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STABILITY OF EQUILIBRIUM AND COMPATIBILITY OF STRUCTURES WITH UNIAXIAL MATERIAL BEHAVIOUR

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Abstract

A generalization of the stability analysis is presented: analogously to the stability analysis of equilibrium, based on the potential energy, stability analysis of compatibility, based on the complementary energy will be introduced.

Nonlinearly inelastic material is considered, excluding strain softening and damage but including strain hardening and locking. The concept of tangent and compliance modulus is applied.

Global stability analysis is investigated, related to the total domain of the state variables, distinguishing the dead and rigid type of load, and their relation with the basic variational principles.

Discrete structural model of frame structures with uniaxial nonlinear and nonelastic material behaviour will be analysed.

Numerical illustrations can be studied in the author's papers, in (KURUTZ 1989, 1993, 1994).

Keywords: stability, tangent and compliance modulus, tangent stiffness and flexibility matrix, potential and complementary energy.

1. Introduction

The classical elastic stability analysis is based on the variational principle of the potential energy. The dissipative stability theories developed recently need also the potential energy, namely, the increment of it, based on the concept of tangentially equivalent elastic structure. Some authors introduce also the complementary energy in stability analyses. Indeed, in contrast to the elastic problems, the analysis of plastification or, inversely, the locking material behaviour makes the variational principle of the complementary energy reasonable. In order to avoid certain confusions in the theoretical bases, the systematization of the stability relations of the variational principles seems to be necessary.

The dissipative stability analysis needs the concept of the tangent modulus and the concept of linear comparison solid which latter was introduced by HILL (1958). Namely, in the case of a general nonlinear material behaviour, the instantaneously changing tangent modulus is needed.

The wide scope of the recent literature on the classical and nonclassical stability analysis, using different terminologies and the multidirectional approaches of the problem, all require to clear the theoretical bases. The type of loading device, the terms 'dead load' or 'rigid load', as well as the type of control, the terms 'load or displacement control', need also to clear the relation with the energetical description.

In this paper, these principles of the classical and nonclassical stability analysis of discrete systems are presented.

A generalization of the stability analysis is investigated: parallelly to the stability of equilibrium based on the potential energy, the stability of compatibility based on the complementary energy will be introduced. Global stability analyses are investigated, related to the total domain of possible deflections or forces. The type of loading devices and their relation with the basic variational principles will also be dealt with.

A discrete structural model with uniaxial nonlinear and nonelastic material behaviour will be analysed. Numerical examples can be studied in the papers of KURUTZ (1989, 1993, 1994).

2. Discrete Models of the Dual Variational Principles of Stability Analyses

Any discrete structural model needs double finitization: finitization of the geometrical and that of the functional spaces.

As for the *geometrical finitization* and the related *state variables,* as frequently used in stability analyses, let the bent structure be composed by perfectly rigid elements connected to each other by special springs in which the material behaviour is concentrated. Let the behaviour of each spring be characterized by a *one-dimensional nonlinear stress-strain function.*

Let $\boldsymbol{\sigma} = \{\sigma_i\}$ and $\boldsymbol{\varepsilon} = \{\varepsilon_i\}$ be the vectors of *stresses and strains,* resp., where the length of the vectors represents the number of all the stress and strain components over the structure. In our discrete model the variables σ_i and ε_i are forces and deformations of the springs, being work-compatible with each other, according to the internal work or complementary work. Vectors $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \{\sigma_i(\varepsilon_i)\}$ and $\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \{\varepsilon_i(\sigma_i)\}$ contain the *one dimensional functions of the material law.*

Let $\mathbf{F} = \{F_i\}$ and $\mathbf{u} = \{u_i\}$ be the vectors of the given *external force loads* and the unknown *displacements,* moreover, let $\mathbf{v} = \{v_i\}$ and $\mathbf{r} = \{r_i\}$ be the vectors of the given *external displacement loads* and the unknown *reaction forces,* being each F_i, u_i and v_i, r_i pairs in work-compatibility with each other, according to the external work or complementary work.

