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NON-EUCLIDEAN APPROACH OF FLOW ON AN EUCLIDEAN PLANE AND ITS DESCRIPTION WITH COMPLEX VARIABLES

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Abstract

The set of curves giving the general solution of the differential equation $y'' = y/k_B^2$ and describable with exponential equations also has properties characteristic of non-Euclidean geometries in the Euclidean plane. They also allow a non-conventional geometrical determination of trigonometric functions promoting the extension of the representation of complex numbers and variables. Thus relations of seepage flows concerning hydraulics of wells and several conclusions drawn from them can be extended and the surveying of mutual interference of wells will be simpler. This paper also gives example for using non-Euclidean methods in geometrical considerations for technical purposes, in our case for describing plane flows.

Keywords: non-Euclidean flow, complex variable functions.

1. Lines of Exponential Equation in the Euclidean Plane

1.1 Derivation and Determination of the Lines of Exponential Equation

Those $y = y(x)$ lines of a plane plotted in a rectangular coordinate system representing the general solution of the second-order differential equation where $k_B > 0$

$$\frac{d^2y(x)}{dx^2} = \frac{y(x)}{k_B^2}, \tag{1}$$

(the set of lines also comprises coordinate axes and $x =$ const. lines, that is, lines parallel with the y axis as well) are called *lines of exponential equation* or *exponential lines* as they have the shape

$$y(x) = C_1 e^{x/k_B} + C_2 e^{-x/k_B}, \tag{2}$$

where C_1 and C_2 are real numbers.

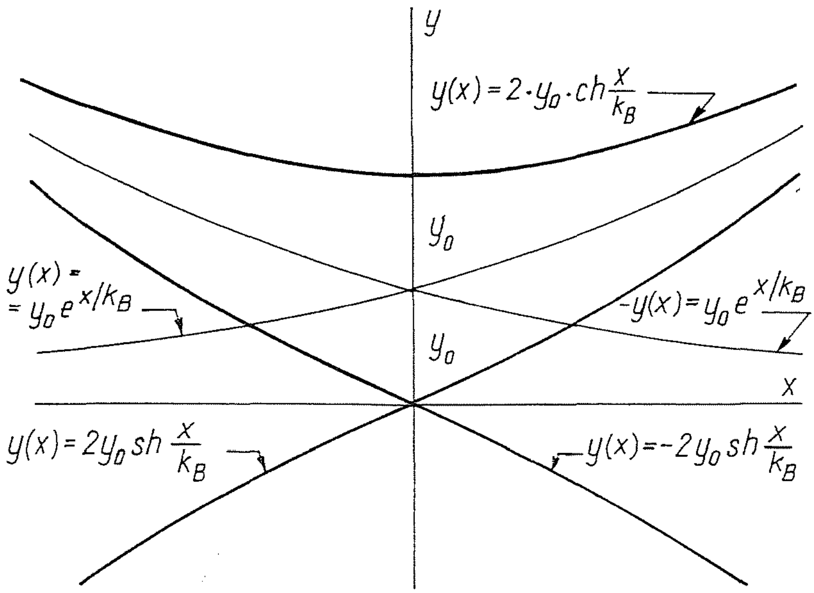


Fig. 1. Addition of lines of exponential equation results in lines of hyperbolic equation

If in *Eq.* (2) either $C_1 = 0$ and $C_2 = y_0 \neq 0$ or $C_1 = y_0 \neq 0$ and $C_2 = 0$ then (for both cases)

$$y(x) = y_0 \cdot e^{\pm x/k_B}, \tag{3}$$

that is, 'simple' exponential lines will be obtained. For further cases, when $C_1/C_2 > 0$, then instead of variable x with shifting the axis by x_0 the variable $(x - x_0)$ can be introduced and with selecting

$$x_0 = \frac{k_B}{2} \cdot \ln \frac{C_1}{C_2} \tag{4}$$

can be achieved that $y_0 = C_1 \cdot e^{-x_0/k_B} = C_2 \cdot e^{x_0/k_B}$ should be from which

$$y(x - x_0) = 2 \cdot y_0 \cdot \text{ch} \frac{x - x_0}{k_B}. \tag{5}$$

In the case of $C_1/C_2 < 0$ with an x_0 selection defined by sign change within the logarithmic expression in *Eq.* (4) $y_0 = C_1 \cdot e^{-x_0/k_B} = -C_2 \cdot e^{x_0/k_B}$, thus

$$y(x - x_0) = 2 \cdot y_0 \cdot \text{sh} \frac{x - x_0}{k_B}. \tag{6}$$

Hence for a general case lines expressible by *Eq.* (2) are cosine hyperbolic or sine hyperbolic curves in a coordinate system of appropriately shifted horizontal axis (*Fig. 1*).

