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ON THE K_0 FACTOR

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Abstract

After the presenting of the significance of the K_0 factor the paper discusses some aspects of the formulae for the coefficient of the earth pressure at rest mainly the well-known Jáky's equitation. The history and the background of this were shown and analyzed. An improvement for the critical Jáky's assumption was introduced and a modified $K_0 \ldots \overline{\varphi}$ relationship was deducted. All the formulae originated on Jáky's idea and some other continuum mechanical or speculative equitations are tested against 153 measured data. As conclusions of these analyses the formulae were qualified and the original Jáky's equitation, the ones from Vierbiczky and Matsuoka-Sakakibara were found to be the best.

Keywords: Earth pressure, K_0 factor, initial stress state, statistical analyses.

1. Introduction: Significance of the K_0 Factor.

Doubtless, the internationally most recognized achievement of soil mechanical research in Hungary has been the formula for the earth pressure at rest

$$K_0 = 1 - \sin \overline{\varphi} \tag{1}$$

due to Professor JÁKY.

The earth pressure at rest or better, its coefficient K_0, has first been interpreted by DONATH (1891), while TERZAGHI (1920) was the first to publish measurement data for K_0. The first theoretical approach to the problem is due to JÁKY (1943), and subsequent modifications of his theory formulated at that time resulted in *Eq.* (1).

At that time, the only importance of this formula was considered to be the determination of earth pressure acting on motionless retaining walls and this view still persists in this country.

On the other hand, in the international literature of the last two decades, the K_0 state was given a broader meaning and a higher importance as natural ('in-situ'), or primary ('undisturbed') soil condition. Namely, in the past two decades, the 'stress history' approach came to prevalence in handling soil behaviour. As a matter of fact, this is a generalization of the long-known fact that soil is a non-linear, inelastic, anisotropic matter, 'remembering' former stress changes. Thus, it is determinant, what had been the initial stress state before stresses were altered by engineering interventions. Neither is it irrelevant, by what stress path the affected soil zone reached the condition previous to intervention.

In up-to-date, computerized FEM geotechnical design procedures, the K_0 factor is usually required as an input, and computations show this value to significantly affect e.g. the safety factor of slope stability (LO and LEE (1973)), or slurry trench wall behaviour (FOURIE and POTTS (1989)). It seems to be proven that anisotropy of shear strength and of deformational characteristics is due to soil formation in K_0 condition. Therefore recent soil models comprise as an initial starting condition the K_0 condition, or closer, its numerical value, the K_0 factor (e.g. OHTA et al. (1985)).

Importance of the K_0 factor has thereby much increased, inducing development of great many new instruments and Procedures for laboratory and field determination of K_0. These methods, however, are not as widely used in Hungarian practice as in more developed countries. Therefore the determination of K_0 still relies on formulae, $K_0 = f(\overline{\varphi})$, which were deduced theoretically (from mechanics of granular continuum), determined empirically (by regression analysis of test results), or formulated speculatively (applying certain considerations). And although since 1948 when JÁKY published *Eq.* (1), a number of such formulae has emerged, the best acknowledged, most frequently used formula has remained to now *Eq.* (1) due to JÁKY.

In an extended study, the Author has considered all the aspects above as well as others of the K_0 condition (SZEPESHÁZI (1993)). Out of them, of course, those discussing *Eq.* (1) by JÁKY, its background, proof and amendment possibilities will be considered here.

2. History of Jáky's Formula

Although *Eq.* (1) is often referred to as the base of comparison in appreciating most of new theoretical or measurement results, in quotations often some doubt or reservation appears. This is likely to arise from the little known origin of the formula, — it is e.g. often quoted as an empirical relationship. Digging into the formula history detected some unexpected facts.

It is little known that JÁKY published his first paper on earth pressure at-rest in the periodical *Technika* No. 9, 1943 (JÁKY (1943)). Its fundamental idea was the same as that to be in the next years paper, but its deduction and final result were different. My analyses showed imperfections in the mathematical deduction, may be responsible for the fact that JÁKY himself did not refer later to this early work, and published another solution in 1944. But even the deduction published therein is little known, it is difficult to explain why KÉZDI's (1972) standard work devotes but a few lines to it, while presenting other deductions in detail. It may have obtained wider publicity in the book by KÉZDI (1961), but by now it is fading into oblivion.