Notice that the force load $\mathbf{F}$ is independent of the displacements $\mathbf{u}$, so it is a constant or *'dead' load,* just like the displacement load $\mathbf{v}$ which is independent of the reactions $\mathbf{r}$, being constant or *'rigid' load.* These terminologies and the related loading devices are detailed in (THOMPSON and HUNT, 1983, p. 188).

By applying *one-parameter load* for the force and for the displacement type loading one-by-one, we have $\mathbf{F}_0 = \lambda\mathbf{F}_0$ and $\mathbf{v}_0 = \mu\mathbf{v}_0$ in which λ is the force and μ is the shape type load parameter, while $\mathbf{F}_0$ and $\mathbf{v}_0$ are the initial values of the loads.

In the recent literature some authors use the terminology: 'load control' and 'displacement control' (BAŽANT and CEDOLIN, 1991, p. 648), related to the basic type of load parameters. If the loading process is controlled by any statical parameter λ then *load control,* and inversely, while controlling the process by a kinematical type μ, *displacement control* is spoken of.

As for the *functional finitization* and the related *generalized coordinates,* assume that the kinematical state of the structure can be characterized by n number of independent kinematical parameters, and, similarly, that the statical state of the structure can be described by m number of independent statical parameters. Then the *kinematical (displacement type) parameters* $\mathbf{q} = \{q_i\}$, $i = 1, 2, \ldots, n$ and the *statical (force type) parameters* $\mathbf{p} = \{p_i\}$, $i = 1, 2, \ldots, m$ are used as the so-called *generalized coordinates* of the given numerical solution.

It is well-known that the *potential energy functional* $\pi(\varepsilon, \mathbf{u})$ requires the compatibility conditions as subsidiary conditions. By introducing the generalized kinematical coordinates $\mathbf{q}$, and by applying the transformations of kinematics

$$\mathbf{u} = \mathbf{u}(\mathbf{q}), \qquad \varepsilon = \varepsilon(\mathbf{u}) = \varepsilon(\mathbf{u}(\mathbf{q})) = \varepsilon(\mathbf{q}),$$

consequently, the functional $\pi(\varepsilon, \mathbf{u})$ is reduced to a scalar function $\pi(\mathbf{q})$ which is kinematically admissible at the same time.

Similarly, by considering the *complementary energy functional* $\pi^*(\sigma, \mathbf{r})$, the required equilibrium conditions are to be fulfilled. By introducing the statical type generalized coordinates $\mathbf{p}$ and by applying the transformations of statics

$$\mathbf{r} = \mathbf{r}(\mathbf{p}), \qquad \sigma = \sigma(\mathbf{r}) = \sigma(\mathbf{r}(\mathbf{p})) = \sigma(\mathbf{p}),$$

the functional $\pi^*(\sigma, \mathbf{r})$ is reduced to the scalar function $\pi^*(\mathbf{p})$ which is statically admissible at the same time.

In any case, as the connection between the *functional and geometrical finitization,* by introducing any type of *generalized coordinates,* it is necessary to transform the concerning *state variables to be work-compatible with the chosen generalized coordinates.*

3. The Tangent and Compliance Modulus

By supposing that the nonlinear and nonelastic structure can be substituted by its tangentially equivalent elastic version (BAŽANT and CEDOLIN, 1991, p. 635), the strain energy $W(\varepsilon)$ at ε can be as the potential function of the stresses $\sigma(\varepsilon)$ considered. Thus, the strain energy is equal to the internal potential, namely $W(\varepsilon) = \pi_{in}(\varepsilon)$. So

$$\delta\pi_{in} = \frac{\mathrm{d}\pi_{in}(\varepsilon)}{\mathrm{d}\varepsilon}\delta\varepsilon = \sigma(\varepsilon)\,\delta\varepsilon \quad \text{and} \quad \delta^2\pi_{in} = \frac{1}{2!}\frac{\mathrm{d}^2\pi_{in}(\varepsilon)}{\mathrm{d}\varepsilon^2}\,\delta\varepsilon^2 = \frac{1}{2!}k_t(\varepsilon)\delta\varepsilon^2$$