1.2 Geometrical Relations in the System of Exponential Lines

1.2.1 In the case of fixed k_B and coordinate axes only one exponential line links up with two given points – $A(x_A, y_A)$ and $B(x_B, y_B)$ – of the Euclidean plane, since from *Eq.* (2):

$$\begin{aligned} y_A &= C_1 \cdot e^{x_A/k_B} + C_2 \cdot e^{-x_A/k_B}, \\ y_B &= C_1 \cdot e^{x_B/k_B} + C_2 \cdot e^{-x_B/k_B}. \end{aligned} \tag{7}$$

The linear equation with two unknowns C_1 and C_2 is unequivocal since the D determinant of the 'coefficients' consisting of the exponential expressions is:

$$D = 2 \cdot \mathrm{sh}\,(x_A - x_B) \neq 0, \tag{8}$$

except when $x_A = x_B$. For this case (if $y_A \neq y_B$ at the same time) the line fitted to the two points and considered to be exponential will be the straight line $x = x_A$ parallel to the y axis also seen in the basic definition.

1.2.2 With a fixed k_B value and coordinate system two not coinciding exponential lines will

(*a*) either cross each other in one point (they have a real common point)

(*b*) or approximate each other asymptotically (they have a common 'infinity point')

(*c*) do not cross each other (have imaginary common points).

All these follow from *Eq.* (7) in the case of $y_A = y_B$ and $x_A = x_B$. The exponential lines $y = y_1(x)$ and $y = y_2(x)$ have the same coordinates in the common point $P(x_p, y_p)$. Let the coefficients of line y_1 be C_{11} and C_{12}, those of y_2 C_{21} and C_{22}. Thus

$$C_{11} \cdot e^{x_P/k_B} + C_{21} \cdot e^{-x_P/k_B} = C_{12} \cdot e^{x_P/k_B} + C_{22} \cdot e^{-x_P/k_B}. \tag{9}$$

After reduction:

$$x_P = \frac{k_B}{2} \cdot \ln \frac{C_{22} - C_{21}}{C_{11} - C_{12}}. \tag{10}$$

Let us mark the fraction in logarithm with G. If

(*a*) $G > 0$ then x_p is real, and so is the crossing point,

(*b*) $G = 0$ then $x_p \to -\infty$,
$1/G = 0$, then $x_p \to \infty$, both versions mean that the two exponential lines meet asymptotically

(*c*) $G < 0$, then x_p is imaginary and so is the crossing point.

The *crossing, asymptotic* or *non-crossing* position of the exponential lines do not correspond to the *crossing* or *parallel* behaviour of Euclidean straights but to the lines classified as 'straights' in the *Bolyai geometry* which really may be in *crossing, 'parallel'*, (here: '*asymptotic*') or *non-crossing* position.

1.2.3 Having a fixed k_B value and coordinate system and the $y = y_1(x)$ equation of at least one exponential line fitting to the point $P(x_p, y_p)$ known the equation of all other exponential lines going across this point can be determined from this.

A selection $x_p = 0$ can be made in the origin of the abscissa without detriment to general validity. Using *Eq.* (9) for two exponential lines fitting to point P the exponential coefficients will be units, thus only the C coefficients remain:

$$C_{11} + C_{21} = C_{12} + C_{22},$$

or

$$C_{21} - C_{22} = C_{12} - C_{11} = C.$$

With this marking

$$\begin{aligned} C_{12} &= C_{11} + C, \\ C_{22} &= C_{21} - C. \end{aligned} \tag{11}$$

Hence if the equation of the exponential line fitting to a point P is known

$$y_1(x) = C_{11} \quad e^{x/k_B} + C_{21} \quad e^{-x/k_B},$$

then the equation of any other exponential line fitting to the same point ($k_B = \text{const.}$):

$$y(x) = (C_{11} + C) \cdot e^{x/k_B} + (C_{21} - C) \quad e^{-x/k_B} = y_1(x) + 2 \quad C \quad \text{sh}\,\frac{x}{k_B}. \tag{12}$$

The exponential equations of the lines going across the same P point only differ in an additive sine hyperbolic function.

1.2.4 Two, crossing exponential lines determine two further ones asymptotical in one direction to the first, in the other direction to the second crossing exponential line.

Let the coefficients of the two crossing exponential lines according to the form (2) be C_{11} and C_{21}, C_{12} and C_{22}, respectively. The coefficients of the further two lines approaching these asymptotically: C_{13} and C_{23}, C_{14}

and C_{24}, respectively. Pursuant to the conditions of *Eq.* (10) if $C_{23} = C_{21}$ and $C_{13} = C_{12}$ then the No. 3 exponential line will approach the No. 1 line at the 'left' side, the No. 2 line at the 'right' side. If $C_{14} = C_{11}$ and $C_{24} = C_{22}$ then the No. 4 exponential line will asymptotically approach the No. 2 line at the 'left' side and the No. 1 line at the 'right' side.

As the condition for the coefficients can always be satisfied our theorem may also be reversed:

There are two exponential lines that can be constructed from an outside point to any exponential line which do not complement each other to a single line and approach the given exponential line one in the first, the other in the second direction.