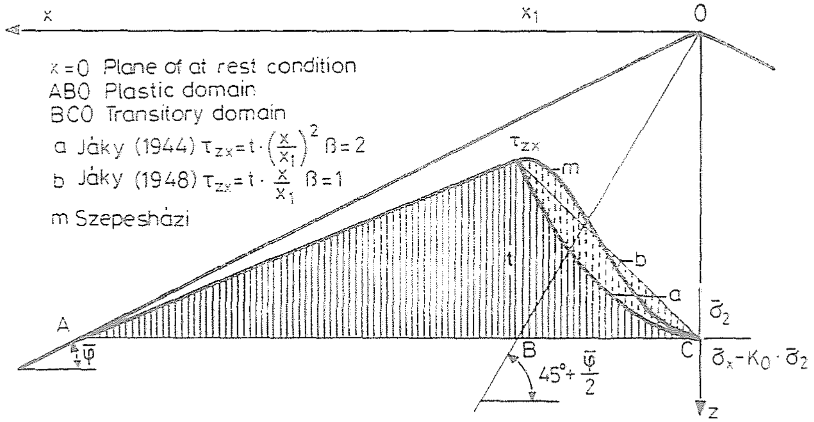

Fig. 1. JÁKY's soil wedge and the different functions $\tau_{zx} = f(x)$

In his 1944 paper, JÁKY proceeded from the stress state of the soil wedge in *Fig. 1* to the formula

$$K_0 = \frac{1 + \frac{2}{3}\sin\overline{\varphi}}{1+\sin\overline{\varphi}}\,[1 - \sin\overline{\varphi}] \tag{2}$$

(Analysis of the deduction follows later.) Examining the fraction value JÁKY found it to hardly differ from 0.9 in the range of $20° < \overline{\varphi} < 40°$, so he recommended the expression

$$K_0 = 0.9\,(1 - \sin\overline{\varphi}) \tag{3}$$

for practical uses. No justification was found for omitting later the factor 0.9. Formula

$$K_0 = 1 - \sin\overline{\varphi} \tag{1}$$

was first applied by JÁKY (1948) at the Conference in Rotterdam 1948, without any explanation.

Hence, unwillingly or not, one has to state: *Eq.* (1) of general use is, in fact, theoretically unjustified, it may be considered at most as a practical simplification of *Eq.* (2).The difference is about 15% that might be acceptable in practice worth the 'grandiousness of simplicity' of *Eq.* (1) but for the science, this problem has to be clarified perhaps right in Hungary, and right in JÁKY's memorial issue.

3. Analysis of the Deduction by Jáky

In his deduction, JÁKY examined the soil wedge bounded by a slope of natural inclination according to *Fig. 1* since in its vertical symmetry plane at rest condition prevails. This assumption is hardly debatable. Stress state in the plastic domain ABO of the wedge is characterized according to JÁKY by

$$\sigma_x = t\,\frac{1+\sin^2\overline{\varphi}}{\sin\overline{\varphi}\,\cos\overline{\varphi}}, \tag{4}$$

$$\sigma_z = t\,\mathrm{ctg}\,\overline{\varphi}, \tag{5}$$

$$\tau_{zx} = t, \tag{6}$$

where

$$t = z\,\gamma\,\sin\overline{\varphi}\,\cos\overline{\varphi} - x\,\gamma\cdot\sin^2\overline{\varphi}. \tag{7}$$

OB is the domain boundary and, at the same time, a principal direction, expressed as:

$$x_1 = z\,\mathrm{tg}\,(45 - \overline{\varphi}/2). \tag{8}$$

Obviously, τ_{zx} linearly increases between A and x_1, in conformity with the diagram, and at $x = x_1$:

$$\tau_{zx1} = x_1\,\gamma\,\sin\overline{\varphi} = z\,\gamma\,\sin\overline{\varphi}\,\mathrm{tg}\,(45 - \overline{\varphi}/2). \tag{9}$$

Transition domain BCO is in an unknown stress state. For its determination, JÁKY assumed shear stress to vary along a line $z =$ const. as a quadratic parabola as seen in *Fig. 1*:

$$\tau_{zx} = \tau_{zx1}\left(\frac{x}{x_1}\right)^2 \tag{10}$$

Hence JÁKY making use of Cauchy's equilibrium equations determined stresses σ_x and σ_x acting along OC, and the K_0 factor as their quotient.

The only critical, debatable element of the deduction is the arbitrary assumption made for the variation of τ_{zx}. It is therefore justified to reconsider the validity of *Eq.* (10) and how this assumption affects the final result.