in which

$$k_t(\varepsilon) = \frac{\mathrm{d}^2\pi_{in}(\varepsilon)}{\mathrm{d}\varepsilon^2}$$

is the *tangent* modulus related to the *linear comparison solid* introduced by HILL (H. 1958). This tangent modulus is the function $k_t(\varepsilon)$ as the actual tangent of the stress-strain function $\sigma(\varepsilon)$. Indeed, the classical tangent modulus aims to give a simple Hooke's law like relation between the stress and strain increments

$$\mathrm{d}\sigma = K_t(\varepsilon)\,\mathrm{d}\varepsilon \quad \text{where} \quad K_t(\varepsilon) = \begin{cases} k_0 & \text{if} \quad \mathrm{d}\varepsilon < 0, \quad \text{(unloading)}, \\ k_t(\varepsilon) & \text{if} \quad \mathrm{d}\varepsilon \geq 0, \quad \text{(loading)}, \end{cases}$$

where k_0 is the initial modulus of elasticity and $k_t(\varepsilon)$ is the actual modulus (actual tangent) at ε of the stress strain curve $\sigma(\varepsilon)$. By using the hypothesis of the tangentially equivalent elastic behaviour, the dual phenomenological theories for the hyperelastic materials can also be used. Here $W^*(\sigma)$ is the complementary strain, or the 'stress' energy, which can be considered, as the complementary 'potential' function of the strains that is $W^*(\sigma) = \pi^*_{in}(\sigma)$.

Thus, the idea arises that similarly to the stability of equilibrium analysed on the basis of the potential energy and the concept of tangent modulus, the stability of compatibility can be investigated, on the basis of the complementary energy and the concept of compliance modulus. For the concept of the compliance tangent modulus, we assume the validity of the Legendre transformation

$$W(\varepsilon) + W^*(\sigma) = \sigma\,\varepsilon\,.$$

Thus, by excluding strain softening, the compliance modulus is restricted to the domain of nonnegative tangent moduli only.

In this way, we can specify the first and second order increments of the smooth complementary energy as follows

$$\delta\pi_{in}^* = \frac{\mathrm{d}\pi_{in}^*(\sigma)}{\mathrm{d}\sigma}\delta\sigma = \varepsilon(\sigma)\,\delta\sigma \quad \text{and} \quad \delta^2\pi_{in}^* = \frac{1}{2!}\frac{\mathrm{d}^2\pi_{in}^*(\sigma)}{\mathrm{d}\sigma^2}\delta\sigma^2 = \frac{1}{2}c_t(\sigma)\,\delta\sigma^2,$$

where

$$c_t(\sigma) = \frac{\mathrm{d}^2\pi_{in}^*(\sigma)}{\mathrm{d}\sigma^2}$$

is the *compliance tangent modulus* of the linear comparison solid. Thus, by the compliance tangent modulus, for strain and stress increments, a Hooke's law like relation can be given

$$\mathrm{d}\varepsilon = C_t(\sigma)\,\mathrm{d}\sigma\,, \qquad \text{where} \qquad C_t(\sigma) = \begin{cases} c_0 & \text{if} \quad \mathrm{d}\sigma < 0, \\ c_t(\sigma) & \text{if} \quad \mathrm{d}\sigma \geq 0, \end{cases}$$

where $c_t(\sigma) = 1/k_t(\varepsilon)$ is the actual tangent of the function $\varepsilon(\sigma)$ and $c_0 = 1/k_0$ is the inverse of the initial elastic modulus k_0.

Similarly to the tangent modulus, the compliance tangent modulus changes instantaneously during the material change of state. This makes the analysis complicated.

4. Reduction of the State Variables to be Work-compatible with the Generalized Coordinates of Potential and Complementary Energy The Structural Tangent and Compliance Modulus

4.1 Reduction of the Stresses and Forces The Structural Tangent Modulus

By using the principle of the potential energy, the kinematical type generalized coordinates $\mathbf{q}$ are introduced. Thus, both the external and the internal potential are expressed in term of the parameters $\mathbf{q}$. Consequently, all the statical type quantities, the stresses σ and the force loads $\mathbf{F}$ are reduced to be work-compatible with them.