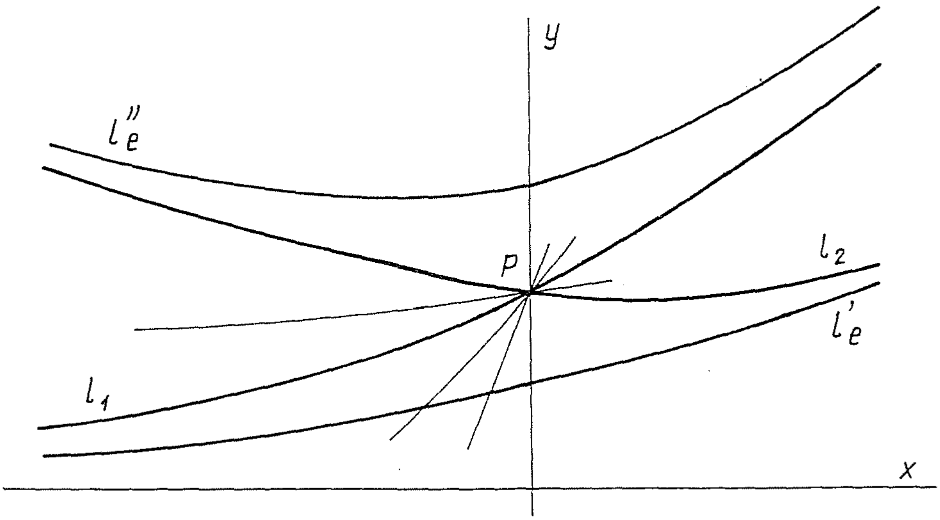


Fig. 2. To any exponential line l_e' or l_e'', through the point P outside them, two exponential lines l_1 and l_2, asymptotical to them, can be coordinated.

This statement is identical with the Bolyai axiom of parallelism if the concept of 'parallelism' is identical with the concept of 'asymptotic approach' (*Fig. 2*). Although theorems stated in paragraph 1.2 for exponential lines in the Euclidean plane and also in the Euclidean rectangular coordinate system correspond to the theorems of the Bolyai geometry the geometry of exponential lines described this way is not a Bolyai geometry as we cannot identically define the 'congruence' axiom. The geometry of exponential lines, however, corresponds to the Bolyai geometry in many theorems and several approaches may point to further uniformities.

1.3 Trigonometric Interpretations in the System of Exponential Lines

Three points in a plane not coinciding with the same exponential line define three exponential lines (according to 1.2.1). These three exponential lines surround an *exponential triangle.* This is called *vertical side triangle* if one side is a straight parallel to the y axis. The exponential triangle of vertical side is *orthogonal* if its base is the line of the x axis. The orthogonal triangle is *central* if its vertex, opposite to the right angle and being on the x axis, is in the origo.

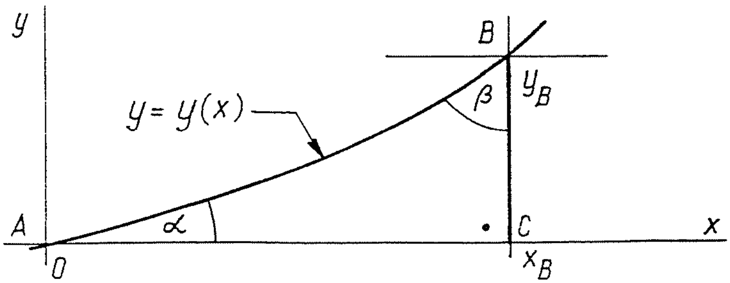


Fig. 3. Central, orthogonal, exponential triangle

Suppose the vertexes and their coordinates of the central orthogonal exponential triangle be $A(0,0)$, $B(x_B, y_B)$, $C(x_B, 0)$. One right angle side of the triangle is the part of the x axis extending from the origo to an x_B distance, the other a vertical straight of y_B height started from the C axis point with x_B abscissa, its hypotenuse is an exponential line going through the starting point (A) and the B point with the equation: (*Fig. 3*)

$$y(x) = \frac{y_B}{\operatorname{sh} \frac{x_B}{k_B}} \cdot \operatorname{sh} \frac{x}{k_B}. \tag{13}$$

The differential function of this:

$$\frac{dy(x)}{dx} = \frac{\frac{y_B}{k_B}}{\operatorname{sh} \frac{x_B}{k_B}} \cdot \operatorname{ch} \frac{x}{k_B}. \tag{14}$$

$x = 0$ at the vertex of the triangle and for this point the value of the differential function is the tangent of the vertex angle belonging to point A:

$$\operatorname{tg} \alpha = \frac{\frac{y_B}{k_B}}{\operatorname{sh} \frac{x_B}{k_B}}. \tag{15}$$

Hence for an exponential – central, orthogonal – triangle the tangent function is not defined by the ratio of the right angle sides opposite and adjacent to it but with the modified form of this ratio involving the constant k_B and a hyperbolic function as well.

