAS – MUELLER, 1967).

An arbitrary internal point in the given area is suitable to be chosen as zero point of the co-ordinate frame (VÖLGYESI, 1993).

To this local frame the co-ordinate transformation

$$x' = (x - x_0)\cos\mu + (y - y_0)\sin\mu \ ,$$
$$y' = -(x - x_0)\sin\mu + (y - y_0)\cos\mu$$

can be used to convert points; – where μ denotes grid convergence at zero point of the new (local) frame, x_0 and y_0 are co-ordinates of zero point of the local frame in the old frame.

2.2 Torsion Balance Measurements

In our country torsion balance was used to measure a good many number of stations; very large part of the area of Hungary – first of all flatland and hilly areas of moderate height – were covered with network of torsion balance stations. As far as torsion balance measurements are considered Hungary is the most well-measured country in the world. Earlier torsion balance measurements were carried out mainly for the purpose of prospecting and for this end only W_{zx} and W_{zy} horizontal gradients were processed. Nevertheless W_Δ and W_{xy} second derivatives are known at each measurement station also, which can be used in geodesy. Torsion balance measurement data are available for the area of Hungary in Eötvös Loránd Geophysical Institute.

Torsion balance measurement points' location in our test area is displayed in *Fig. 2*. Stations were not located with the same density because – as we have mentioned – observations were carried out with a greater density of points in 'disturbed' areas of rugged topography.

To evaluate deflections of the vertical so-called anomalies of

$$\Delta W_\Delta = W_\Delta - U_\Delta \ ,$$
$$\Delta W_{xy} = W_{xy} - U_{xy}$$

should be used instead of measured W_Δ, W_{xy} values, where U_Δ and U_{xy} denote normal values of the second derivatives.

The above mentioned ΔW_Δ and ΔW_{xy} anomalies of second derivatives were determined in the test area for each station and isoline maps in

Fig. 3 and *Fig. 4* were plotted using these values. Isoline values plotted are in units of $10^{-9}\mathrm{s}^2$, that is 1 E (Eötvös).

The following accuracies are characteristics of torsion balance measurement data, with the previous notations,

$$\begin{aligned} \mu^2_{W_\Delta} &\approx 1.7 \,, \\ \mu^2_{W_{xy}} &\approx 1.5 \,, \\ C_{W_\Delta, W_{xy}} &= 0 \end{aligned}$$

based on the detailed studies of (BIRÓ – FÖLDVÁRINÉ – HAZAY – HOMORÓDI, 1965) and (BADEKAS, 1967), that is standard error of the W_Δ values is ± 1.3 E, of the W_{xy} values is ± 1.2 E and the correlation coefficient can be treated as zero.

2.3 Corrections to Torsion Balance Measurements

Second derivatives measured by torsion balance include many kinds of effects. For further processing – depending on the task torsion balance measurements are to be used for – different effects have to be considered and corresponding corrections may be applied to ΔW_Δ and ΔW_{xy} second derivatives.

Mainly neighbouring topography and its density inhomogeneities have a considerable effect on the results of torsion balance measurements. It is usual to compute the effect of neighbourhood in two or three steps, e.g.: (BADEKAS – MUELLER, 1967). There is not a uniform agreement as to the limits of computation, – we deal with the corrections according to the following division:

1. The effect of immediate neighbourhood of the measurement point up to 100 m – the so-called *terrain correction* (δW^s),
2. the effect of masses in the range between 100 and 5000 m – the so-called *topographic correction* (δW^t),
3. the effect of masses beyond range of 5000 m the so-called cartographic correction (δW^c).

Spirit levelling height data of the immediate neighbourhood are needed to determine *terrain correction*. The ground is usually made flat inside a circle of two–three meters of diameter around the torsion balance measurement point and it is usual to level symmetrically along 8 directions at 1.5, 2, 3, 5, 10, 20, 30, 40, 50 m distance from the measurement point. Levelling beyond the 50 m range is only for a greatly rugged terrain to a maximum of 100 m distance. Terrain correction is usually determined from levelling data by using graphs or tables.

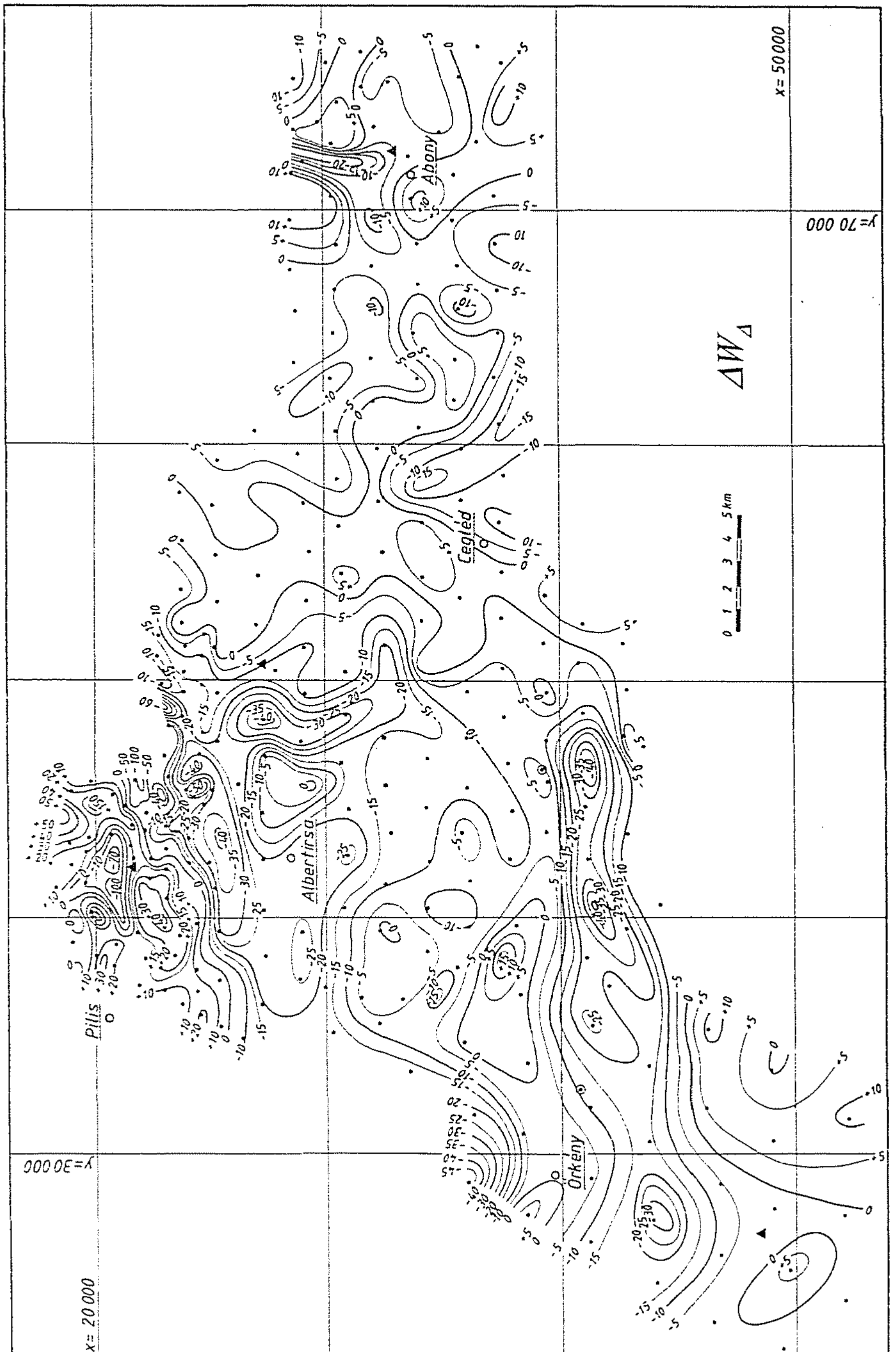
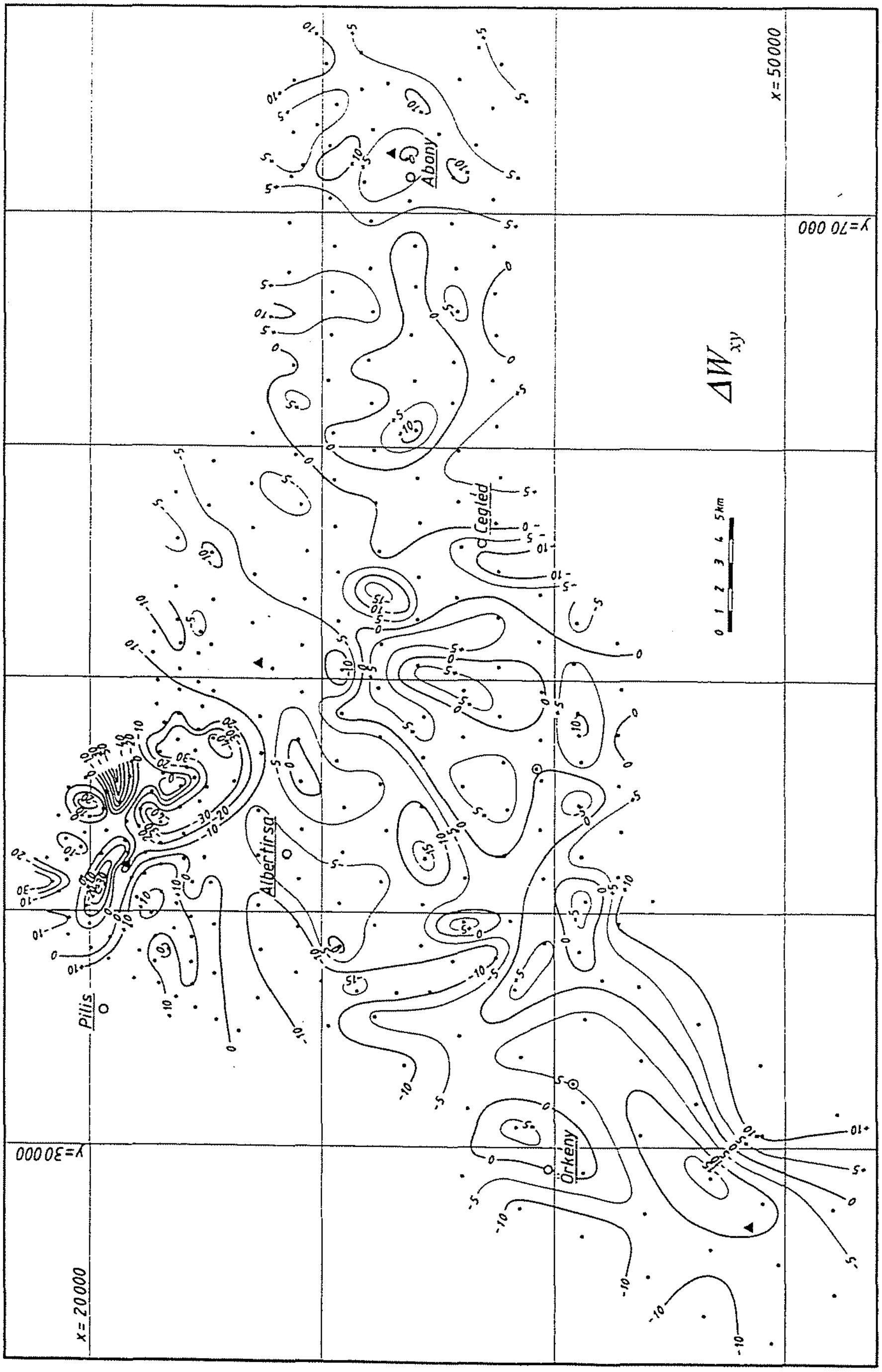

x= 50000
y=70 000
Abony
ΔW_{Δ}
Cegléd
0 1 2 3 4 5 km
Albertirsa
Pilis
Örkeny
y=30 000
x= 20 000

Fig. 3.