It is imperative to have $\tau_{zx} = 0$ in the vertical, $x = 0$ plane z — a plane of symmetry — being also a principal direction. It cannot be considered as imperative but it seems justified to have here also $d\tau_{zx}/dx = 0$, made by JÁKY to be met by *Eq.* (10). Yet there is nothing to justify τ_{zx} to vary just according to a quadratic function, since the requirement of horizontal tangent at the centre is met for any exponent higher than 1.

Let us consider how the exponent β of a power function of a more general form than (10) affects the K_0 value.

$$\tau_{zx} = \tau_{zx1} \left(\frac{x}{x_1}\right)^{\beta} \tag{11}$$

Deduction after JÁKY (1944) leads to:

$$K_0 = \frac{1 + \frac{2}{\beta+1} \sin \overline{\varphi}}{1 + \sin \overline{\varphi}} [1 - \sin \varphi] \tag{12}$$

For $\beta = 2$ this of course yields JÁKY's *Eq.* (2), while $\beta = 1$ leads curiously just to *Eq.* (1). As seen in *Fig. 1*, exponent $\beta = 1$ means linear variation of τ_{zx} in domain BCO. This is the most trivial variation ('the shortest path between two points being the straight'), arguing for *Eq.* (1). For $\beta = 1$, however, at $x = 0$, the valid requirement of $d\tau_{zx}/dx = 0$ is not satisfied. While β is more than unity but tends to unity (from above), then *Eq.* (12) tends to *Eq.* (1), and this assumption is by no means inferior to JÁKY's $\beta = 2$. Even, it may be less considered as arbitrary, since $\beta = 1$ was seen to lead to the most trivial transition. Thus, it may be concluded that *Eq.* (1) of general use can be deduced, in fact, from JÁKY's theory, at a slight, justified modification.

4. A Possibility to Improve Jáky's Solution

Variations of τ_{zx} both after JÁKY and according to (11) are objectionable in that along the OB line they do not fit tangentially the up to then linear variation, although break is not justified by anything. For instance, CHOWDURRY (1978) published curves in *Fig. 2* obtained by FEM for a similar stress variation inside an embankment.

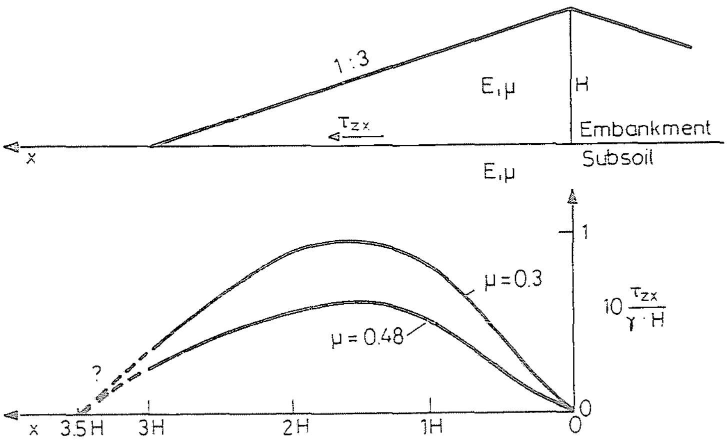

Fig. 2. Variation of stress τ_{zx} under a slope, according to FEM results by Chowdurry (1978)

Pondering this fact, JÁKY's assumption and my former statements it seems justified to look for a new τ_{zx} function, that meets — in addition to the three boundary conditions specified by JÁKY. — also the requirement of tangential function at x_1, the least deviate in the domain $0 < x < x_1$ from the most trivial linear transition τ_{zx}. Advisably, a function of the form

$$\tau_{zx} = A\, z\, x^B\, e^{c\,x} \tag{13}$$

or its Mac-Laurin set

$$\tau_{zx} = A\, z\, x^B\, (1 + C\, x) \tag{14}$$

will be tried, automatically meeting the first two requirements, values of parameters A and C as a function of B may be expressed from the third and fourth requirements, hence, B may be determined from the fifth requirement.

Factor K_0 has been determined from the assumed function τ_{zx} plotted in *Fig. 1.* Omitting details, the final result:

$$K_0 = (1 - \sin\overline{\varphi}) \left[\frac{\sin\overline{\varphi}}{(1 + \sin\overline{\varphi}) \cdot (\sin\overline{\varphi} + \sqrt{4.5 + 4\,\sin\overline{\varphi}} + 3)} \right]. \tag{15}$$

In the range $20^\circ < \overline{\varphi} < 40^\circ$ this intricate formula may be properly approximated (with a max. error of 1.2%) as:

$$K_0 = 0.95\,(1 - \sin\overline{\varphi}) \tag{16}$$

Curiously, it yields exactly the 'average' from JÁKY's formulae (1) and (3).