Using identities $\sin\alpha = \frac{\mathrm{tg}\,\alpha}{\sqrt{1+\mathrm{tg}^2\,\alpha}}$ and $\cos\alpha = \frac{1}{\sqrt{1+tg^2\alpha}}$

$$\sin\alpha = \frac{\frac{y_B}{k_B}}{A} \text{ and } \cos\alpha = \frac{\mathrm{sh}\,\frac{x_B}{k_B}}{A} \tag{16}$$

can be written, where

$$A = \sqrt{\mathrm{sh}^2\frac{x_B}{k_B} + \frac{y_B^2}{k_B^2}}$$

Therefore all trigonometric function definitions will be changed compared to the definitions of the Euclidean geometry. This is not a surprise for those who know spherical geometry and Bolyai geometry, however, *Eqs.* (15) and (16) do not meet the trigonometric function definition of either of the mentioned geometries.

2. Representation of Complex Numbers on the Euclidean Plane, in the Geometrical System of Exponential Lines

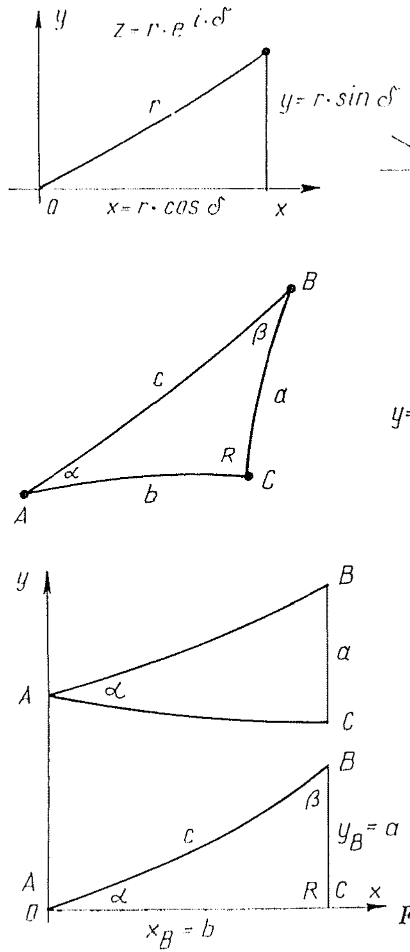
2.1 Interpretation of a Complex Number According to the Euler Theorem

The complex number composed by summing the real and imaginary parts in the form $z = x + iy$ can also be expressed in exponential form or with trigonometric functions according to the Euler theorem:

$$z = r\ e^{i\cdot\vartheta} = r\ (\cos\vartheta + i\cdot\sin\vartheta), \tag{17}$$

where $r = \sqrt{x^2+y^2}$ absolute value
$\vartheta =$ arcus or argumentum
$i = \sqrt{-1}$ imaginary unit.

Eq. (17) makes possible to represent the complex number by including it in a rectangular triangle in an orthogonal coordinate system according to the interpretation of the Euclidean geometry. The $A(0,0)$ vertex point of the triangle is the starting point, its $C(x,0)$ point is on the x axis, the $B(x,y)$ point represents the complex number which is now also given by the hypotenuse of r length and by the ϑ angle of the hypotenuse and the positive x axis.



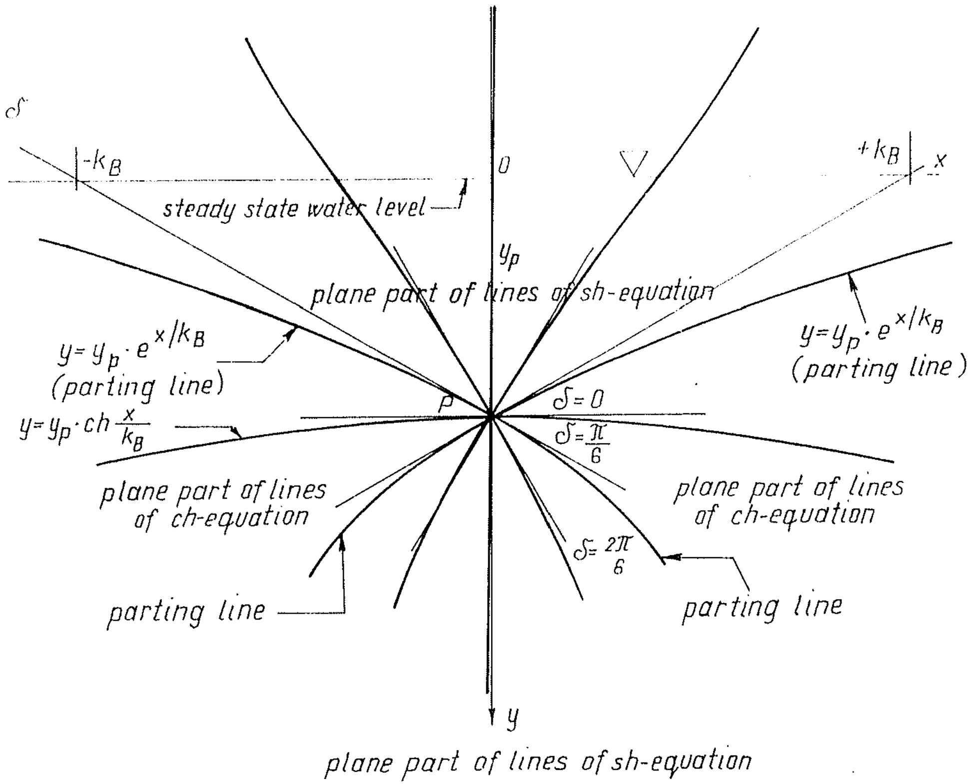


Fig. 4. An 'exponential set of radii' crossing the point P, which is the description of the complex variable transformation $w = \ln(z - z_0)$ in the system of exponential lines

