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DYNAMIC ANALYSIS OF STRUCTURES SUPPORTING A MOVING MASS, EXPOSED TO EXTERNAL AND INTERNAL DAMPING

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Abstract

In the dynamic analysis of structures supporting a moving load, for the sake of simplification, the mass either of the moving load or the supporting structures is neglected. In earlier papers, examples for reckoning with the timely variation of the mass are found only for simple structures. It is well known that dynamic stress values are influenced by external and internal damping. Their combined effects are only reckoned with in free vibration and in exciting by harmonic forces, in cases where the dynamic system has a constant mass matrix. There is an adequate numerical method for the analysis of structures with several degrees of freedom, with permanent mass matrix, under external damping. An algorithm has been presented in this paper for the analysis of dynamic excess displacements of structures, for cases both the effects of moving mass and of internal friction have to be reckoned with. The developed algorithm and numerical method have been tested on examples. The mentioned factors showed important effects, justifying to be reckoned with in the analysis of real structures.

Keywords: moving mass, internal damping, vibration of structures.

1. Introduction

An important problem of the dynamic analysis of structures is to determine stresses in a structure due to a moving load. To simplify the procedure, in knowledge of parameters involved in the problem, the mass either of the moving load or the supporting structures is neglected. Examples for the approximate consideration of both effects are found (FRÝBA, 1972) only for simple structures (simple beams). Values of dynamic stresses are affected by external and internal damping. A suggestion has been made to reckon with their combined effect (GYÖRGYI, 1985), but only for case of free vibration where the dynamic system has a constant mass matrix. This paper presents a method of analysis for structures with several degrees of freedom, timely variation of the mass matrix, exposed to external and internal damping.

2. Application of Direct Integration

2.1 Reckoning of Moving Mass under External Damping

The second-order linear differential equation $\mathbf{M\ddot{u}}+\mathbf{C\dot{u}}+\mathbf{Ku}=\mathbf{r}$ describing the displacement of structures expresses the dynamic equilibrium at any time in the considered time range. Forces of inertia are expressed by $\mathbf{M\ddot{u}} = \mathbf{f}_I(t)$, damping forces by $\mathbf{C\dot{u}} = \mathbf{f}_D(t)$, stiffness forces by $\mathbf{Ku} = \mathbf{f}_E(t)$, while $\mathbf{r}(t)$ is the vector of external forces. (Matrices are of order n). The dynamic analysis is intended to solve the matrix differential equation under initial conditions $\dot{\mathbf{u}}_0$, $\mathbf{u}_0$ and $\ddot{\mathbf{u}}_0$ at a time t_0, and in knowledge of displacements, to compute the dynamic stresses.

The initial value problem is solved advisably by the Wilson θ method (BATHE and WILSON, 1976). Wilson assumes a linearly varying acceleration between times t and $t+\theta\Delta t$. (For $\theta = 1.4$, the procedure is certain convergent.)

In this case:

$$\ddot{\mathbf{u}}_{t+\tau} = \ddot{\mathbf{u}}_t + \frac{\tau}{\theta\Delta t}\left(\ddot{\mathbf{u}}_{t+\theta\Delta t} - \ddot{\mathbf{u}}_t\right) , \tag{1}$$

$$\dot{\mathbf{u}}_{t+\tau} = \dot{\mathbf{u}}_t + \ddot{\mathbf{u}}_t\tau + \frac{1}{2}\frac{\tau^2}{\theta\Delta t}\left(\ddot{\mathbf{u}}_{t+\theta\Delta t} - \ddot{\mathbf{u}}_t\right) , \tag{2}$$

$$\mathbf{u}_{t+\tau} = \mathbf{u}_t + \dot{\mathbf{u}}_t\tau + \frac{1}{2}\ddot{\mathbf{u}}_t\tau^2 + \frac{1}{6}\frac{\tau^3}{\theta\Delta t}\left(\ddot{\mathbf{u}}_{t+\theta\Delta t} - \ddot{\mathbf{u}}_t\right) . \tag{3}$$