Abony
Cegléd
Albertirsa
Pilis
Örkény
x = 50000
y = 70 000
y = 30 000
x = 20000
ΔW_xy
0 1 2 3 4 5 km

Fig. 4.

The accuracy of terrain correction is influenced mainly by three factors:

- accuracy of measured height differences,
- error of the approximate density value used in the computation,
- departures of the real and its approximating model surface of the ground.

Considering all the three error sources the standard error of *terrain correction* of both second derivatives is

$$\mu_{\delta W^s_\Delta} \approx \mu_{\delta W^s_{xy}} \approx \pm 3 \text{ E}$$

according to investigations of (BADEKAS – MUELLER, 1967).

Needed height data for computing *topographic correction* can be taken from topographic maps. The same method can be used to compute corrections as for the terrain correction. The same investigation of (BADEKAS – MUELLER, 1967) shows the standard error of topographic corrections:

$$\mu_{\delta W^t_\Delta} \approx \mu_{\delta W^t_{xy}} \approx \pm 2 \text{ E} .$$

Height data needed to compute *cartographic correction* can also be taken from maps. The very same method can be used to compute correction as for terrain or topographic corrections. The same investigation of (BADEKAS – MUELLER, 1967) shows the standard error of cartographic corrections:

$$\mu_{\delta W^c_\Delta} \approx \mu_{\delta W^c_{xy}} \approx \pm 1 \text{ E} .$$

In our test area in Hungary the values

$$\Delta W^t_\Delta = W_\Delta - U_\Delta - \delta W^s_\Delta - \delta W^t_\Delta$$

and

$$\Delta W^t_{xy} = W_{xy} - U_{xy} - \delta W^s_{xy} - \delta W^t_{xy}$$

were available at each measurement point as well, and isoline maps of *Figs. 5* and *6* were drawn using these data. Labels of contour lines are in Eötvös unit. There was no reason for computing cartographic corrections in our test area because these corrections are negligibly small.

It is because the test area surrounding Cegléd is flat the ΔW^t_Δ and ΔW^t_{xy} values can be considered practically – using a designation by Eötvös – subsurface anomalies.

For the sake of simplicity ΔW^t_Δ and ΔW^t_{xy} values are *corrected*; and as for ΔW_Δ and ΔW_{xy} are *uncorrected* second derivatives they will be called later on.

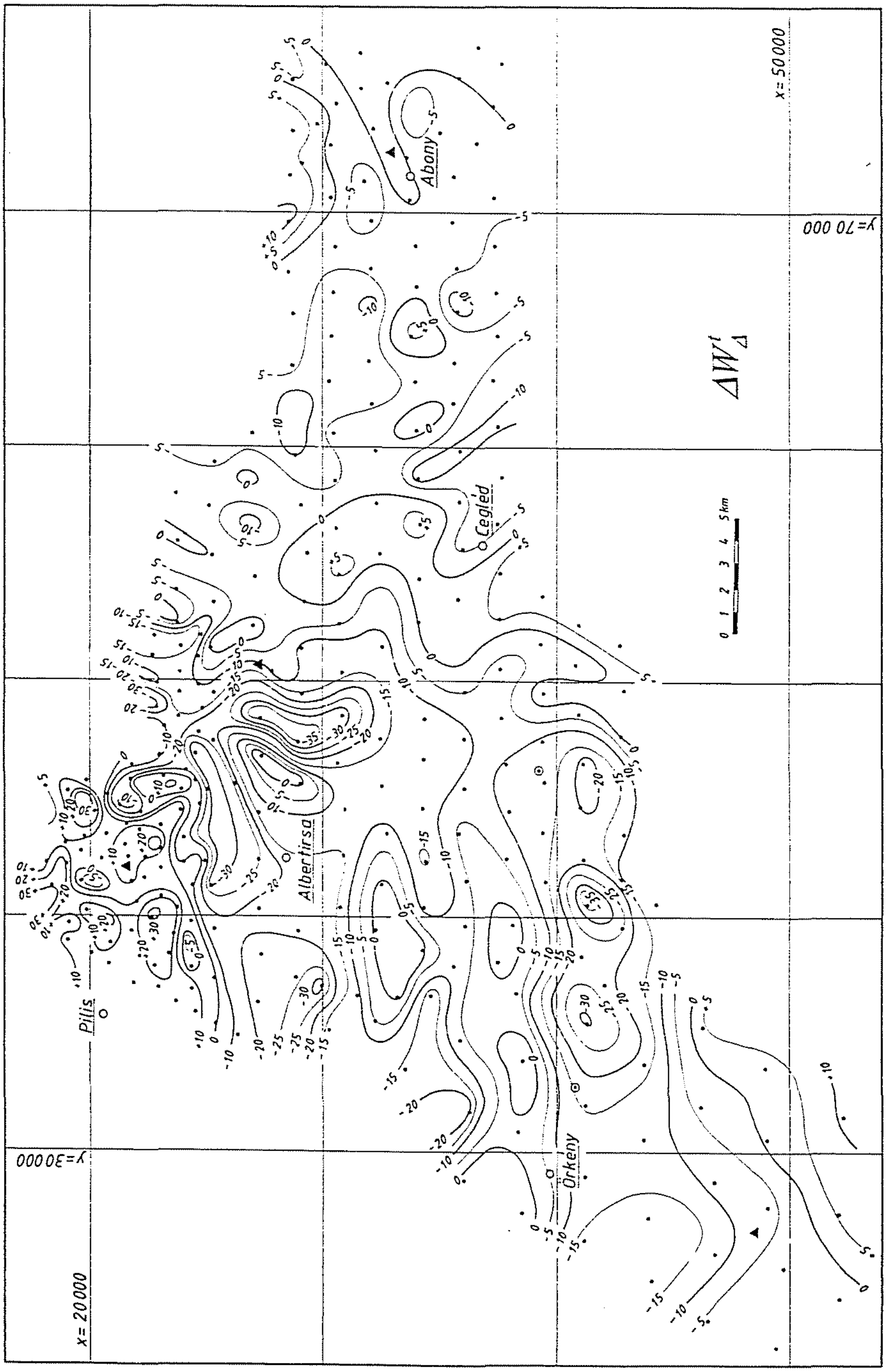

Abony
Cegléd
Albertirsa
Pilis
Örkény
x= 50000
x= 20000
y= 70000
y= 30000
0 1 2 3 4 5 km

Fig. 5.

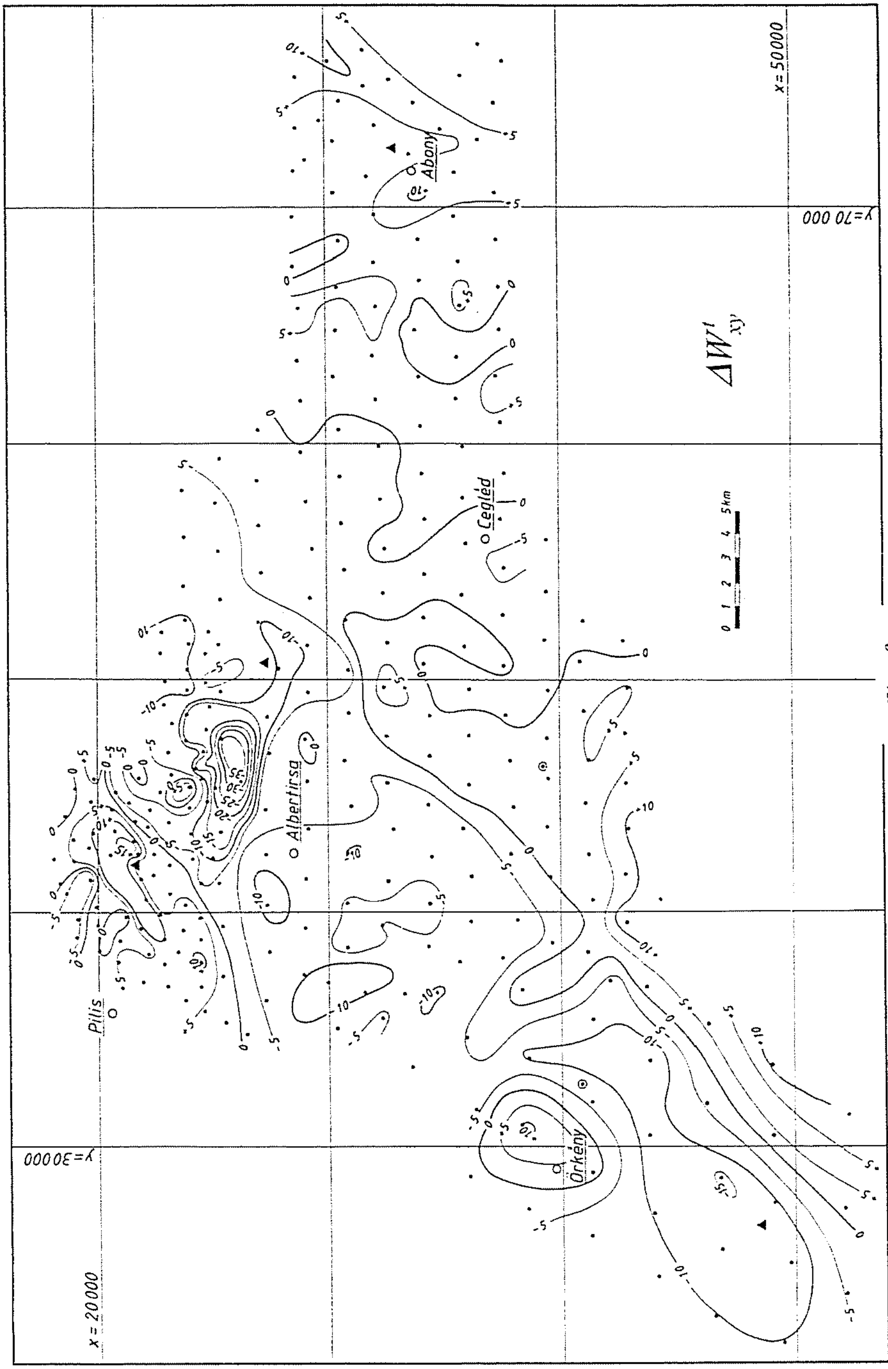

Abony
Cegléd
Albertirsa
Pilis
Örkény
x=50000
x=20000
y=70 000
y=30000
ΔW^t_{xy}
0 1 2 3 4 5 km

Fig. 6.

Using the law of error propagation,

$$\mu_{\Delta W^t_\Delta} \approx \pm 4.3 \text{ E} ,$$
$$\mu_{\Delta W^t_{xy}} \approx \pm 4.1 \text{ E}$$

values are yielded for standard errors of corrected second derivatives.