It is essential, however, that trigonometric functions of the Euler theorem given by (17) are *mathematical functions*, the Euler theorem itself can also be deduced from function series disregarding the geometrical sense. This involves that if the trigonometric functions in (17) can also be given *another geometrical interpretation* than the Euclidean the complex number may also be represented according to this interpretation. If alternative interpretations are equally possible in the Euclidean plane, then the samé points in the same Euclidean plane may obtain different complex number interpretations, that is, there are different possibilities for complex number representation even in the same coordinate system.

The interpretation of the tg, sin, cos functions were previously given, in *Eqs.* (15) and (16), for central, orthogonal triangles composed of exponential lines. Having a fix k_B basic length in the existing system of exponential lines, by giving a new interpretation to the trigonometric functions in the Euler theorem the representation of the complex number may also be given a new interpretation (*Fig. 4*).

2.2 Representation of the Complex Numbers

Replacing trigonometric functions in *Eq.* (17) according to *Eq.* (16) derived from *Eq.* (15), including that $r = A$ and immediately reducing

$$z = A \quad e^{i \cdot \vartheta} = \operatorname{sh} \frac{x}{k_B} + i \quad \frac{y}{k_B}. \tag{18}$$

Hence the general $P(x, y)$ point represents the z complex number to be reckoned from *Eq.* (18) in the system of exponential lines with a selected k_B.

The locus of the points represented by complex numbers of $z = x + iy$ form in the Euclidean plane and Euclidean system of geometry is a straight of ϑ angle to the positive x axis if for any conjugate (x, y) values:

$$\frac{y}{x} = \text{const.} = \operatorname{tg} \vartheta. \tag{19}$$

Argumentum change yields a series of radii going through the centre. A series of radii passing through an arbitrary $P(0, y_P)$ point being without detriment to general validity on the y axis will be formed by the series of points determined by the complex numbers $z = y + i(y - y_P)$ for which:

$$\frac{y - y_P}{x} = \text{const.} = \operatorname{tg} \vartheta. \tag{20}$$

The locus of the points represented by the complex numbers interpreted by (18) in the geometrical system of *exponential lines* given with a defined k_B

value in the Euclidean plane is a 'series of radii' going through the origo with a starting point tangent forming a ϑ angle with the positive x axis if for any x, y – see *Eq.* (15) –

$$\frac{\frac{y}{k_B}}{\operatorname{sh}\frac{x}{k_B}} = \text{const.} = \operatorname{tg}\vartheta. \tag{21}$$

The equation for the series of exponential radii going across the origo can also be formed by expressing y from (21) according to (12) and (13). Sh curves play the role of the 'straights' of the Euclidean geometry in the geometry of exponential lines if the 'series of radii' is going across the origo.

The ch curve is one line of the series of exponential radii passing across point $P(0, y_p)$, different from the origo, symmetrical to the y axis, that is, the line with the equation:

$$y_1(x) = \frac{1}{2}\ y_P \cdot (e^{x/k_B} + e^{-x/k_B}). \tag{22}$$

Now its ordinates are increased with the

$$y(x) = k_B \cdot \operatorname{tg}\vartheta\ \frac{1}{2}\ (e^{x/k_B} - e^{-x/k_B}), \tag{23}$$

system of equations giving the sh function in exponential form, derived from (21) according to (12). Hence a general exponential series of radii may be composed, depending on the actual coefficients, that is, on ϑ, of purely exponential, ch and sh curves as well. As the derivative of the function selected for base in *Eq.* (22) is equal to zero in the locus $x = 0$, that is, in the P point, it will have no effect on tangent relations of *Eq.* (23) even after composition. Therefore the equation

$$\frac{\frac{y-y_P}{k_B}}{\operatorname{sh}\frac{x}{k_B}} = \text{const.} = \operatorname{tg}\vartheta$$

will be true for every (x, y) point of the exponential series of radii going across point $P(0, y_P)$.

2.3 Representation of Complex Variable Functions

If the $w = \varphi + i\psi$ complex variable is a function of the $z = x + iy$ complex variable, thus $w = f(z)$, then this function relation means that the series of lines defined by the f operation and fulfilling the conditions $\varphi =$ const. and $\psi =$ const. can be expressed in the coordinate system of variables x and y belonging to z by functions of form $y = y(x)$ and represented in the *existing geometrical system.*

The z complex variable can also be expressed in a polar coordinate system:

$$z = r \quad e^{i \cdot \varphi}$$

if the origo is selected for a pole. If the point representing the complex number x_0 is the pole then

$$(z - z_0) = r \cdot e^{i \cdot \vartheta}.$$

Let us see the complex variable function $w = \ln(z - z_0)$. After separating real and complex variables and selecting for w $\varphi =$ const. and $\psi =$ const., equations

$$r = e^{\varphi} \quad \text{and} \quad \vartheta = \psi \tag{25}$$

will be obtained. The second equation will be examined further in this paper.