Hence:

$$\ddot{\mathbf{u}}_{t+\theta\Delta t} = \frac{6}{(\theta\Delta t)^2}\left(\mathbf{u}_{t+\theta\Delta t} - \mathbf{u}_t\right) - \frac{6}{\theta\Delta t}\dot{\mathbf{u}}_t - 2\ddot{\mathbf{u}}_t , \tag{4}$$

$$\dot{\mathbf{u}}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t}\left(\mathbf{u}_{t+\theta\Delta t} - \mathbf{u}_t\right) - 2\dot{\mathbf{u}}_t - \frac{\theta\Delta t}{2}\ddot{\mathbf{u}}_t . \tag{5}$$

Assuming $\mathbf{r}(t)$ to linearly vary during this period:

$$\mathbf{r}_{t+\theta\Delta t} = \mathbf{r}_t + \theta(\mathbf{r}_{t+\Delta t} - \mathbf{r}_t) .$$

Displacements at time $t+\theta\Delta t$ result from:

$$\left(\mathbf{K} + \frac{6}{(\theta\Delta t)^2}\mathbf{M} + \frac{3}{\theta\Delta t}\mathbf{C}\right)\mathbf{u}_{t+\theta\Delta t} =$$

$$= \mathbf{r}_{t+\theta\Delta t} + \mathbf{M}\left(\frac{6}{(\theta\Delta t)^2}\mathbf{u}_t + \frac{6}{\theta\Delta t}\dot{\mathbf{u}}_t + 2\ddot{\mathbf{u}}_t\right) + \mathbf{C}\left(\frac{3}{\theta\Delta t}\mathbf{u}_t + 2\dot{\mathbf{u}}_t + \frac{\theta\Delta t}{2}\ddot{\mathbf{u}}_t\right) . \tag{6}$$

Thereupon, displacements, velocities and accelerations may be computed at time $t + \Delta t$. For constants **M**, **C**, **K**, coefficient matrix

$$\mathbf{A} = \left(\mathbf{K} + \frac{6}{(\theta\Delta t)^2}\mathbf{M} + \frac{3}{\theta\Delta t}\mathbf{C}\right)$$

has to be decomposed but once after having assumed time interval Δt, while the right-hand vector is time-dependent. If computation has also to reckon with the moving mass, the mass matrix $\breve{\mathbf{M}}(t) = \mathbf{M}+\mathbf{M}_1(t)$ has to be applied. In this case matrix **A** will be time-dependent, and for every time interval, time-consuming decomposition has to be performed. To avoid it, it is suggested to formulate the dynamic problem as $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \breve{\mathbf{r}}$, where $\breve{\mathbf{r}} = \mathbf{r} - \mathbf{M}_1\ddot{\mathbf{u}} = \mathbf{r} - \mathbf{r}_1$.

In this case displacements result from:

$$\left(\mathbf{K} + \frac{6}{(\theta\Delta t)^2}\mathbf{M} + \frac{3}{\theta\Delta t}\mathbf{C}\right)\mathbf{u}_{t+\theta\Delta t} =$$

$$= \breve{\mathbf{r}}_{t+\theta\Delta t} + \mathbf{M}\left(\frac{6}{(\theta\Delta t)^2}\mathbf{u}_t + \frac{6}{\theta\Delta t}\dot{\mathbf{u}}_t + 2\mathbf{u}_t\right) + \mathbf{C}\left(\frac{3}{\theta\Delta t}\mathbf{u}_t + 2\dot{\mathbf{u}}_t + \frac{\theta\Delta t}{2}\ddot{\mathbf{u}}_t\right), \tag{7}$$

where

$$\breve{\mathbf{r}}_{t+\theta\Delta t} = \mathbf{r}_{t+\theta\Delta t} - \mathbf{M}_{1_{t+\theta\Delta t}}\left(\frac{6}{(\theta\Delta t)^2}(\mathbf{u}_{t+\theta\Delta t} - \mathbf{u}_t) - \frac{6}{\theta\Delta t}\dot{\mathbf{u}}_t + 2\ddot{\mathbf{u}}_t\right). \tag{8}$$

Now, vector $\mathbf{u}_{t+\theta\Delta t}$ is seen to have appeared also in the right-hand of the equation, to be computed by iteration. Now, the solution is accelerated, but there is only convergency if the accessory mass is not too big compared to the structural mass, or the integration spacing is not too small.