If isoline maps of our test area in Hungary, ΔW_Δ in *Fig. 3*, ΔW^t_Δ in *Fig. 5*, ΔW_{xy} in *Fig. 4*, and ΔW^t_{xy} in *Fig. 6* are compared it can be seen that the relatively small corrections resulting from the nearly flat terrain do not affect considerably the appearance of ΔW_Δ and ΔW_{xy} second derivatives, they only simplify the picture a little. More significant variations can only be seen in the vicinity of Pilis where corrections are greater due to the more rugged topography of the southern extension part of Gödöllő Hills.

2.4. ξ and η Values for Starting and Checking the Interpolation

Six points can be found in the area of Hungary in *Fig. 2* where ξ, η deflections of the vertical are known. Each of these points is such that gravimetric (approximately absolute) deflections of the vertical are available based on gravity data; four of them (originally points labelled 1, 2 and 3, and later 27) are astrogeodetic points. Points 1, 2 and 3 are starting and closing ones of interpolation lines, points 13, 14 and 27 served the purpose of checking interpolated values. It is noted that only components of gravimetric deflections of the vertical were known at point 27 previously, however, during the time of our test computations astronomic position determination was carried out by the Cartographic Institute of the Hungarian Army and so meanwhile this point became astrogeodetic stations as well. Relative deflections of the vertical provided by the Documentation Department of FMI refer to a relative geodetic datum.

The accuracy of relative deflections of the vertical at the astrogeodetic stations 1, 2, 3 and 27 can be described by the standard error of astronomic position determinations, which is, according to (BIRÓ – FÖLDVÁRINÉ – HAZAY – HOMORÓDI, 1965)

$$\mu_{\xi o} \approx \mu_{\eta o} \approx \pm\, 0.2'' .$$

Further examination is required if the accuracy of relative deflections of the vertical at astrogeodetic (checking) points 13 and 14 is needed, because gravimetric deflections of the vertical are directly available at these points instead of relative deflections of the vertical. If relative deflections of the vertical are also required at these points, one should know both relative and absolute deflections of the vertical at least at three neighbouring points in order to determine the mutual position of the relative and absolute ellipsoid along an area. Then relative (astrogravimetric) deflections of the vertical corresponding to known gravimetric deflections can be computed at these points.

In the present case at astrogeodetic points 1, 2, 3 and 27 both relative and gravimetric deflections of the vertical are available and thus relative deflection components at points 13 and 14 were determined from gravimetric ones. The accuracy of relative (transformed) values that were determined so at points 13 and 14, depends mainly on the accuracy of gravimetric deflections of the vertical, which is according to (BIRÓ – FÖLDVÁRINÉ – HAZAY – HOMORÓDI, 1965)

$$\mu_{\xi gr} \approx \mu_{\eta gr} \approx \pm\, 0.5'' \, .$$

Because of the gravity effect of surface topography is also included in the known values of deflection of the vertical at the above listed points; and because of interpolation computations were also performed during one process of our test computations with W_Δ and W_{xy} second derivatives provided with topographic corrections also, hence topographic correction was necessary for known components of deflection of the vertical as well. This correction was determined according to the method of (RENNER, 1952) by considering surface topography of the innermost neighbourhood around computation points.

3. Test Results and Implications

It was the first important task of our test computations to test different methods of solution of interpolation and their intercomparison. The most suitable method can be chosen then as the result of this test. After this the problem of weighting is treated, and tried to throw light on the matter how the accuracy of interpolation is affected by the geometrical configuration of interpolation networks, i.e. what is the optimal geometric arrangement of networks.

3.1 Intercomparison of Different Methods of Solution

Practical solutions of interpolation can be classified to the following two main groups (VÖLGYESI, 1993): in the case 'A' $\Delta\xi$, $\Delta\eta$ differences of components of deflection of the vertical are treated as unknowns, and in the case 'B' ξ, η components of deflection of the vertical at points themselves are the unknowns to be determined.

In case of solutions of the group 'A' (i.e. when $\Delta\xi$, $\Delta\eta$ differences between points are the unknowns) there are three possibilities of the interpolation:

A1: the *complete* coefficient matrix, formed by the coefficients of $4n-6$ equations of type

$$\Delta\xi_{ji}\sin\alpha_{ij} - \Delta\eta_{ji}\cos\alpha_{ij} = T_{ij} \;,$$
$$\Delta\xi_{ji} + \Delta\xi_{kj} + \Delta\xi_{ik} = 0 \;,$$
$$\Delta\eta_{ji} + \Delta\eta_{kj} + \Delta\eta_{ik} = 0 \;,$$
$$\sum_{i=1}^{n-1} \Delta\xi_{i+1,i} = \xi_n - \xi_1$$

and

$$\sum_{i=1}^{n-1} \Delta\eta_{i+1,i} = \eta_n - \eta_1$$

has to be inverted, i.e. $4n-6$ unknown $\Delta\xi$, $\Delta\eta$ values are computed (VÖLGYESI, 1993).

A2: instead of $4n-6$ unknowns we deal only with the absolutely necessary $2n-2$ unknown $\Delta\xi$, $\Delta\eta$ values and invert the corresponding coefficient matrix of smaller size,

A3: unknowns $\Delta\xi$, $\Delta\eta$ are determined stepwise (by successive elimination).

When solving adjustment problems of a rather large size – because of the unavoidable accumulation of rounding errors during the computation – it is in any case feasible to look after some method which leads to the solution of an equation system with minimum number of unknowns (VÖLGYESI, 1979). If we do not want the unnecessary work of determining unknowns that are not required and we do not want to risk the accuracy of solution due to accumulation of rounding errors, then there is nothing to say of case A1 later on since after all the same ξ, η deflections of the vertical are determined in the case A2 but through the computation of considerably fewer unknowns $\Delta\xi$, $\Delta\eta$.

The interpolation method elaborated by RENNER (1952, 1956, 1957) also belongs to the group A1, where in its original form $\Delta\xi$, $\Delta\eta$ differences between network points were chosen as unknowns and it was required inversion of the complete coefficient matrix as well. Because of the above mentioned facts we will not deal with this method here. Though we performed test computations based on an adapted form of Renner's method –

where not $\Delta\xi$, $\Delta\eta$ differences are unknown but direct ξ, η values of interpolation points – but these results will be reported later on in the section dealing with suitable geometrical figures of interpolation networks.

Special computer programs were developed for cases A2 and A3. Only absolutely necessary $\Delta\xi$ and $\Delta\eta$ unknowns between points of interpolation chain connecting two astrogeodetic points are regarded as unknowns by both programs and they are determined by method of matrix orthogonalization by program FGVSUORT (VÖLGYESI, 1980) and by successive elimination (BADEKAS and MUELLER, 1967) by program FUGGOSUC.

Three different programs were made for handling case B; these are FUGGOVON, FUGGOOLD and FUGGOORT. All three programs are suitable for computing interpolation networks of arbitrary shape by determining direct values of deflection components at network points by considering ξ, η fixed values of arbitrary number and distribution. Conventional adjustment process is utilized by FUGGOVON and FUGGOOLD to calculate the unknowns while numerically more stable matrix orthogonalization process is used by FUGGOORT. Program FUGGOOLD is a more efficient version of FUGGOVON which computes itself the sides of interpolation networks automatically – on the contrary to program FUGGOVON.

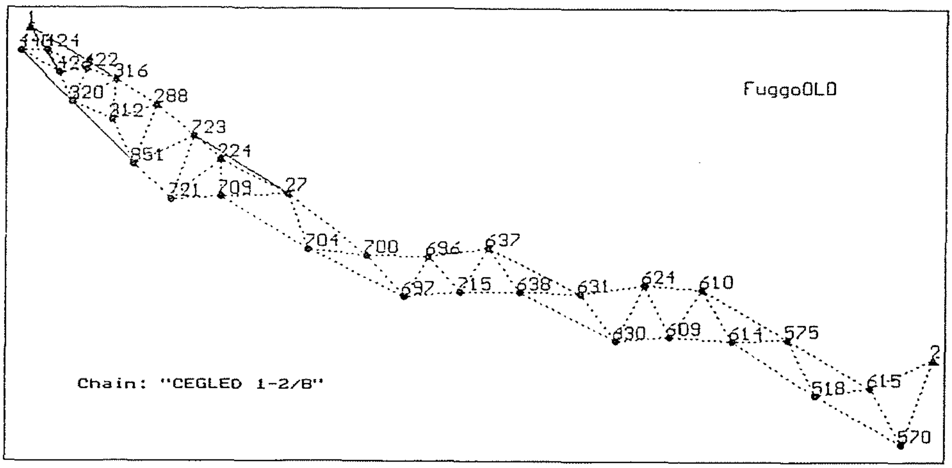


Fig. 7.

Several test computations were performed by all above mentioned interpolation programs to examine and compare practical methods of solution of the interpolation. Computation results of the interpolation network connecting astrogeodetic points 1 and 2 of our test area (*Fig. 7*) will

Table 1

Point numb.	FUGGOSUC ξ''	η''	FGVSUORT ξ''	η''	FUGGOVON ξ''	η''	FUGGOOLD ξ''	η''	FUGGOORT ξ''	η''
1	2.20	4.00	2.20	4.00	2.20	4.00	2.20	4.00	2.20	4.00
440	1.85	3.98	2.04	4.04	2.05	4.04	1.76	3.98	2.05	4.04
424	1.80	4.19	2.01	4.04	2.01	4.04	1.67	4.24	2.01	4.04
426	1.56	4.29	1.91	4.05	1.92	4.05	1.47	4.31	1.92	4.05
422	1.92	4.94	2.26	4.55	2.26	4.55	1.82	4.75	2.26	4.55
320	1.68	4.18	2.18	3.88	2.19	3.88	1.91	4.08	2.19	3.88
316	1.83	5.48	2.22	4.95	2.22	4.95	1.74	4.49	2.22	4.95
312	1.83	5.57	2.41	5.07	2.41	5.06	2.30	4.60	2.41	5.06
288	1.91	6.01	2.41	5.30	2.42	5.29	2.15	4.58	2.42	5.29
851	2.73	5.20	3.49	4.58	3.50	4.58	3.38	3.96	3.50	4.58
723	3.15	5.71	3.78	4.82	3.78	4.81	3.59	4.03	3.78	4.81
721	3.35	4.97	4.24	4.17	4.24	4.16	4.22	3.44	4.24	4.16
224	3.65	5.73	4.36	4.71	4.37	4.70	4.23	3.84	4.37	4.70
709	3.57	5.58	4.41	4.55	4.42	4.55	4.38	3.68	4.42	4.55
27	3.74	6.84	4.57	5.59	4.58	5.58	4.53	4.53	4.58	5.58
704	4.10	6.75	5.06	5.43	5.07	5.42	5.17	4.31	5.07	5.42
700	4.24	6.75	5.23	5.27	5.24	5.26	5.36	4.02	5.24	5.26
697	4.49	6.54	5.46	5.07	5.47	5.06	5.60	3.83	5.47	5.06
696	4.35	6.38	5.31	4.91	5.32	4.90	5.45	3.67	5.32	4.90
715	4.58	6.13	5.50	4.68	5.51	4.67	5.63	3.46	5.51	4.67
637	4.45	6.04	5.37	4.62	5.38	4.61	5.54	3.42	5.38	4.61
638	4.67	5.96	5.51	4.56	5.52	4.55	5.64	3.38	5.52	4.55
631	4.67	5.77	5.51	4.47	5.53	4.47	5.64	3.38	5.53	4.47
630	4.98	5.60	5.63	4.41	5.64	4.40	5.66	3.40	5.64	4.40
624	4.73	5.55	5.49	4.44	5.50	4.44	5.65	3.51	5.50	4.44
609	5.07	5.61	5.61	4.59	5.62	4.58	5.64	3.72	5.62	4.58
610	4.85	5.58	5.50	4.66	5.51	4.66	5.65	3.88	5.51	4.66
614	5.12	5.67	5.53	4.84	5.54	4.84	5.55	4.14	5.54	4.84
575	5.00	5.59	5.41	4.95	5.42	4.95	5.44	4.41	5.42	4.95
518	5.60	5.39	5.70	4.85	5.71	4.85	5.55	4.40	5.71	4.85
615	5.46	4.52	5.59	4.20	5.60	4.20	5.47	3.94	5.60	4.20
570	6.40	4.03	6.16	3.86	6.17	3.87	5.82	3.73	6.17	3.87
2	5.20	3.40	5.20	3.40	5.20	3.40	5.20	3.40	5.20	3.40