The equation $\vartheta = \psi$ yields for different ψ value selections a 'series of radii', namely in the Euclidean system of straights series of radii composed of straights and, in the system of exponential lines: 'series of radii' composed of exponential equation lines.

Suppose that $z_0 = i \cdot y_P$, that is, the pole is on the y axis in y_P distance from the origo. The line of the exponential line system according to *Eq.* (2) and given by the coefficients $C_1 = C_2 = y_P/2$

$$y_1(x) = \frac{y_P}{2} \quad (e^{x/k_B} + e^{-x/k_B}) = y_P \quad \text{ch} \, \frac{x}{k_B} \tag{26}$$

will in any case go across this point. The sh functions of *Eqs.* (12) and (23) will be added to this. $\vartheta = 0$, according to (26).

Let us see the special case $C = \pm y_P/2$, then

$$y = y_P \quad e^{\pm x/k_B}. \tag{27}$$

Here

$$\text{tg}\,\vartheta = \frac{y_P}{k_B}. \tag{28}$$

Further C values can be reckoned with selecting for $\psi = \vartheta$ between 0 and π regularly $\pi/n, 2\pi/n \ldots k\pi/n$ $[k = 1, 2 \ldots n]$.

The 'exponential set of radii' crossing the P point contains lines of ch equations and sh equations as well. The border between them is the line of the simple exponential curves, one asymptotic to the positive, the other to the negative x axis. Lines of ch equation do not cross the horizontal axis, but all exponential lines crossing it are of sh equation.

The transformation of the complex variable function $w = \ln(z - z_0)$ in the system of exponential lines not only differs from the Euclidean sense transformation but also has an independent meaning for flows to be described below.

3. Description of Seepage Flow in the Geometrical System of Exponential Lines

3.1 Water Table Depression of Wells or Relief Drains

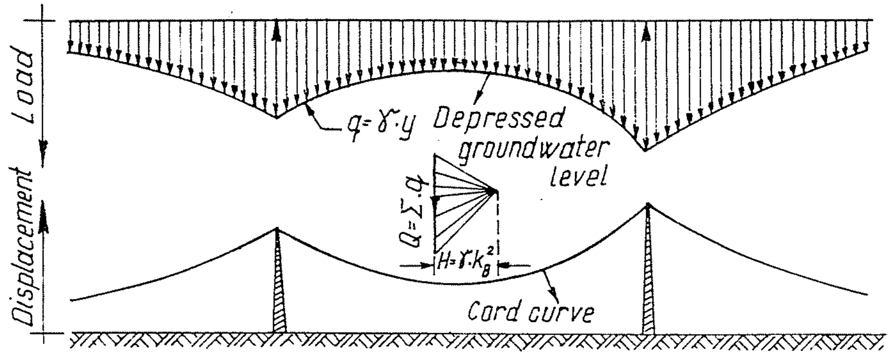


Fig. 5. Water table depression is equal to the load of a cord or chain which is proportional to the displacement

The depression of groundwater surface can be dynamically treated like a cord or chain loaded with a distributed force system where the acting force ('upward force' here) is proportional to the displacement of the cord or chain due to loading. In this case conditions of *Eq.* (1) are satisfied as the equilibrium form of the cord curve is given by its second derivative proportional to load. The general solution of the (1) differential equation which also was available from the former $z = \ln(z - z_0)$ transformation in the geometrical system of exponential lines can approximately be obtained from the known well-hydraulic equations (*Fig. 5*). For single percolation wells this means that depression calculated from steady-state groundwater level for an x distance to the centre of the well is:

$$y(x) = y(0) \cdot e^{\pm/k_B}, \tag{29}$$

where the depression of the $y(0)$ origo has to be substituted according to measurement data, on the base of depression 'outside the well' instead of 'inside the well'. The k_B value – 'Bolyai inflexion' of length dimension of the depression system – can be determined empirically, depending on the quality of the water supplying layer. This is in relation through the equation $k_B = R/e$ [$e = 2.718\ldots$] with the R value of the theoretically finite 'depression radius' determinable from different theories. The marking k_B indicates Bolyai geometry origin to discern it from the k_D Darcy depression coefficient of velocity dimension being also used in this paper (*Fig. 6*).