2.2 Taking a Moving Mass and a Proportional Internal Damping into Consideration

In the relationships above, physical purport of matrix **C** has not been considered. For an external damping, the matrix can be assembled in knowledge of single damping elements related to the structure. For a damping due to frequency-independent internal friction, the matrix of equivalent external damping — for different damping parameters of single structural units — may be assumed in knowledge of complex stiffness matrix $\mathbf{K}_u+i\mathbf{K}_v$, in form $\mathbf{C} = \mathbf{MV}\left\langle \frac{1}{\omega_{ru}} \right\rangle \mathbf{V}^*\mathbf{K}_v$ using eigenvectors normed to $\mathbf{M}$ of the eigenvalue problem $\mathbf{K}_u\mathbf{v}_r = \omega_{ru}^2\mathbf{M}\mathbf{v}_r$ (GYÖRGYI, 1985). (Now, in the matrix differential equation of vibration $\mathbf{K}$ will be replaced by $\mathbf{K}_u$.)

For structural units with the same damping parameters (proportional damping) the equivalent damping matrix $\mathbf{C} = v\mathbf{MV}\left\langle \frac{1}{\omega_{ru}} \right\rangle \mathbf{V}^*\mathbf{K}$ and $\mathbf{K}_u = u\mathbf{K}$, where

$$v = \frac{4\gamma}{4+\gamma^2}; \qquad u = \frac{4-\gamma^2}{4+\gamma^2}; \qquad \omega_{ru} = \frac{\omega_r}{\sqrt{1+\frac{\gamma^2}{4}}}; \qquad \gamma = \frac{\vartheta}{\pi}.$$

Here ϑ is the logarithmic decrement of damping, ω_r may be obtained from the r-th eigenvalue of the eigenvalue problem $\mathbf{Kv} = \omega^2\mathbf{Mv}$ for the undamped case, while $\mathbf{V}$ is a matrix containing eigenvectors normed for $\mathbf{M}$. Obviously, in case of internal damping, the direct integration problem has to be preceded by solving an eigenvalue problem. All these argue for taking it into consideration in selecting the solution method of the dynamic problem, and to try to apply modal analysis.

3. Applying Modal Analysis

3.1 Reckoning with a Moving Mass without Damping

The problem is to solve differential equation $\mathbf{M\ddot{u}}+\mathbf{Ku} = \mathbf{r}-\mathbf{M}_1\mathbf{\ddot{u}}$. Solution is wanted in form $\mathbf{u} = \mathbf{Vx}$, in knowledge of eigenvalues and eigenvectors normed to $\mathbf{M}(\mathbf{V}^*\mathbf{MV} = \mathbf{E})$ of the eigenvalue problem $\mathbf{Kv} = \omega^2\mathbf{Mv}$. (Actually, initial conditions for $\mathbf{x}$ are $\mathbf{x}_0 = \mathbf{V}^*\mathbf{Mu}_0$, $\mathbf{\dot{x}}_0 = \mathbf{V}^*\mathbf{M\dot{u}}_0$). After substitution and multiplying from the left by transposed matrix $\mathbf{V}^*$:

$$\mathbf{V}^*\mathbf{MV\ddot{x}} + \mathbf{V}^*\mathbf{KVx} = \mathbf{q}, \tag{9}$$

where

$$\mathbf{q} = \mathbf{V}^*\mathbf{r} - \mathbf{V}^*\mathbf{M}_1\mathbf{V\ddot{x}} = \mathbf{f} - \mathbf{B\ddot{x}}. \tag{10}$$

Due to orthogonality, theoretically, n single-unknown equations may be considered. It is known that in solving real technical problems, in the solution computed on the basis of eigenvectors it is sufficient to involve a certain number ($m < n$) of eigenvectors, computable by convenient procedures (e.g. subspace iteration) even for extended systems. Equation r:

$$\ddot{x}_r + \omega_r^2 x_r = q_r. \tag{11}$$

According to those above:

$$\left(\omega_r^2 + \frac{6}{(\theta\Delta t)^2}\right) x_{r_{t+\theta\Delta t}} = q_{r_{t+\theta\Delta t}} + \frac{6}{(\theta\Delta t)^2} x_{r_t} + \frac{6}{\theta\Delta t}\dot{x}_{r_t} + 2\ddot{x}_{r_t}\,, \tag{12}$$

where:

$$\begin{aligned} q_{r_{t+\theta\Delta t}} &= f_{r_{t+\theta\Delta t}} - \mathbf{b}^*_{r_{t+\theta\Delta t}} \ddot{\mathbf{x}}_{t+\theta\Delta t}\,, \\ f_{r_{t+\theta\Delta t}} &= f_{r_t} + \theta\left(f_{r_{t+\theta\Delta t}} - f_{r_t}\right)\,, \\ \mathbf{b}_{r_{t+\theta\Delta t}} &= \mathbf{b}_{r_t} + \theta\left(\mathbf{b}_{r_{t+\theta\Delta t}} - \mathbf{b}_{r_t}\right)\,. \end{aligned} \tag{13}$$