5. Formulae $K_0 = f(\overline{\varphi})$ Confronted with Measurement Results

153 measured data pairs $K_0 \ldots \overline{\varphi}$ have been collected from the literature, to be compared with values obtained from various formulae among them those presented above.

Mean and standard deviation of quotients of K_0 factors measured and computed from the different formulae have been determined. *Table 1* contains 'rating data' for formulae discussed above, and for two other formulae found to be good in the investigation. The 153 measurement data and the tested relationships have been plotted in *Fig. 3.*

MATSUOKA et al. (1987) deduced their formula from a rather promising soil model (based on continuum mechanics), while VIERZBICZKY (see RYMSZA (1979)) applied the Rankine factor, assuming a mobilization of two thirds of the internal friction angle (i.e. by speculative means) to obtain the formula quoted in the *Table 1.* (Justification of this latter assumption was supported by the Author in his Thesis, demonstrating that in K_0 state even according to several soil failure criteria about 60 to 70 % of the shear strength is mobilized).

Table 1

K_0-formula		$K_{0\text{meas}}/K_{0\text{comp}}$	
Author	Form	Mean	Standard deviation
Jáky (1944) (3)	$0.9\,(1-\sin\overline{\varphi})$	1.113	0.191
Jáky (1948) (1)	$1-\sin\overline{\varphi}$	1.020	0.171
Szepesházi (16)	$0.95\,(1-\sin\overline{\varphi})$	1.073	0.181
Matsuoka et al	$1/(1+2\sin\overline{\varphi})$	0.956	0.132
Vierzbiczky	$\mathrm{tg}^2(45-\overline{\varphi}/3)$	1.006	0.128

Table 1 and the diagrams lead to the following conclusions. Maybe against expectations, the formula by VIERZBICZKY proved the best. Practically, it has no standard error ($k \approx 1.0$) and it has the lowest standard deviation among all. Another advantage is to be of the same form as the Rankine factors. Formula by MATSUOKA and SAKAKIBARA is hardly inferior. It somewhat underestimates the K_0 value — according to *Fig. 3*, mainly in the range $20° < \overline{\varphi} < 25°$. It is of a form somewhat cumbersome to handle, but this fact is offset by its theoretical exactness. It also appears from *Fig. 3* that formulae by VIERZBICZKY and MATSUOKA-SAKAKIBARA obtained

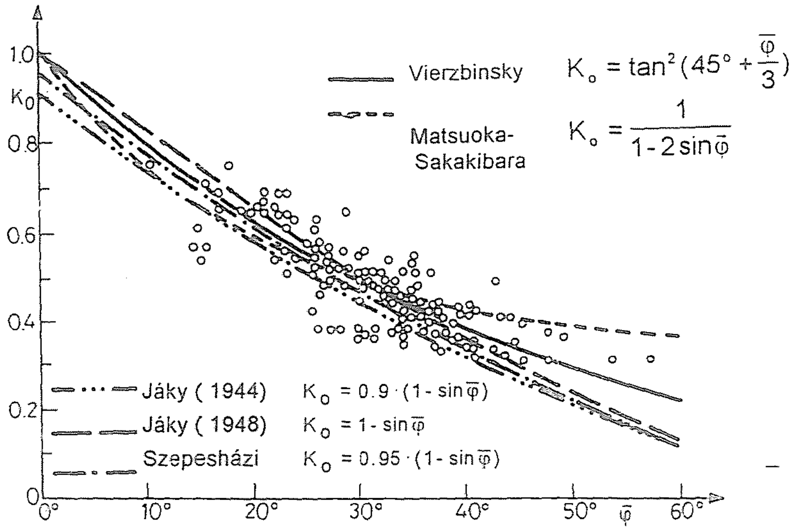

Fig. 3. The highest rated relationships $K_0 = f(\overline{\varphi})$ according to measurements

the best statistical rating since in the range $\overline{\varphi} > 35°$ they yield mildly decreasing values, again, well fitting measurements.