1 and 2 = given points;
27=checking point: ($\xi_{27} = 4.80$, $\eta_{27} = 5.42$)

be presented in more detail. Computation results of different interpolation programs can be compared in tabular form for chain 'Cegléd 1–2/B' in *Fig. 7*. Output results of programs FUGGOSUC, FGVSUORT, FUGGOVON, FUGGOOLD and FUGGOORT can be found in *Table 1*. It can be seen that some results were provided by FUGGOVON and FUGGOORT and values computed by FGVSUORT differ only few hundredth of an arc second by these.

Interpolated values by FUGGOVON and FUGGOORT at checkpoint 27 are the most close to known ξ, η values (difference of ξ is $-0.22''$ and of η is $+0.16''$) but results are not much worse for FGVSUORT (differences here are $-0.23''$ and $+0.17''$). The interpolation network which can be seen on *Fig. 7* is a good example of cases where program FUGGOOLD must not be used. It is discernible that network points are substantially more distant from each other near the closing point 2 than near the other closing point 1 . Hence if such a distance is prescribed so as not a single point should be left where points are more distant from each other, then also redundant network sides are created where points are closer to each other along which the change of ΔW_Δ and ΔW_{xy} values are no more linear. Such network sides can be seen on *Fig. 7* next to astrogeodetic point 1 which were connected by continuous lines by the computer. Therefore – in accord with our expectations – differences are greater at checkpoint 27 with respect to known ξ, η values (differences are $-0.27''$ and $+0.89''$).

Interpolated values by program FUGGOSUC differ more sharply from previously mentioned computation results and from known deflection components at point 27. Difference of ξ is $-1.06''$ and of η is $+1.42''$ at point 27.

Of course different interpolation methods were examined through computation of not only the chain 1–2/A but also through a number of other ones. Results of these computations are summarized in *Table 2*.

In this table differences of computed and known ξ, η values are indicated at checkpoints of different interpolation networks by the successive elimination method FUGGOSUC and programs FGVSUORT, FUGGOVON and FUGGOORT. Standard deviations were computed using above mentioned differences that are characteristic to the accuracy of each interpolation method.

Table 2

Sign of network	check. point	FUGGOSUC $\delta\xi$	FUGGOSUC $\delta\eta''$	FGVSUORT $\delta\xi''$	FGVSUORT $\delta\eta''$	FUGGO-VON/ORT $\delta\xi''$	FUGGO-VON/ORT $\delta\eta''$
1–2/A	27	−0.53	−0.75	−0.54	−0.74	−0.54	−0.74
1–2/B	27	−1.06	+1.72	−0.23	+0.17	−0.22	+0.16
2–3/A	13	−0.57	−2.34	−0.62	−2.51	−0.63	−2.54
2–3/B	13	+1.75	+4.70	+0.71	+0.93	+0.70	+0.89
2–3/B	14	+2.50	+2.07	+1.31	+0.38	+1.26	+0.27
3–1/A	14	−0.31	−0.68	−0.10	−0.48	−0.09	−0.58
3–1/B	14	+0.31	−0.10	+0.65	+0.15	+0.71	+0.26
Standard deviations:		±1.27	±2.23	±0.70	±1.08	±0.69	1.09
		±1.81		±0.91		±0.91	

According to the results of our tests which are summarized in *Table 1* and *Table 2* to examine and compare different interpolation methods it can be stated that in most cases deflection of the vertical computed by successive elimination differed greatly from values yielded by other methods and in many cases they differed sharply from known checkpoints' values. There are nearly two times as great errors in values of deflection of the vertical computed by successive elimination as the interpolated values of different methods which fact can be seen from the data of *Table 2*.

The accuracy of results yielded by programs FGVSUORT, FUGGOVON and FUGGOORT can be taken as practically identical but unfortunately the possibility to apply FGVSUORT is somewhat limited because – similarly to the program FUGGOSUC – this program can exclusively be used in such interpolation chains the starting and closing points of which are astrogeodetic points. The chain can contain no other point of restraint besides these two astrogeodetic points. Any other known values of deflection of the vertical inside the network besides the two extreme points can only be used for checking.

Output results of program FUGGOOLD are not presented in *Table 2* because it is useful to compute with different distance limits depending on point distribution and so by jointly comparing them would not have yielded a true, uniform representation of facts. Our examinations showed that the accuracy of programs FUGGOVON and FUGGOORT can be attained by program FUGGOOLD only if interpolation points are homogeneously distributed and the distance limit is appropriately small. Hence – even if it is very comfortable – only if our points are homogeneously distributed it is worth creating network sides by the computer.

We haven't met such a task during our test computations for which it was possible to give preference to any of the two programs FUGGOVON and FUGGOORT depending on the accuracy of interpolated values. Nevertheless it is a crucial issue for preferring program FUGGOORT that – on the contrary to program FUGGOVON – it can be used to compute interpolation networks of a very large size.

If final conclusions should be drawn from the results of our comparative tests it can be stated that the errors of deflections of the vertical computed by successive elimination are nearly twice as big, hence it is not suitable to apply method of successive elimination (to use program FUGGOSUC). In our test area when interpolation *chains* are computed programs FGVSUORT, FUGGOVON and FUGGOORT yield results of practically the same accuracy, but the application of program FGVSUORT is somewhat limited because it can only be used for such interpolation chains the extreme points of which are astrogeodetic ones. In our examinations same results were provided by programs FUGGOVON and FUGGOORT in all respects for chains with relatively few points expect that in networks of larger size FUGGOORT is numerically more stable according to other tests, and it runs also if the number of unknowns is greater. Finally it can be concluded that it is very comfortable to use the program FUGGOOLD instead of FUGGOVON but it is worth creating network sides by this program only where our points are homogeneously distributed.

3.2 The Problem of Weighting

According to theoretical considerations (VÖLGYESI, 1993) during the adjustment process of interpolation two simple approximations were adopted: on one side since T_{ij} fictive measurements computed through *Eq.* (1) can be regarded as uncorrelated, hence the weight matrix is a diagonal one; on the other side since there are terms in the main diagonal of our weighting coefficient matrix which are proportional to squares of the side lengths therefore it comes out from the inverse square of distance.

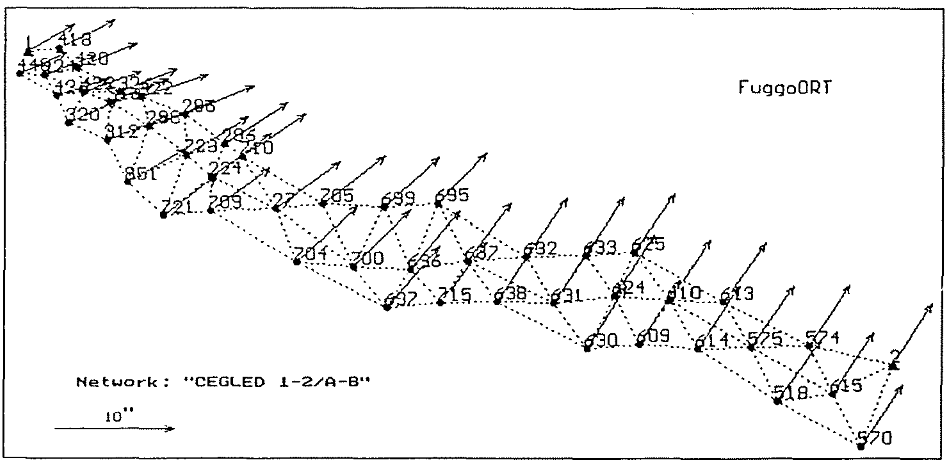


Fig. 8.

During test computations we tried to get an answer to the question whether accuracy of the interpolated values can be increased by this weighting. To this end computation of deflections of the vertical for networks on *Figs. 8, 9* and *10* were carried out without weights (using unit weights) and with weights previously mentioned.

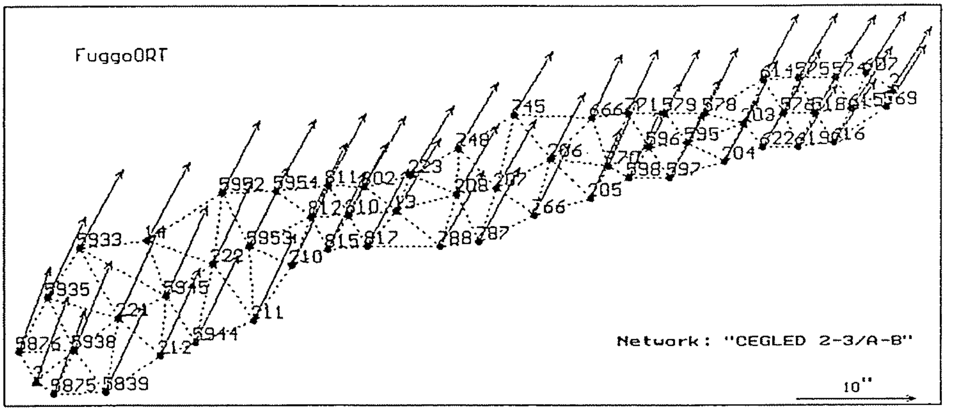

FuggoORT
Network: "CEGLED 2-3/A-B"
10''

Fig. 9.

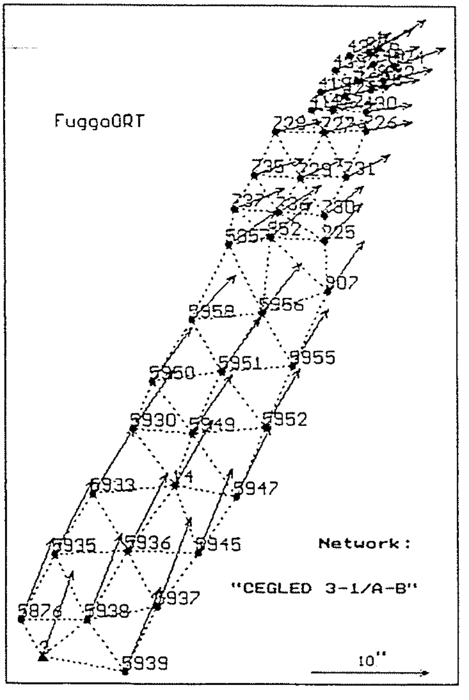

FuggoORT
Network:
"CEGLED 3-1/A-B"
10''

Fig. 10.