To compute $q_{r_{t+\theta\Delta t}}$ requires $\ddot{\mathbf{x}}_{t+\theta\Delta t}$, to be computed exactly in knowledge of values $x_{r_{t+\theta\Delta t}}$ from a relationship similar to (4). For a guaranteed convergency, iteration may be applied also here.

The problem may also be solved without iteration. In this case, vectors and/or matrices of size m corresponding to the number of eigenvectors involved into the analysis lead to the equation system:

$$\left[\mathbf{D} + \frac{6}{(\theta\Delta t)^2}\mathbf{B}_{t+\theta\Delta t}\right]\mathbf{x}_{t+\theta\Delta t} =$$

$$= \mathbf{f}_{t+\theta\Delta t} + \left(\mathbf{B}_{t+\theta\Delta t} + \mathbf{E}\right)\left(\frac{6}{(\theta\Delta t)^2}\mathbf{x}_t + \frac{6}{\theta\Delta t}\dot{\mathbf{x}}_t + 2\ddot{\mathbf{x}}_t\right)\,.$$

Here $\mathbf{D}$ is a diagonal matrix with element r being $\omega_r^2 + \frac{6}{(\theta\Delta t)^2}$, while $\mathbf{E}$ is a unit matrix. Although now an equation system has to be solved, it lasts much less time than to decompose matrix $\mathbf{A}$ for every time step.

3.2 Reckoning with a Moving Mass in Case of Proportional Internal Damping

For proportional internal damping, differential equation of motion becomes:

$$\mathbf{M}\ddot{\mathbf{u}} + \left(v\mathbf{M}\mathbf{V}\left\langle\frac{1}{\omega_{ru}}\right\rangle\mathbf{V}^*\mathbf{K}\right)\dot{\mathbf{u}} + u\mathbf{K}\mathbf{u} = \mathbf{r} - \mathbf{M}_1\ddot{\mathbf{u}}\,. \tag{14}$$

In knowledge of eigenvalues and eigenvectors normed for $\mathbf{M}$ of the eigenvalue problem $\mathbf{K}\mathbf{v} = \omega^2\mathbf{M}\mathbf{v}$, solution may be sought for in form $\mathbf{u} = \mathbf{V}\mathbf{x}$. After substitution and multiplying by transposed matrix $\mathbf{V}^*$ from the left:

$$\mathbf{V}^*\mathbf{M}\mathbf{V}\ddot{\mathbf{x}} + v\mathbf{V}^*\mathbf{M}\mathbf{V}\left\langle\frac{1}{\omega_{ru}}\right\rangle\mathbf{V}^*\mathbf{K}\mathbf{V}\dot{\mathbf{x}} + u\mathbf{V}^*\mathbf{K}\mathbf{V}\mathbf{x} = \mathbf{q}\,, \tag{15}$$

where

$$\mathbf{q} = \mathbf{V}^*\mathbf{r} - \mathbf{V}^*\mathbf{M}_1\mathbf{V}\ddot{\mathbf{x}} = \mathbf{f} - \mathbf{B}\ddot{\mathbf{x}}\,. \tag{16}$$

Because of orthogonality, also now, n single-unknown equations may be considered. Equation r:

$$\ddot{x}_r + \gamma\omega_{ru}\dot{x}_r + \omega_{ru}^2 x_r = q_r\,. \tag{17}$$