According to the *Table 1*, among formulae relying on JÁKY's theory, *Eq.* (1) is the best, with parameters hardly inferior to the former ones. According to the diagram for $\overline{\varphi} < 35°$ it may supersede them, this global rating is only impaired by underestimation for higher $\overline{\varphi}$ values, but this range has a lesser practical significance. *Eq.* (16) resulting from the Author's theoretical improvement yields a somewhat better curve for the range $\overline{\varphi} < 35°$, but because of deviations beyond $\overline{\varphi} > 35°$, the overall rating is inferior to that for *Eq.* (1).

Appreciation of this rating has to be completed by stating that most of the collected data originate from publication on new measurement methods. In appreciating measurements the reference was mostly *Eq.* (1) by JÁKY and fair coincidence between measured values and K_0 factors computed by *Eq.* (1) was generally meant as demonstrating reliability of the measurements, as if calibrating them. This fact impairs somewhat the good rating of *Eq.* (1), and puts a higher rating on formulae by VIERZBICZKY and MATSUOKA-SAKAKIBARA.

6. Summary

Presentation of the overall, increasing importance of the K_0 factor beyond computation of earth pressure at rest is followed by analysis of the solution by Prof. JÁKY for the K_0 factor. The generally used *Eq.* (1) was published by JÁKY without proof, probably as simplification of *Eq.* (3), and even this deduction comprised an arbitrary assumption. It is clear therefore that *Eq.* (1) is more justifiable than *Eq.* (3), the former having resulted from an assumption less arbitrary than that for deducing *Eq.* (3). Further improvement of the critical assumption has led to another formula $K_0 = f(\overline{\varphi})$, mean between the two formulae by JÁKY.

Formulae were tested against great many measurements $K_0 \ldots \overline{\varphi}$ and *Eq.* (1) was found to be correct mainly in the range $\overline{\varphi} < 35°$. JÁKY's original *Eq.* (3) yielded less accurate results by about 10%, while the Author's formula (16) by 5%, *Eq.* (1) was superseded by speculative and continuum mechanical relationships by VIERZBICZKY and by MATSUOKA-SAKAKIBARA, respectively.

References

DONATH, A. D. (1891): Untersuchungen über Erddruck auf Stützwände, *Zeitschrift für Bauwesen*, Berlin.

CHOWDURRY, R. N. (1978): Slope Analysis. Developments in Geotechnical Engineering. Elsevier Scientific Publ. Co., Amsterdam, Oxford, New York.

FOURIE, A. B. – POTTS, D. M. (1989): Comparison of Finite Element and Limiting Equilibrium Analyses for an Embedded Cantilever Retaining Wall. *Geotechnique*, Vol. 39, No 2.

JÁKY, K. (1943): Natural Earth Pressure (in Hungarian). *Technica*, No. 9.

JÁKY, J. (1944): Coefficient of Earth Pressure at rest. (In Hungarian), *Magyar Mérnök és Építész Egylet Közlönye*, No. 22.

JÁKY. J. (1948): Pressure in Silos. *Proc. 2nd Int. Conf. on Soil Mech. and Found. Eng.*, Vol. 1, Rotterdam.

KÉZDI, Á. (1961): Erddrucktheorien. Springer-Verlag, Berlin.

KÉZDI, Á. (1972): Soil Mechanics. I. (in Hungarian), 4th ed., Tankönyvkiadó, Budapest.

LO, K. Y. – LEE, C. F. (1973): Analysis of Progressive Failure in Clay Slopes. *Proc. 8th Int. Conf. on Soil Mech. and Found Eng.*, Vol. 1, Moszkva.

MATSUOKA, H. – SAKAKIBARA, K. (1987): A constitutive Odel for Sands and Clays Evaluating Principal Stress Rotation. *Soils and Foundation*, Vol. 27, No. 4.

OHTA, H. – NISHIHARA, A. – MORITA, Y. (1985): Undrained Stability of K_0 Consolidated Clays. *Proc. 11th Int. Conf. on Soil Mech. and Found. Eng.*, Vol. 1, San Francisco.

RYMSZA, B. (1979): Earth Pressure at Rest in Design of Retaining Structure. *Proc. 7th Eur. Conf. on Soil Mech. and Found.* Vol. 1, Bringhton.

SZEPESHÁZI, R. (1993): Characteristics, Methods of Determination, and Consequences of Initial (K_0) Stress States of Soils (in Hungarian), Doctor's Thesis, Technical University of Budapest.

TERZAGHI, K. (1920): Old Earth Pressure Theories and New Test Results. *Engineering NewsRecord*, Vol. 85, No. 14.