Table 3

Sign of network	check. point	without weights $\delta\xi''$	$\delta\eta''$	using weights $\delta\xi''$	$\delta\eta''$
1–2/AB	27	−0.85	+0.30	−0.81	+0.11
2–3/AB	13	+0.88	+0.79	+0.80	+0.58
3–1/AB	14	+0.09	−0.21	+0.22	−0.03
Standard		±0.71	±0.50	±0.67	±0.34
deviations:		±0.61		±0.53	

Computation results are summarized in *Table 3* where it is shown how large deviations resulted at checkpoints of different interpolation networks between *known* and *computed* ξ, η values which come from unit weights and inverse square weighting. Also mean square deviations were calculated by these deviations which are characteristic to the accuracy of these two methods.

By reason of these tabulated data it can be stated that the accuracy of interpolated values can be increased only by a little by introducing weights. In our test area the increase is about $0.08''$.

A vectorial picture of values computed by weighting was drawn in *Figs. 8*, *9* and *10* as well. These vectors can either be considered as horizontal force components or direct $\Theta = \sqrt{\xi^2 + \eta^2}$ values of deflections of the vertical. (The first differs from the second only by a factor of the vector g).

According to our interpretation deflections of the vertical can be considered as vectors when for positive direction of vectors the direction from ellipsoidal zenith towards astronomical zenith is chosen and for length of vector the absolute value of Θ is chosen at the point under question. Hence by using an appropriate scale either values of deflection of the vertical or horizontal force components can be read from the same figure.

3.3 Geometry of Interpolation Networks

Coefficients a_1 and c_1 in the case of successive elimination method depend only on the network geometry whereas coefficients b_1 and d_1 are functions of partly the geometrical arrangement of network and partly of ΔW_Δ and ΔW_{xy} second derivatives (VÖLGYESI, 1993). In the first place let us examine how standard deviations

$$\mu_{\xi_i} = \pm \left[\mu_{\xi_0}^2 + \left(\sum_{k=1}^{i} a_k \right)^2 \mu_u^2 + \mu_{\Sigma b}^2 \right]^{\frac{1}{2}}$$

and

$$\mu_{\eta_i} = \pm \left[\mu_{\eta_0}^2 + \left(\sum_{k=1}^{i} c_k \right)^2 \mu_u^2 + \mu_{\Sigma d}^2 \right]^{\frac{1}{2}}$$

will be developed according to different geometrical arrangement of networks.

To this end let us compute how much misclosures can be expected by elimination method for deflections of the vertical determined according to the following three versions:

- in the first version: *when ξ values are fixed*, when ξ components at both closing points are given and η component is given at only one of closing points; then parameter u is yielded by the expression

$$\Delta\xi_{n1} = \sum_{i=1}^{n-1} a_i u + \sum_{i=1}^{n-1} b_i \tag{3}$$

- in the second version: *when η values are fixed*, when η components at both closing points are given and ξ component is given at only one of closing points; then parameter u is yielded by the expression

$$\Delta\eta_{n1} = \sum_{i=1}^{n-1} c_i u + \sum_{i=1}^{n-1} d_i \tag{4}$$

- and finally in the third version: *when both ξ and η values are fixed*, when both ξ and η components are given at both closing points and parameter u is yielded through the expression

$$u = \frac{\left(\sum_{i=1}^{n-1} a_i\right)^2 \mu_{\Sigma d}^2 u_\xi + \left(\sum_{i=1}^{n-1} c_i\right)^2 \mu_{\Sigma b}^2 u_\eta}{\left(\sum_{i=1}^{n-1} a_i\right)^2 \mu_{\Sigma d}^2 + \left(\sum_{i=1}^{n-1} c_i\right)^2 \mu_{\Sigma b}^2} \tag{5}$$

by an adjustment process.

In the above cases misclosures w_ξ and w_η of ξ and η components of deflection of the vertical are defined through the equations

$$w_\xi = \Delta\xi_{nm} - \Delta\xi'_{nm} \ , \tag{6}$$

$$w_\eta = \Delta\eta_{nm} - \Delta\eta'_{nm} \ , \tag{7}$$

where $\Delta\xi_{nm}$ and $\Delta\eta_{nm}$ are differences of deflection components at astrogeodetic points n and m; $\Delta\xi'_{nm}$ and $\Delta\eta'_{nm}$ denote the sum computed by (4) or (3) in the case of fixed ξ or η values, respectively.

Let the error of Σb_i in (3) and (4) be ε_b and the error of Σd_i be ε_d. Then the value of u_0 free from error can theoretically be computed from (3) through the equation

$$\sum_{i=m}^{n} a_i u_0 + \sum_{i=m}^{n} b_i + \varepsilon_b = \xi_{nm}$$

as

$$u_0 = \frac{\xi_{nm} - \sum_{i=m}^{n} b_i}{\sum_{i=m}^{n} a_i} - \frac{\varepsilon_b}{\sum_{i=m}^{n} a_i} = u - \frac{\varepsilon_b}{\sum_{i=m}^{n} a_i} \; . \tag{8}$$

On the other side it comes from (8) and considering (7) yields

$$w_\eta = \frac{\sum_{i=m}^{n} c_i}{\sum_{i=m}^{n} a_i}\varepsilon_b - \varepsilon_d \; .$$

It can be proved (BADEKAS and MUELLER, 1967) that

$$\frac{\sum_{i=m}^{n} c_i}{\sum_{i=m}^{n} a_i}\varepsilon_b = \tan\alpha_{nm} \; ,$$

where α_{nm} denotes azimuth of the line that connects two astrogeodetic points, hence

$$w_\eta = \varepsilon_b \tan\alpha_{nm} - \varepsilon_d \; . \tag{9}$$

Similarly

$$w_\xi = \varepsilon_d \cot\alpha_{nm} - \varepsilon_b \; . \tag{10}$$

It comes from expressions (9) and (10) that

$$\alpha_{nm} \longrightarrow 0^\circ(180^\circ) \begin{cases} w_\xi = \infty \\ w_\eta = -\varepsilon_d \end{cases}$$

and

$$\alpha_{nm} \longrightarrow 90^\circ(270^\circ) \begin{cases} w_\xi = -\varepsilon_d \\ w_\eta = \infty \end{cases}$$

that the closing error w_ξ of component ξ is advantageously reduced when the α_{nm} azimuth of the line connecting initial and endpoint of interpolation network approaches 90° or 270° but it becomes infinitely large as α_{nm}

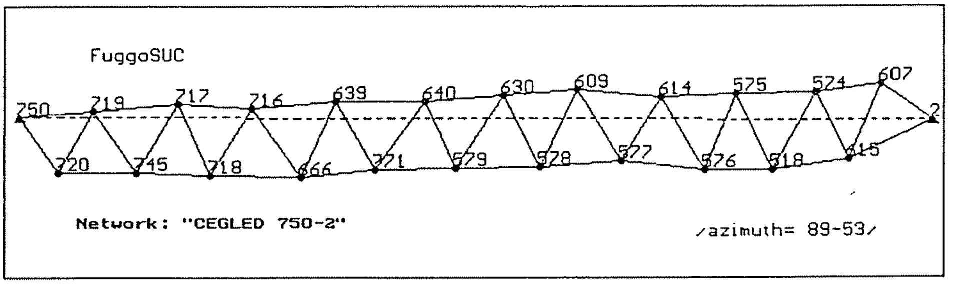


Fig. 11.

azimuth approaches 0° or 180°. In the latter case even values of component ξ computed through adjustment (5) are extremely unreliable. (We face a completely similar but reversed situation for w_η values and components of η).

Computations were performed to test the above statements along a chain, which can be seen in *Fig. 11* between stations 750 and 2 – where the azimuth of the line connecting starting and closing points of the chain (very close to 90°), $\alpha_{750-2} = 89°38'$ was altered between the values of $89°23'$ and $89°53'$ by a movement of point 750 by some 10 m in the x direction. Computations were carried out in both directions 750 $\rightarrow$ 2 and 2 $\rightarrow$ 750 as well with fixed ξ_{nm}, fixed η_{nm} values; and with fixed ξ_{nm} and η_{nm} values according to adjustment (5). Misclosure of component η with fixed ξ_{nm} values are very high ($w_\eta = 15.15''$) in agreement with theoretical considerations and also standard deviations of interpolated values are very high. Computation results can be studied also in *Fig. 12* where it can easily be seen that as the azimuth of the fictitious line connecting starting and closing points approaches 90° the reliability of interpolation declines much (in the case of two fixed ξ at both endpoints and one fixed η component at only one of endpoints).

A very similar defect arises for the case $\alpha_{nm} = 0°$ (180°) hence our test computations for this case have no detailed presentation.

We have tried to find an answer to the question in our test computations which arrangement of points of interpolation network is optimal so as to compute deflections of the vertical of the highest accuracy. To this end we compared the accuracies of interpolated values for single and double chains, for an arbitrary area and for point distribution by Renner. These computations were performed by the program FUGGOORT in each case and by weighting discussed in the previous chapter in order to be fully comparable.

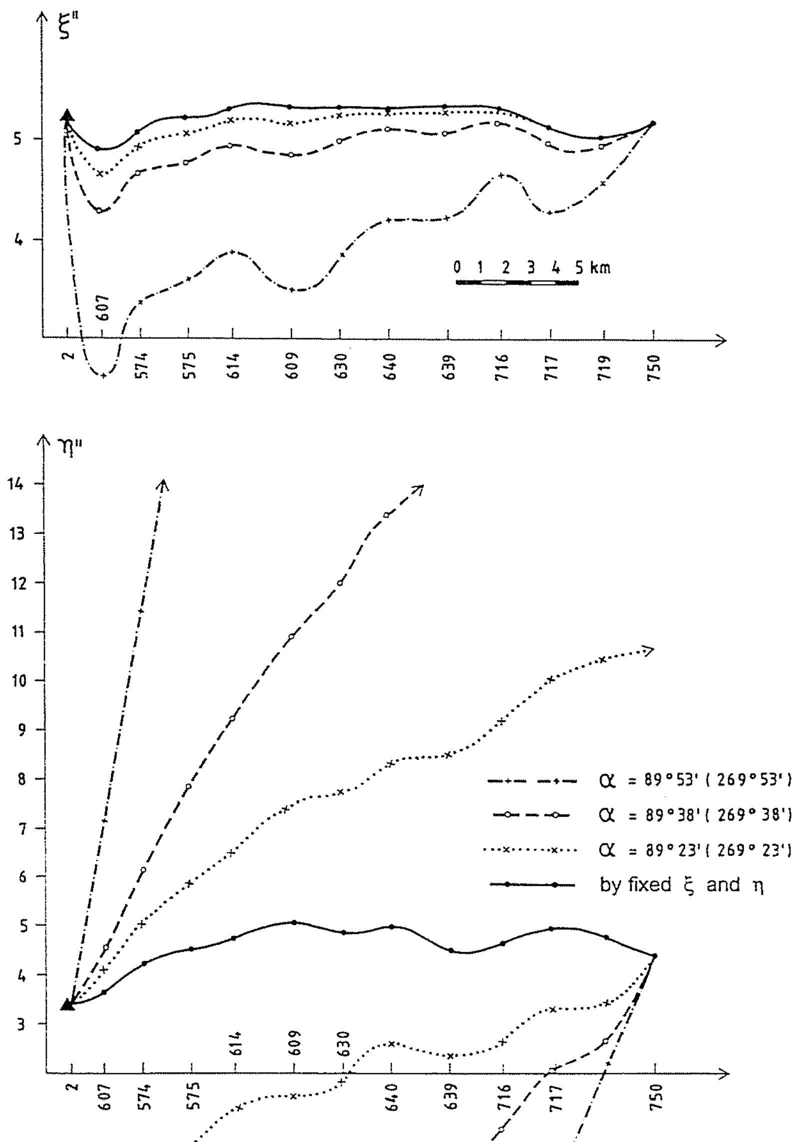


Fig. 12.