According to those above:

$$\left(\omega_{ru}^2 + \frac{6}{(\theta\Delta t)^2} + \frac{3}{\theta\Delta t}\gamma\omega_{ru}\right) x_{r_{t+\theta\Delta t}} =$$

$$= q_{r_{t+\theta\Delta t}} + \frac{6}{(\theta\Delta t)^2}x_{r_t} + \frac{6}{\theta\Delta t}\dot{x}_{r_t} + 2\ddot{x}_{r_t} + \gamma\omega_{ru}\left(\frac{3}{\theta\Delta t}x_{r_t} + 2\dot{x}_{r_t} + \frac{\theta\Delta t}{2}\ddot{x}_{r_t}\right). \tag{18}$$

Here $q_{r_{t+\theta\Delta t}}$ is the same as for the undamped case (13), comprising vector $\ddot{\mathbf{x}}_{t+\theta\Delta t}$, so requiring again an iteration procedure. If necessary, the problem will be the direct solution of the equation system of order m:

$$\left[\breve{\mathbf{D}} + \frac{6}{(\theta\Delta t)^2}\mathbf{B}_{t+\theta\Delta t}\right]\mathbf{x}_{t+\theta\Delta t} =$$

$$= \mathbf{f}_{t+\theta\Delta t} + (\mathbf{B}_{t+\theta\Delta t} + \mathbf{E})\left(\frac{6}{(\theta\Delta t)^2}\mathbf{x}_t + \frac{6}{\theta\Delta t}\dot{\mathbf{x}}_t + 2\ddot{\mathbf{x}}_t\right) +$$

$$+\mathbf{G}\left(\frac{3}{\theta\Delta t}\mathbf{x}_t + 2\dot{\mathbf{x}}_t + \frac{\theta\Delta t}{2}\ddot{\mathbf{x}}_t\right),$$

where $\breve{\mathbf{D}}$ and $\mathbf{G}$ are diagonal matrices, elements r of them $\omega_r^2 + \frac{6}{(\theta\Delta t)^2} + \frac{3}{\theta\Delta t}\gamma\omega_{ru}$ and $\gamma\omega_{ru}$.

3.3 Computation for Other than Proportional Internal Damping or for Composite Internal and External Damping

In this general case, the damping matrix cannot be diagonalized by means of eigenvectors for the undamped solution, so inasmuch as diagonalization is to be made in the left-hand side of the matrix equation, damping forces $\mathbf{C}\dot{\mathbf{u}} = \mathbf{f}_D(t)$ obtained by means of the damping matrix involving the effect of external damping and equivalent internal damping have to appear in the right-hand side of the matrix equation. Also now, the dynamic equation may be written in the form $\mathbf{V}^*\mathbf{M}\mathbf{V}\ddot{\mathbf{x}} + \mathbf{V}^*\mathbf{K}\mathbf{V}\mathbf{x} = \mathbf{q}$ but

$$\mathbf{q} = \mathbf{V}^*\mathbf{r} - \mathbf{V}^*\mathbf{M}_1\mathbf{V}\ddot{\mathbf{x}} - \mathbf{V}^*\mathbf{C}\mathbf{V}\dot{\mathbf{x}} = \mathbf{f} - \mathbf{B}\ddot{\mathbf{x}} - \mathbf{H}\dot{\mathbf{x}}. \tag{19}$$

Solution may be obtained from *Eq.* (12) but the purport of $q_{r_{t+\theta\Delta t}}$.

Now:

$$q_{r_{t+\theta\Delta t}} = f_{r_{t+\theta\Delta t}} - \mathbf{b}^*_{r_{t+\theta\Delta t}}\ddot{\mathbf{x}}_{t+\theta\Delta t} - \mathbf{h}^*_r\dot{\mathbf{x}}_{t+\theta\Delta t}. \tag{20}$$

Vectors $\ddot{\mathbf{x}}_{t+\theta\Delta t}$, $\dot{\mathbf{x}}_{t+\theta\Delta t}$ depend on vector $\mathbf{x}_{t+\theta\Delta t}$ of elements $x_{r_{t+\theta\Delta t}}$ in (12), requiring an iteration procedure. Unknowns belonging to the subspace may be obtained from an equation system of order m if needed:

$$\left[\breve{\mathbf{D}} + \frac{6}{(\theta\Delta t)^2}\mathbf{B}_{t+\theta\Delta t} + \frac{3}{\theta\Delta t}\mathbf{H}\right]\mathbf{x}_{t+\theta\Delta t} =$$

$$= (\mathbf{B}_{t+\theta\Delta t} + \mathbf{E})\left(\frac{6}{(\theta\Delta t)^2}\mathbf{x}_t + \frac{6}{\theta\Delta t}\dot{\mathbf{x}}_t + 2\ddot{\mathbf{x}}_t\right) +$$

$$+\mathbf{G} + \mathbf{H}\left(\frac{3}{\theta\Delta t}\mathbf{x}_t + 2\dot{\mathbf{x}}_t + \frac{\theta\Delta t}{2}\ddot{\mathbf{x}}_t\right) + \mathbf{f}_{t+\theta\Delta t}.$$