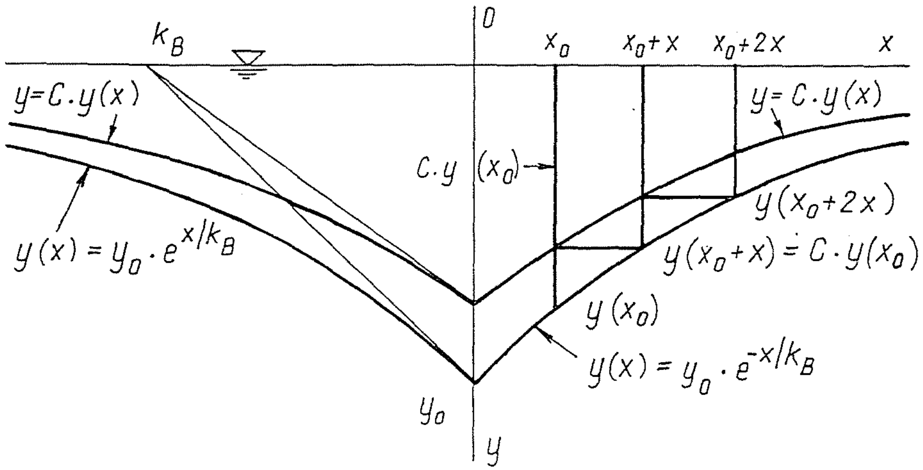


Fig. 6. Depression of a single percolation well by a water table line of exponential equation

From the derivative function of the depression water surface line also follows that the surface slope corresponding to the derivative of the function (I) is,

$$I = \frac{dy(x)}{dx} = \pm\frac{1}{k_B} \quad e^{\pm x/k_B} = \frac{y}{k_B}. \tag{30}$$

From this:

$$y = k_B \quad I, \tag{31}$$

which is similar to the well-known equation of the filtration velocity determinable from Darcy's law:

$$v = k_D \quad I. \tag{32}$$

On the ground of (30) and (31):

$$v = \frac{k_D}{k_B} \cdot y, \tag{33}$$

thus filtration velocity can also be determined from the depression in knowledge of the two k factors. Hence equation for wells' discharge is also obtainable:

$$Q = 2 \cdot \pi \cdot r \cdot m \cdot \frac{k_D}{k_B} \cdot y_0, \tag{34}$$

where $Q =$ discharge
$r =$ radius of the well
$m =$ filtration head
$y_0 =$ depression outside the well.

3.2 Groundwater Depression of Well Groups

The total depression of two co-operating wells can be determined by algebraic summing of the suctions of the single wells – if not only depression but also filling is meant here summing of signs is to be included. As in spaces between the two wells according to (29) the exponential function values of positive and negative signs will be summed by the same k_B there will always be an $x = 0$ starting point on the x axis to which either

$$y(x) = 2 \cdot y_0 \quad \text{ch} \frac{x}{k_B} = y_{\text{ch}}(x), \tag{35}$$

or

$$y(x) = 2 \cdot y_0 \cdot \text{sh} \frac{x}{k_B} = y_{\text{sh}}(x), \tag{36}$$

will be valid for the water table depression line. It is known that, disregarding, constants, the functions $y_{\text{ch}}(x)$ and $y_{\text{sh}}(x)$ are mutually derivatives of each other, so the I water slope values may also be determined from the actual values of the 'other' function. Hence on the analogy of (31):

$$\begin{aligned} y_{\text{ch}} &= k_B \cdot I_{\text{sh}}, \\ y_{\text{sh}} &= k_B \quad I_{\text{ch}}, \end{aligned} \tag{37}$$

and, similarly to (33), for the ch-type water table lines:

$$v = \frac{k_D}{k_B} \quad y_{\text{sh}}, \tag{38}$$

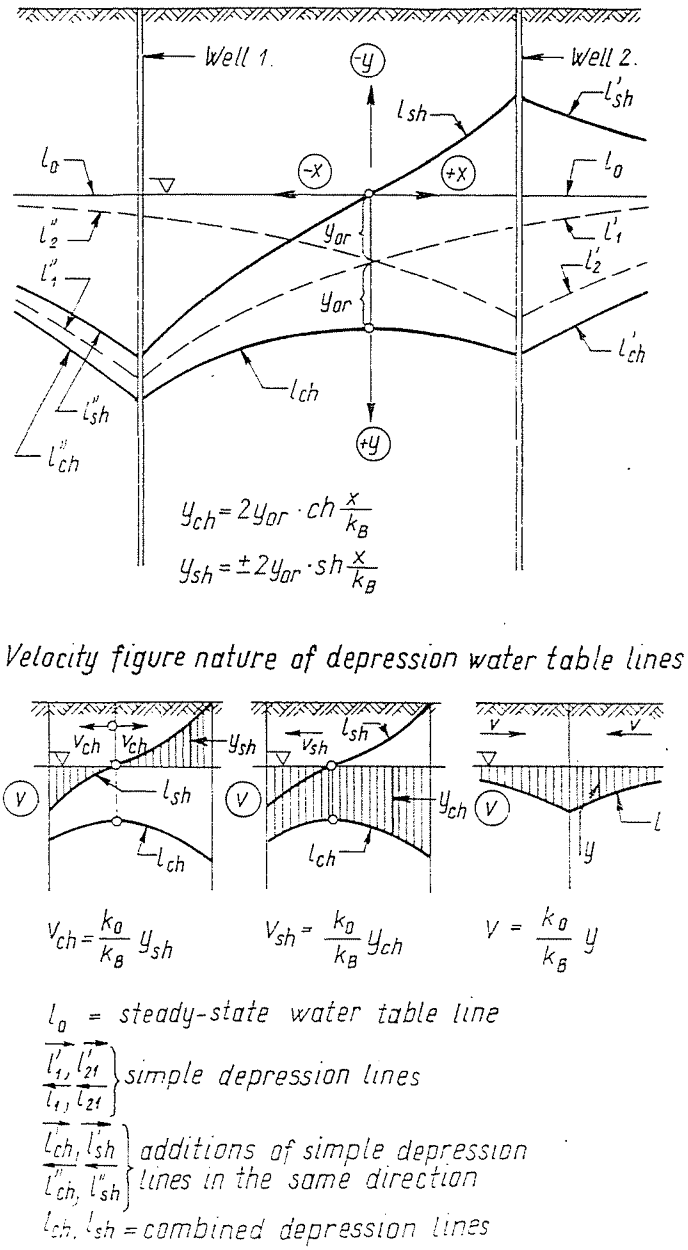


Fig. 7. Water table lines and velocity figures of two wells due to joint depression (filling combined with depression)

for the sh-type water table lines:

$$v = \frac{k_D}{k_B} \cdot y_{\text{ch}} . \tag{39}$$

Hence the 'simple' depression lines are at the same time velocity-figures when proportioned with k_D/k_B. The 'combined' lines will turn with the same factor to velocity figure of their contrary type equivalent (*Fig. 7*).

3.3 Interaction of Wells

The water table depression lines of two or more depression wells reach each other, mutually increasing depression effect, thus a higher depression will occur than would for each well if they supplied the same amount of water individually.

Suppose for two cooperating wells total depression at the No. 1 well be y_1 composed of the well's own y_{11} depression and the y_{12} depression due to the other well. The total y_2 depression of the No. 2 well is similarly composed of an own y_{22} depression and an other y_{21} depression. Known are only the y_1 and y_2 depression values, the Bolyai inflexion base length and the x distance of the wells, thus two equations can be set:

$$\begin{aligned} y_1 &= y_{11} + y_{12} = y_{11} + y_{22} \cdot e^{-x/k_B}, \\ y_2 &= y_{21} + y_{22} = y_{11} \cdot e^{-x/k_B} + y_{22}. \end{aligned} \tag{40}$$

Solution for y_{11} and y_{22}:

$$y_{11} = \frac{y_1 - y_2 \ e^{-x/k_B}}{1 - e^{-2x/k_B}} \quad \text{and} \quad y_{22} = \frac{y_2 - y_1 \cdot e^{-x/k_B}}{1 - e^{-2x/k_B}}. \tag{41}$$

The mutual effect of two wells is defined from the quotient of the summed 'own' depressions and the sum of the 'own' and 'other' depression. The latter 'total', that is, actually obtainable depression values are selected to be identical for every well, so $y_1 = y_2$ will drop out from the fraction (*Fig. 8*)

$$\eta = \frac{y_{11} + y_{22}}{y_1 + y_2} = \frac{1}{1 + e^{-x/k_B}}. \tag{42}$$

If two wells are infinitely near and $x = 0$, efficiency of the cooperation will be 0.5, and, if they are infinitely far and $x \to \infty$ then this efficiency will be unit.

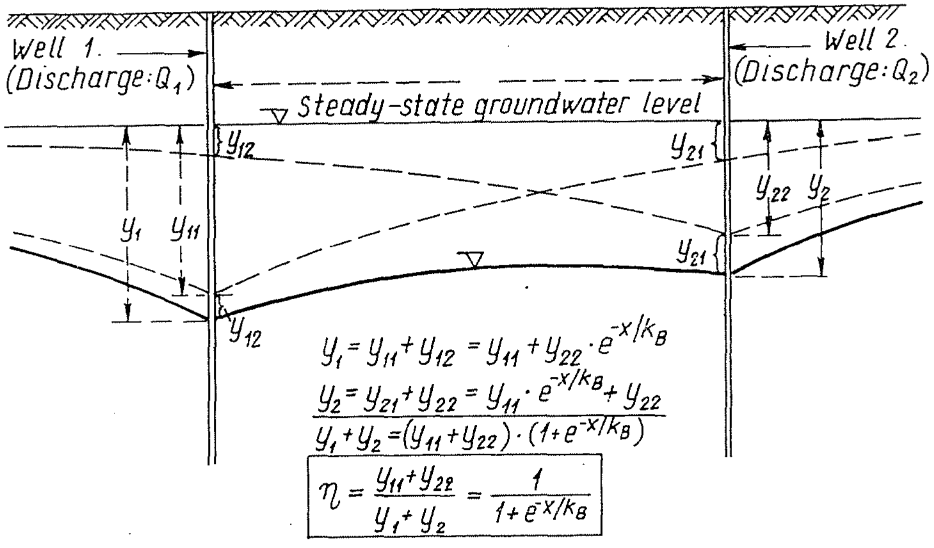


Fig. 8. Mutual effect of two depression wells and determination of the efficiency of the cooperation

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