First computations were carried out along single and double chains as they can be seen in *Figs. 8, 9* and *10*. Single chains are: in *Fig. 8* between astrogeodetic points 1–2 the upper 'A' and the lower chain 'B', in *Fig. 9* between astrogeodetic points 2–3 the upper 'B' and the lower chain 'A', and in *Fig. 10* between points 3–1 left side chain 'A' and right side chain 'B'. Double 'AB' chains are: complete networks in *Figs. 8, 9* and *10*.

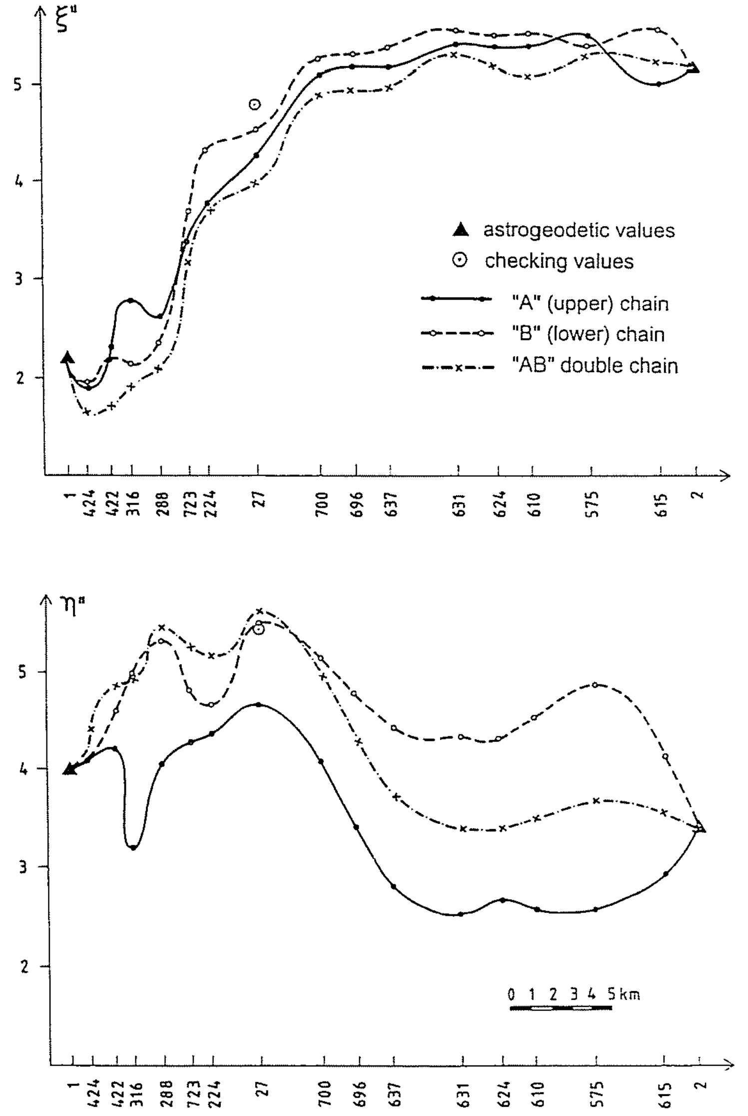


Fig. 13.

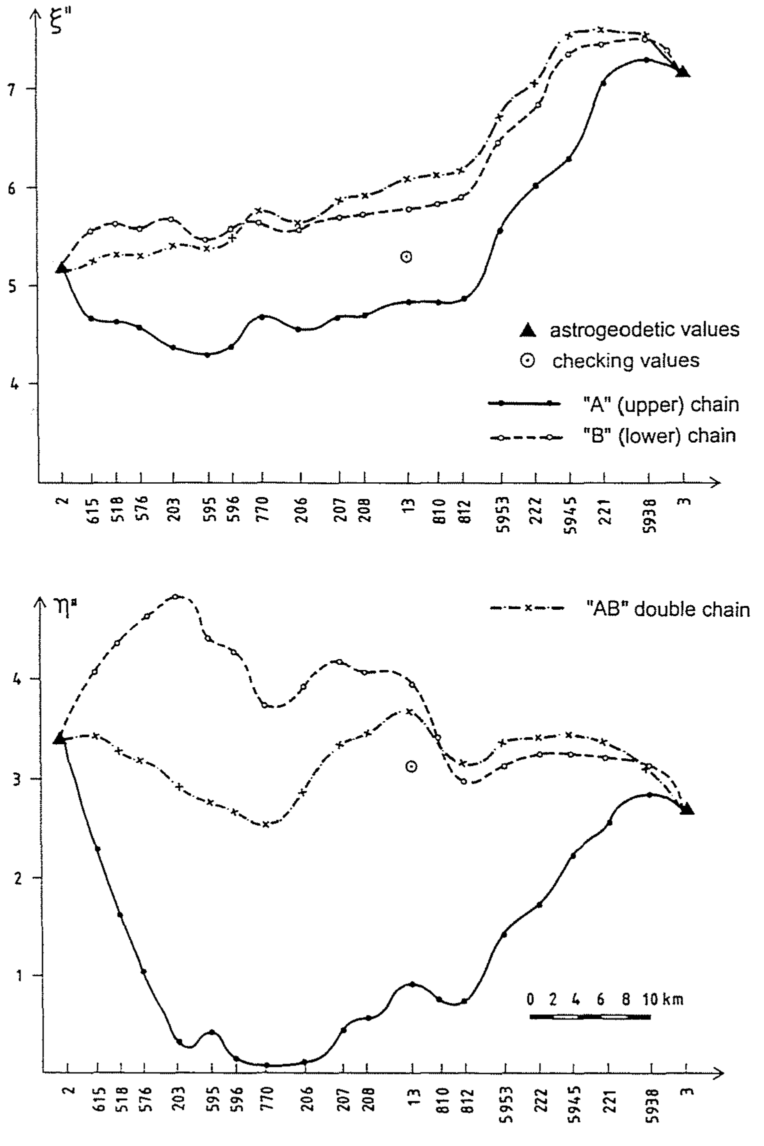


Fig. 14.

Interpolated results of single 'A' and 'B' chains and of double 'AB' networks as the combination of these two can be compared in *Figs. 13, 14* and *15*. Chains 'A' and 'B' that are side by side to each other contain

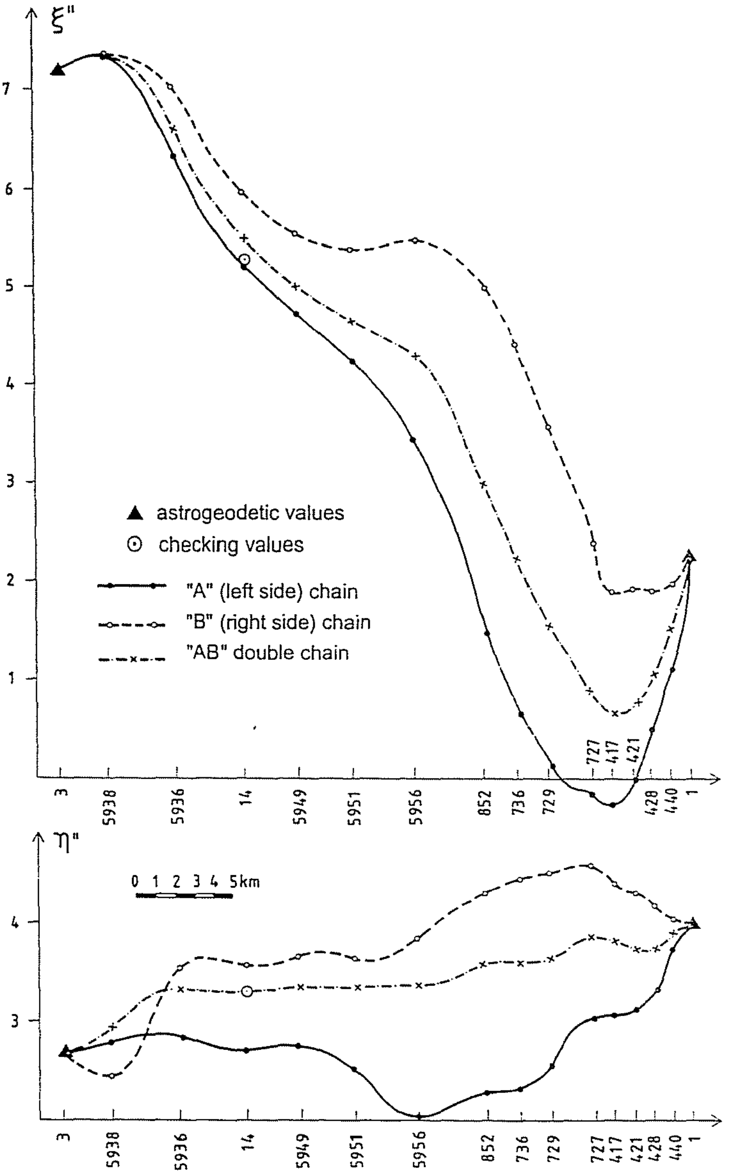


Fig. 15.

identical points for which the computation should give theoretically the same ξ and η values computed either along chains 'A' or 'B'. Interpolated values of identical points of neighbouring chains are connected by simple or broken lines in *Figs. 13, 14* and *15*. It can be seen that interpolated values of identical points of chains that are next to each other (simple and dotted lines) differ by much in most cases, sometimes the difference exceeds the value of $3''$. It can be noted as well the greater the deviations in identical points of two neighbouring ('A' and 'B') chains, the more favourable the picture is when a combined ('AB') network from two chains is interpolated, because these latter lines (denoted in these figures by dashed-dotted lines) run in the midst of the two greatly departed extreme values and in most cases they approximate better given ξ, η values at checkpoints as well.