4. Numerical Experience

4.1 The Examined Structure

In numerical experiences, computations referred to a realistic structure. For a bridge spanning 30 m, types *a* and *b* simulate a bridge with reinforced concrete, and with steel structure, respectively.

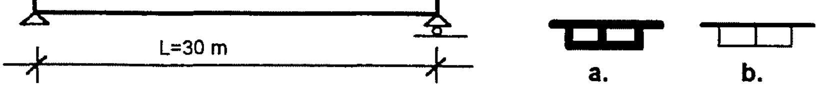

Fig. 1. Arrangement of the examined structure

Rigidity and material characteristics are seen in *Table 1*.

The load moving on the bridge amounts to 800 kN. In different cases, the moving load velocities have been assumed in the range from 0 to 50 m/s. Internal damping had a factor γ of 0.1.

Table 1
Rigidity and material characteristics of the structure

Characteristic	Type a	Type **b**
cross-section area	$3.12\,m^2$	$0.40\,m^2$
moment of inertia of cross-section	$2.13\,m^4$	$0.35\,m^4$
elastic constant of the material	$20.000.000\,kN/m^2$	$200.000.000\,kN/m^2$
Poisson's ratio	0.166	0.3
weight per unit volume	$25\,kN/m^3$	$150\,kN/m^{3*}$

* including the accessory weight of the bridge deck pavement.

4.2 The Applied Numerical Method

The dynamic problem has been solved by modal analyses. A numerical experiment has been made to determine the number of eigenvectors required for a solution of the needed accuracy. Displacement of the structure midpoint has been tested by taking an ever increasing number of eigenvectors into consideration. Percentages of excess displacements due to dynamic effect in structure of type a, for different velocities are seen in *Table 2.*

Table 2
Excess displacements due to dynamic effect

Num. of eigenvectors → velocity [m/s] ↓	1	3	5	7
10	2.4	3.9	4.1	4.3
20	4.6	6.0	6.2	6.3
30	11.3	12.3	12.6	12.8
50	16.5	18.7	18.8	18.8

Apparently, reckoning with five eigenvectors yields excess dynamic displacements at an adequate accuracy. In this case it is no problem to solve the equation system with five unknowns in every time step, that may be avoided by application of the iterational procedure within a given time step. The time interval in the problem has been assumed as the shortest vibration time belonging to eigenvectors in the solution, of a value $\Delta t = \frac{T_p}{10}$ as recommended in literature. This procedure proved to be convenient even for three eigenvectors. By the way, reckoning only the first, so- called fundamental vibration, and computing the time interval as $\Delta t = \frac{T_1}{10}$ (mainly for $v < 20\,m/s$) the displacement is much less than that for the given vibration pattern applying an exacter computation.

Also in this case, applying interval $\Delta t = \frac{T_1}{100}$ yields an adequate result. This remark is, however, only a theoretical one, namely, applying at least five eigenvectors, an interval still less than this critical one was obtained.

Adequacy of the iteration procedure depends on the number of eigenvectors reckoned with in the analysis, on the size and velocity of the moving mass. Applying only the first eigenvector corresponding to the fundamental vibration, the procedure is convergent even for a moving load of 800 kN. As it was seen, in the given problem advisably five eigenvectors are involved to achieve the desired accuracy. Now, convergency subsists even for a load of 300 kN. In the iteration process the number of iterations in a given step depends on the load velocity. For a lesser velocity this number is lower. The statements above are illustrated in *Table 3*.

Table 3
Number of iterations required in given time step of the iteration process

Force kN		100			300			500			800		
Veloc. m/s		10	20	50	10	20	50	10	20	50	10	20	50
	1	3	4	4	6	6	8	9	10	11	24	28	36
Num.	3	5	6	7	12	13	16	53	42	76	–	–	–
of	5	7	8	9	32	34	34	–	–	–	–	–	–
eig. v.	7	11	11	13	–	–	–	–	–	–	–	–	–

4.3 Excess Dynamic Displacements along the Structure

In the structure, dynamic effects cause dynamic displacements in excess to those static displacements, depending on the velocity and on the structural rigidity. In engineering practice a dynamic coefficient is applied, and the dynamic effect is replaced by static analysis applying a load multiplied by the dynamic coefficient. In fact, however, there is a system of dynamic coefficient as seen in *Table 4*. This table shows percentages of dynamic excess displacements of different structural cross-sections for both types of structures, at various velocities, for a load of 800 kN.

This paper was intended to develop a computation method taking both the effect of the moving load mass and the internal damping of the structure into consideration. *Table 5* shows what it means for dynamic excess displacements percentages to ignore the damping effect of internal damping, or to omit the mass of the moving load (800 kN).

Table 4
Percentages of dynamic excess displacements

Place of displacement	0.1 L		0.2 L		0.3 L		0.4 L		0.5 L	
Vel. [m/s] ↓ type of st.	a	b	a	b	a	b	a	b	a	b
5	0.1	–	1.0	–	1.3	0.2	2.1	1.3	2.4	2.0
10	2.1	–	3.0	0.5	3.8	0.8	4.5	1.7	4.1	2.2
20	4.0	1.0	4.8	1.8	5.7	3.0	6.5	4.1	6.2	4.2
30	9.1	5.0	9.9	5.9	10.9	6.6	12.1	6.4	12.6	5.2
50	24.0	11.7	24.3	14.4	23.3	14.4	21.3	13.1	18.8	10.8
Statical displac. [mm]	3.29	2.02	6.27	3.85	8.62	5.30	10.1	6.23	10.7	6.55

Table 5
Effect of the moving load mass and of the internal damping

Type of structure	a			b		
Vel. [m/s] ↓ γ, $M1$ →	γ, $M1$	$\gamma = 0$, $M1$	γ, $M1 = 0$	γ, $M1$	$\gamma = 0$, $M1$	γ, $M1 = 0$
5	2.4	5.6	1.8	2.0	4.1	1.9
10	4.2	7.4	2.8	2.2	3.4	1.9
20	6.2	11.1	4.3	4.2	8.4	2.4
30	12.9	17.8	7.9	5.2	10.0	4.9
50	18.8	22.5	6.9	10.8	16.3	10.0

Examinations showed omission of internal damping to result in a significant overestimation of the dynamic effect, while omission of the moving load mass to be a neglect to the detriment of safety.

4.4 Displacement Diagrams

Displacements of three different points of the structure (type **b**) due to a moving load at velocity $v = 30\,\mathrm{m/s}$ are seen in *Fig. 2*.

Displacements at the mid-point of the structure for various velocities are seen in *Fig. 3*. Apparently, in different cases some forms of vibration appear with different weights in the solution. *Fig. 4* shows the effect of neglecting the mass of moving load on displacements at the mid-point of the structure for velocity $50\,\mathrm{m/s}$. Apparently, reckoning with the damping and with the moving mass much affect the timely variation of displacements of the given point.

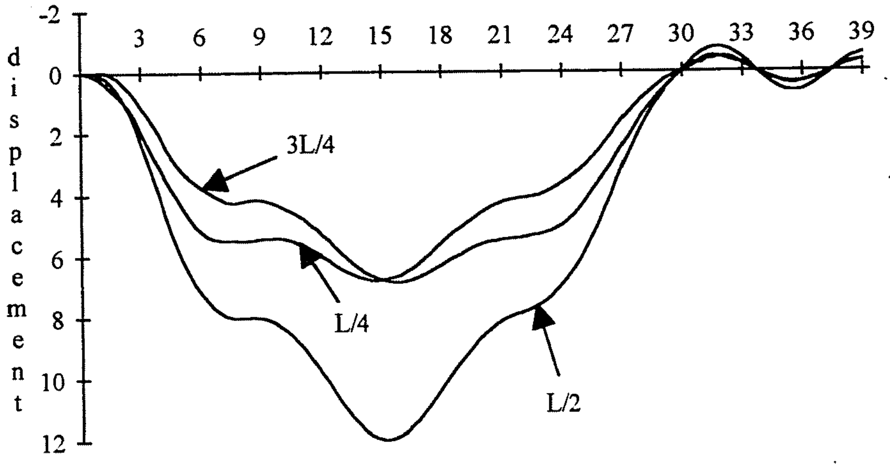

Fig. 2. Displacement [mm] at points L/4, L/2, 3L/4 place of moving force [m]