Table 4

Part of network	check. point	chain 'A' $\delta\xi''$	chain 'A' $\delta\eta''$	chain 'B' $\delta\xi''$	chain 'B' $\delta\eta''$	common 'AB' $\delta\xi''$	common 'AB' $\delta\eta''$
1 − 2/AB	27	−0.54	−0.74	−0.22	+0.16	−0.81	+0.11
2 − 3/AB	13	−0.63	−2.54	+0.70	+0.89	+0.80	+0.58
3 − 1/AB	14	−0.09	−0.58	+0.71	+0.26	+0.22	−0.03
Standard deviations:		+0.48	+1.56 ±0.91	+0.59	+0.54	+0.67	+0.34 ± 0.53

Results of computations were summarized also in *Table 4*. We have made a comparison in this Table between *computed* and *known* ξ, η values that resulted in various networks at checkpoints. Standard deviations that are characteristic to the accuracy of interpolation were also determined based on the above differences for different kind of chains.

It can be deduced from these tabulated data and from *Figs. 13, 14* and *15* that more accurate ξ, η values can be gained by computing along a *double chain* rather than along *single chains*.

It was mentioned previously that test computations were also carried out by the interpolation method of Renner. We think proper the report on these results here as well because our conclusions on this matter are basically related to the network geometry. As the principle of the method of Renner requires a square network of 1.5 km length it was applied to cover our test area (*Fig. 16*). It is because there are no torsion balance measurements at grid points of this network second derivatives ΔW_Δ and ΔW_{xy} were taken at these points as readings from isoline maps of *Figs. 3–6*. Hence basically a linear interpolation was applied to get second derivatives from torsion balance measurement at corner points of the square shaped network. (Empty squares were used to indicate such network points in *Fig. 16* where no torsion balance measurements were made and second derivatives were determined by the above process.) As it can be seen

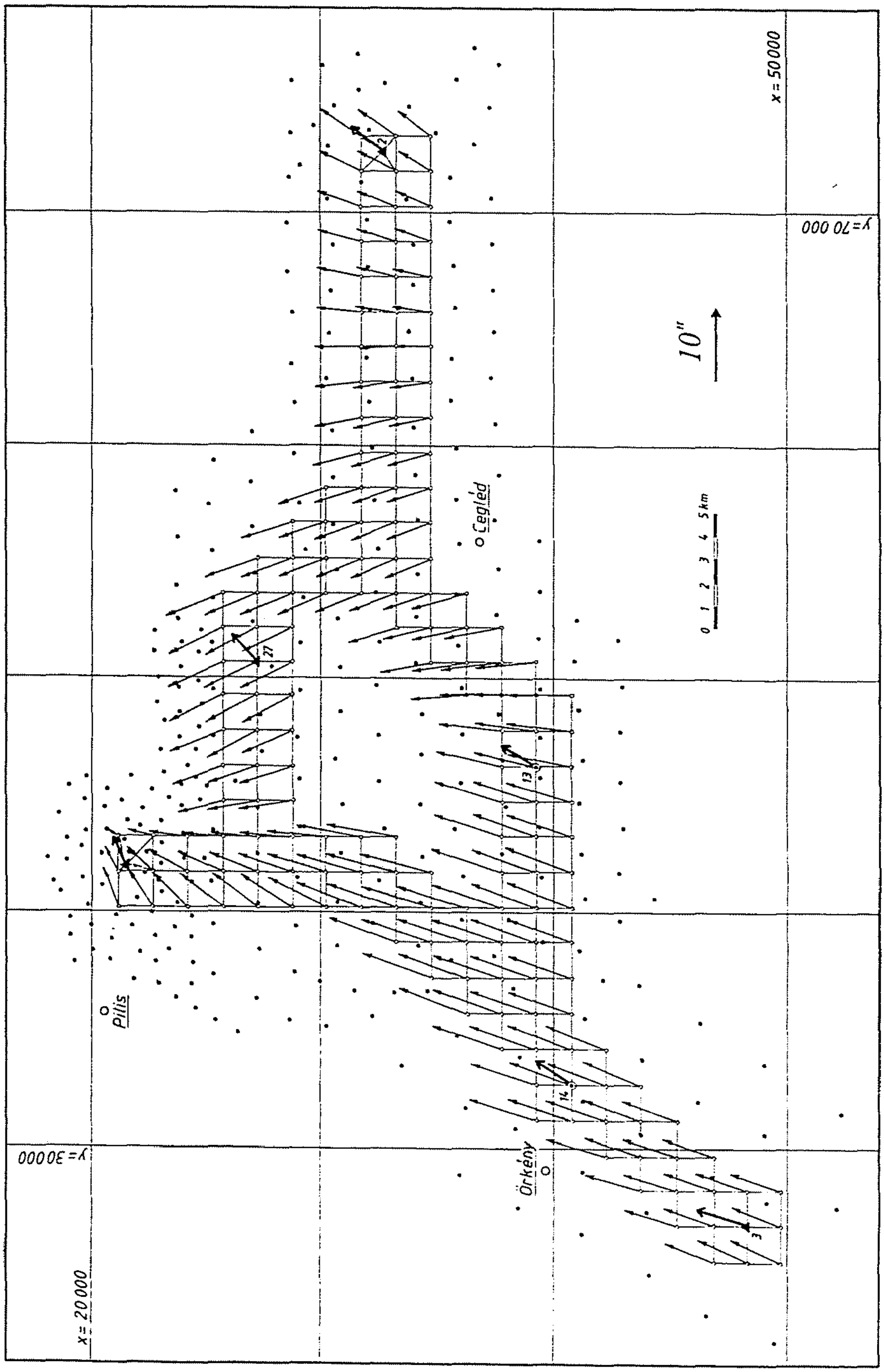

x = 50000
y = 70000
10″
Cegléd
0 1 2 3 4 5 km
27
13
Pilis
14
Örkény
y = 30000
3
x = 20000

Fig. 16.

in *Fig. 16* the square-shaped network was located so that astrogeodetic station 3 is its corner point and 1 and 2 were attached to the network by triangles. Control points 13, 14 and 27 are corner points of the square-shaped network so interpolated deflection values can be checked directly.

This network has 177 points altogether and 174 of these are points of unknown deflection values. There are altogether 348 unknowns because there are two unknown components of deflection of the vertical at each point and 542 equations can be written in all.

In *Fig. 16* $\Theta = \sqrt{\xi^2 + \eta^2}$ deflection of the vertical which resulted from the computation was visualized in a vectorial form previously discussed. We may see in this figure that deflections of the vertical interpolated by the method of Renner differ considerably from known values at checkpoints drawn as thick line vectors.

Table 5

Checking point	Renner's method		program FUGGOORT	
	$\delta\xi''$	$\delta\eta''$	$\delta\xi''$	$\delta\eta''$
27	+2.89	−9.62	−0.69	−0.51
13	+3.84	−1.24	+0.54	+0.96
14	+3.95	−0.26	+0.55	+0.29
Standard	±3.59	±5.60	±0.60	±0:65
deviations:	±4.70		±0.62	

Computation results were summarized in *Table 5* as well. For the moment let us consider only the first part of this table where a comparison was made. Large deviations resulted between *known* ξ, η values and that of *computed* according to the Renner's method. Standard deviations $m_\xi = \pm 3.59''$ and $m_\eta = \pm 5.60''$ unfortunately prove that this method – at least in our test area – should not be applied. According to our investigations these large standard deviations and the inapplicability of the method under discussion are resulted from the following two main sources of error:

1. The grid distance of the square-shaped interpolation network by Renner is an unvaried value for the whole area. This may cause deep problems for areas where grid constant is larger than the maximum distance for which the differences of ΔW_Δ and ΔW_{xy} values can be treated as linear. The linearity of differences of ΔW_Δ and ΔW_{xy} values between two points is but the most important prerequisite taken as fundamental equations (1) and (2) were obtained through an integral approximation (VÖLGYESI, 1993). Such areas in our test field is the vicinity of astrogeodetic station 1 where according to *Figs. 3* and

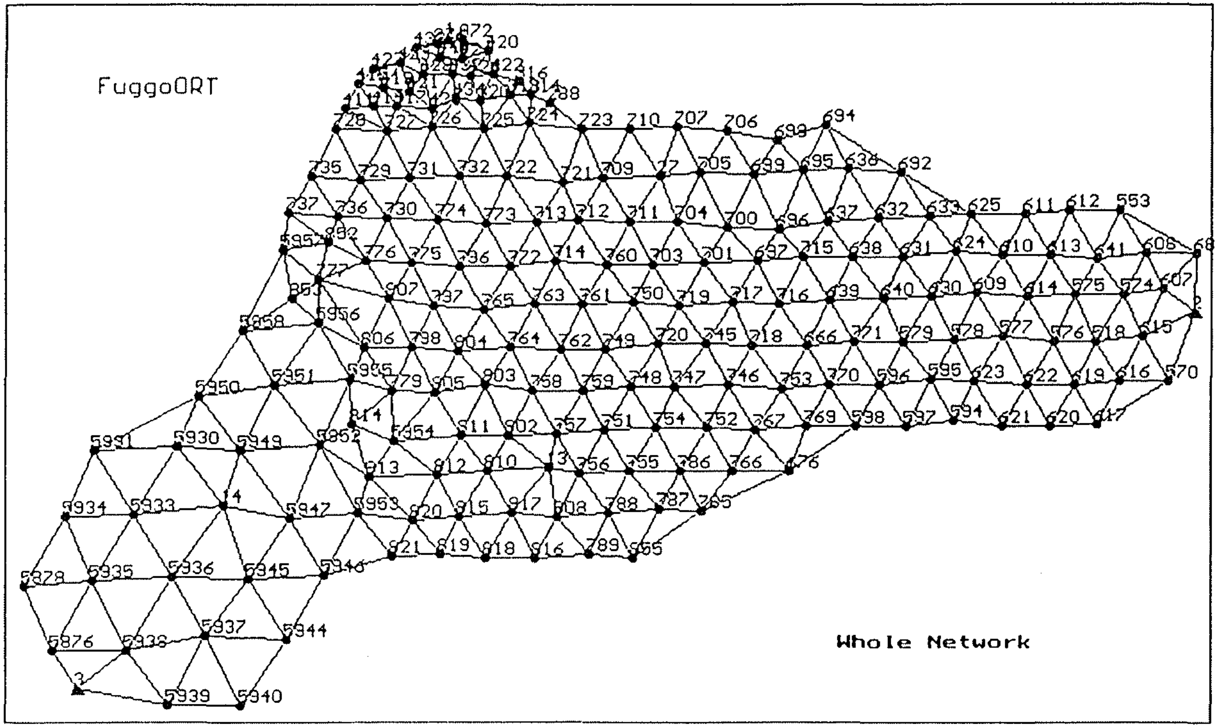


Fig. 17.