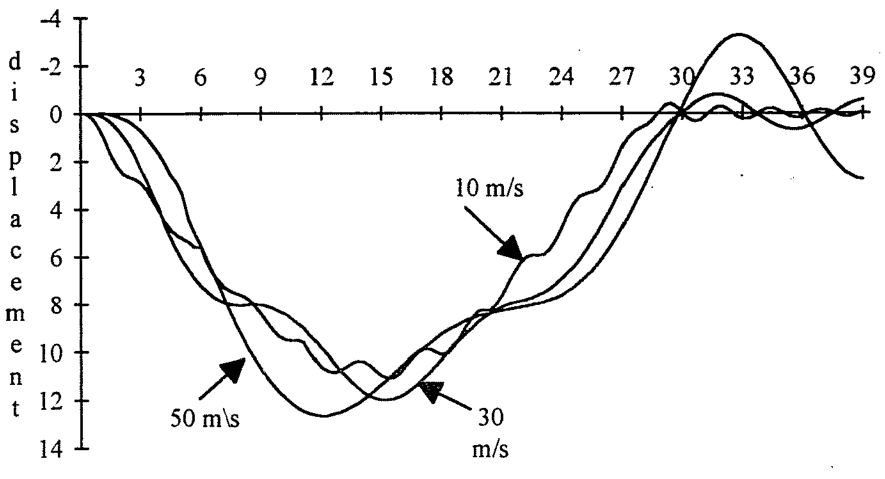

Fig. 3. Displacement [mm] of the mid-point in case of different velocity place of moving force [m]

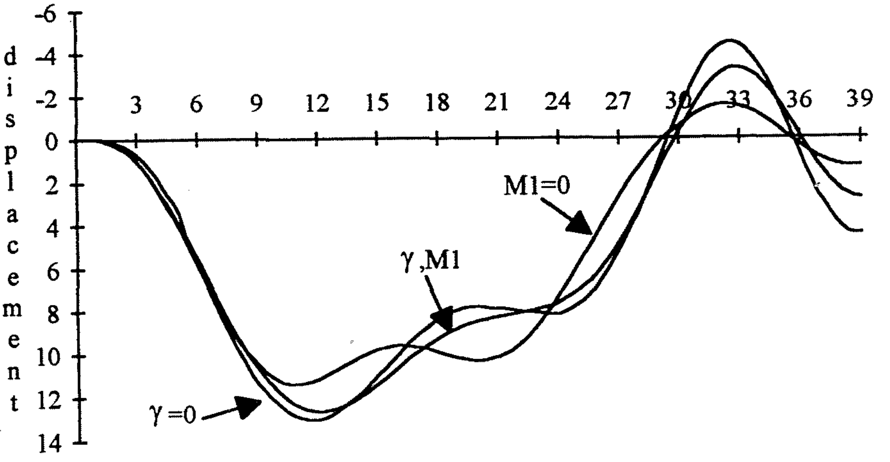

Fig. 4. Displacement [mm] of the mid-point in case of different γ, $M1$ place of moving force [m]

5. Conclusion

An algorithm has been presented for computing dynamic excess displacements of structures, if effects the mass of the moving load and of internal friction are to be taken into consideration. The developed algorithm and the numerical have been tested on actual problems. It may be stated that the mentioned factors have an important effect, justified to be reckoned with in the analysis of real structures.

References

FRÝBA, L. (1972): Vibration of Solids and Structures under Moving Loads. Prague. Publishing House of the Czechoslovak Academy of Sciences. 1972.

GYÖRGYI, J. (1985): Viscous and Hysteretic Damping in Vibration of Structures. *Periodica Polytechnica Ser. Civil Engineering,* Vol. 29, No. 1–2, pp. 23–31.

BATHE, K. J. – WILSON, E. L. (1976): Numerical Methods in Finite Element Analysis. New Jersey. Prentice-Hall, 1976.