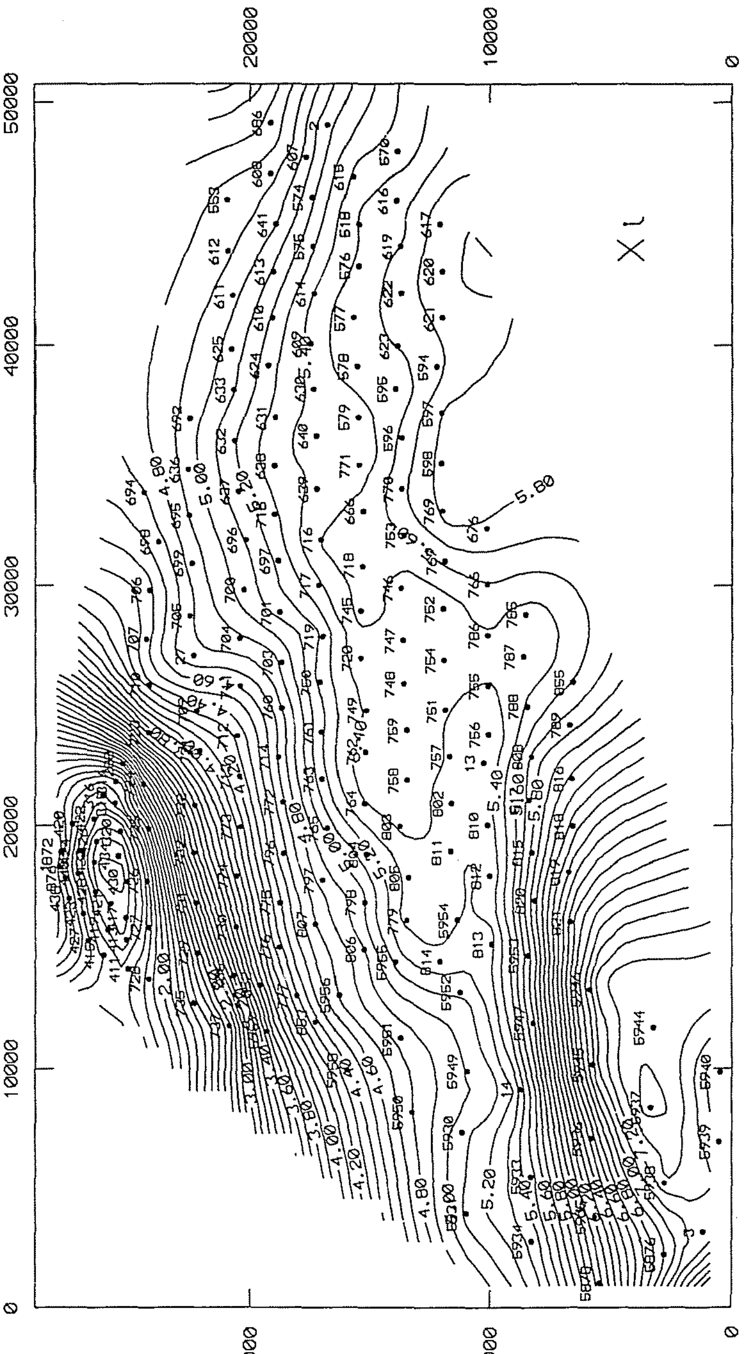


Fig. 18.

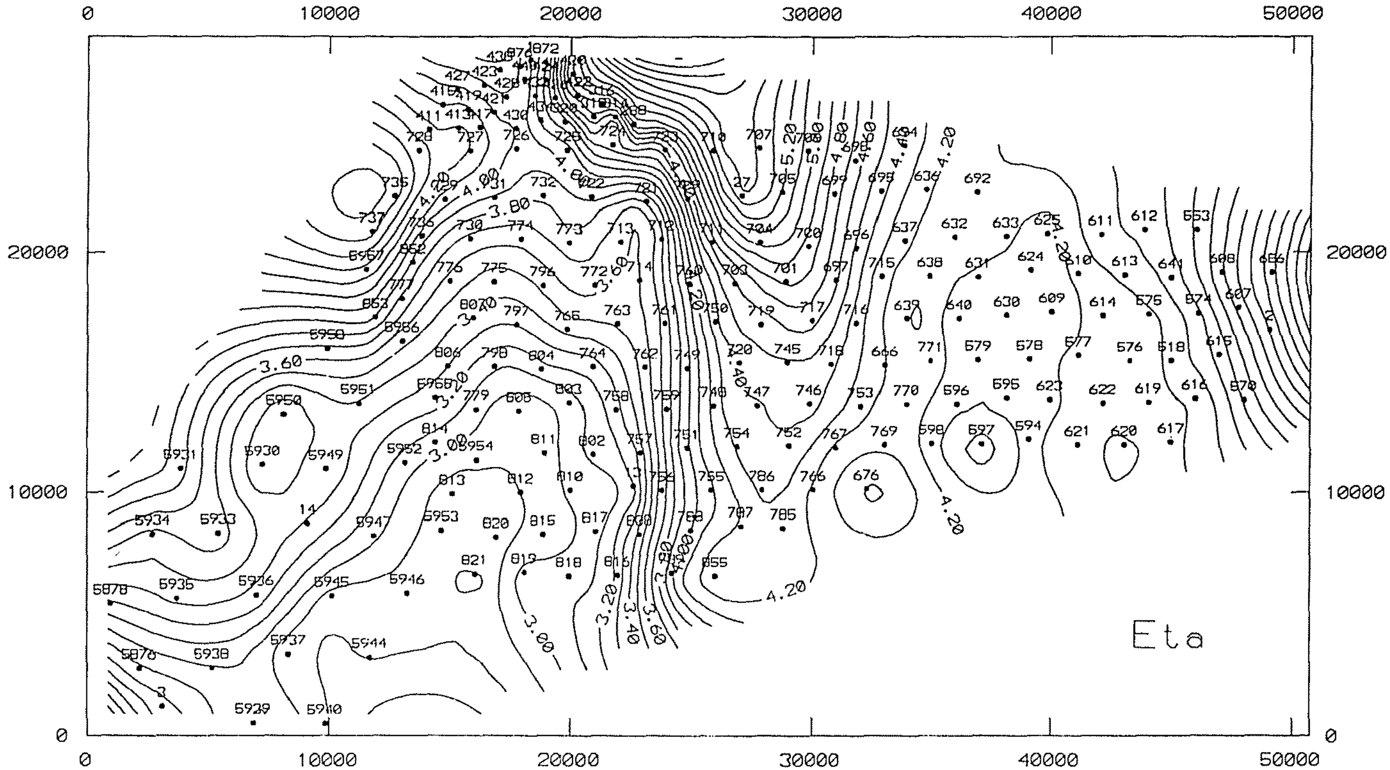


Fig. 19.

4 this requirement evidently has not been met. This difficulty is hard to overcome because if grid constant reduced in the given area, then number of unknowns will increase.

2. Another source of error is the interpolation of ΔW_Δ and ΔW_{xy} values at grid points of the square-shaped network from measurement data of neighbouring points because there are no torsion balance measurements at these corner points. These interpolated second derivatives – mainly at such more 'disturbed' sites as the surrounding area of point 1 – can considerably differ from actual values.

To eliminate the above mentioned two sources of error it is useful to choose torsion balance measurement sites to be points of the interpolation network (in more disturbed areas according to the density of torsion balance measurements) with an increased point density and to interpolate an arbitrary shaped network instead of a regular square-shaped one. Recently, however, by applying modern computer technique there is no need to make our computations more simple by using a regular square-shaped grid.

The interpolation network in *Fig. 17* was created to get rid of these sources of error. This network has 206 points in all and 203 of these are points with unknown deflections. Since there are two unknown components of deflection of the vertical at each point there are 406 unknowns for which 558 equations can be written. In *Figs. 18* and *19* ξ and η components of deflections of the vertical are visualized in isoline maps that resulted from the computation. Besides this it was given in the second part of *Table 5* how large deviations arose with program FUGGOORT at checkpoints between *computed* and *known* ξ, η values. Standard deviations $m_\xi = \pm\ 0.60''$ and $m_\eta = \pm\ 0.65''$ computed from these departures at checkpoints corroborate the fact that even for large continuous ξ, η values of acceptable accuracy can be computed where the interpolation network is suitable.

4. Summary and Conclusions

A program package for mainly PC and partly for larger computers were developed for actual computations which can be used to determine deflections of the vertical by any method of interpolation either along chains or in networks covering arbitrary areas, to draw interpolation network and vectors of interpolated deflections of the vertical.

Test computations were performed in the area surrounding *Cegléd*, extending over some 1200 km^2 and well measured by torsion balance, where both topographic conditions and the densities of torsion balance and astrogeodetic stations reflect average flatland conditions in Hungary; more to this there was a possibility to check calculations because astrogeodetic and astrogravimetric data were available.

During our test computations firstly we made a comparison between some practical methods of interpolation. First a conclusion was drawn that it is not advantageous to use the traditional Eötvös and Renner method because a considerable surplus work is done when $4n-6$ unknown values are dealt with instead of the absolutely necessary $2n-2$ $\Delta\xi$ and $\Delta\eta$ unknown components of deflection of the vertical at n points of the interpolation network. And this affects disadvantageously the accuracy of results in large networks since rounding errors are unavoidable to accumulate. Unfavourable observations were gained through the application of successive elimination method since it was established that deflections of the vertical computed by successive elimination have two times as big errors. Hence it is not practical to use this method as well. The most advantageous among practical solutions of interpolations are those methods of solution where directly ξ, η values are chosen and computed at points instead of $\Delta\xi$, $\Delta\eta$ differences of deflection components between two points. In our test computations we gained the most favourable experiences through the practical application of matrix orthogonalization process.

An answer was searched for during our tests to the question whether the accuracy of interpolated deflections of the vertical can be improved by using appropriate weights. By the analysis of our calculations it can be established that the accuracy of interpolated values slightly increases when the observations, which are based on torsion balance measurements, are provided with a weighting inverse to the square of distance between interpolation points. The accuracy of interpolated values in our test area was improved by a value of $0.08''$ when the above weighting was applied.

We intended also to determine the optimal geometrical arrangement of interpolation networks and it was treated as one of the most important problems. The results of our test computations have shown that the worst geometrical arrangement is to create simple chains between astrogeodetic points. It is extremely disadvantageous when successive elimination is used to create simple chains where the azimuth of the line connecting the starting and closing points of the chain is close to 0° (180°) or 90° (270°). Nearest interpolated values to known deflections of the vertical at checkpoints were obtained when the computation was performed along double chains. In this case the standard deviation $\pm 0.53''$ of interpolated deflections of the vertical was yielded from known deflections at checkpoints. The worst results were provided by the geometrical arrangement of Renner between interpolation points in our test area. There were two major reasons for this as our investigations have shown. First, the same grid constant should not be used for all parts of the area since thus differences of ΔW_Δ and ΔW_{xy}

between neighbouring points will not even approximately remain linear in more 'disturbed' areas. Second, torsion balance measurement data must be interpolated for points of the square grid hence values so determined – especially in more 'disturbed' areas – can deviate considerably from actual values. To eliminate these two error sources it is useful to choose torsion balance measurement sites to be points of the interpolation network (in more disturbed areas according to the density of torsion balance measurements) with an increased point density and to interpolate an arbitrary shaped network instead of a regular square-shaped one. Standard deviations $m_\xi = \pm 0.60''$ and $m_\eta = \pm 0.65''$ computed from deviations at checkpoints in such a network in our test area confirm the fact that even for large continuous areas ξ, η values of acceptable accuracy can be determined where the geometrical arrangement of interpolation network is suitable.

If in some area of Hungary which was surveyed by torsion balance there were astrogeodetic points available with an increased density of points in the future it would be also important to test the optimal density of astrogeodetic points of restraint for interpolation, that is whether it is possible to decrease errors of interpolated deflections of the vertical by increasing number of astrogeodetic points.

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