By applying the generalized kinematical coordinates $\mathbf{q}$, through the compatibility transformations $\varepsilon = \varepsilon(\mathbf{u})$ and $\mathbf{u} = \mathbf{u}(\mathbf{q})$, any strain variable $\varepsilon_i = \varepsilon_i(\mathbf{u}(\mathbf{q}))$ of the internal potential are no more scalar. Consequently, for the first variations of the compound function $\pi_{in}(\mathbf{q}) = \pi_{in}(\varepsilon(\mathbf{u}(\mathbf{q})))$, we obtain

$$\delta\pi_{in} = \frac{\partial \pi_{in}(\mathbf{q})}{\partial q_i}\delta q_i = \frac{\partial \pi_{in}(\varepsilon)}{\partial \varepsilon_l}\,\frac{\partial \varepsilon_l(\mathbf{u})}{\partial u_j}\,\frac{\partial u_j(\mathbf{q})}{\partial q_i}\delta q_i = \boldsymbol{\sigma}^T(\varepsilon)\,\delta\varepsilon = \mathbf{f}^T(\mathbf{q})\,\delta\mathbf{q}$$

in which

$$\mathbf{f}^T(\mathbf{q}) = \{f_i(\mathbf{q})\}^T = \frac{\partial \pi_{in}(\varepsilon)}{\partial \varepsilon_l}\,\frac{\partial \varepsilon_l(\mathbf{u})}{\partial u_j}\,\frac{\partial u_j(\mathbf{q})}{\partial q_i} = \sigma_l(\varepsilon)\frac{\partial \varepsilon_l(\mathbf{u})}{\partial u_j}\,\frac{\partial u_j(\mathbf{q})}{\partial q_i}$$

are the reduced stresses being work-compatible with the generalized coordinates $\mathbf{q}$.

The above derivatives also can be expressed in matrix form, so for the *reduced stresses* we have

$$\mathbf{f}^T(\mathbf{q}) = \boldsymbol{\sigma}^T(\varepsilon)\,\mathbf{A}(\mathbf{u})\,\mathbf{B}(\mathbf{q}) = \boldsymbol{\sigma}^T(\varepsilon)\,\mathbf{M}(\mathbf{q}) = \boldsymbol{\sigma}^T(\varepsilon\,(\mathbf{u}(\mathbf{q})))\;\mathbf{M}(\mathbf{q}),$$

in which the matrices of the derivatives are

$$\mathbf{A}(\mathbf{u}) = \{a_{ij}(\mathbf{u})\} = \left\{\frac{\partial \varepsilon_i(\mathbf{u})}{\partial u_j}\right\} \qquad \text{and} \qquad \mathbf{B}(\mathbf{q}) = \{b_{ij}(\mathbf{q})\} = \left\{\frac{\partial u_i(\mathbf{q})}{\partial q_j}\right\}$$

while

$$\mathbf{M}(\mathbf{q}) = \{m_{ik}(\mathbf{q})\} = \mathbf{A}(\mathbf{u}) \qquad \mathbf{B}(\mathbf{q}) = \left\{\frac{\partial \varepsilon_i(\mathbf{u})}{\partial u_j}\,\frac{\partial u_j(\mathbf{q})}{\partial q_k}\right\}.$$

For the sake of clearness, in the following expressions we neglect the parentheses $\{.\}$ indicating matrices.

For qualifying the stability, we need the second order increment of the internal potential, too

$$\delta^2\pi_{in} = \frac{1}{2!}\delta q_i\frac{\partial^2 \pi_{in}(\mathbf{q})}{\partial q_i \partial q_j}\delta q_j = \frac{1}{2}\delta q_i\Bigg(\frac{\partial u_i(\mathbf{q})}{\partial q_m}\,\frac{\partial \varepsilon_m(\mathbf{u})}{\partial u_n}\,\frac{\partial^2 \pi_{in}(\varepsilon)}{\partial \varepsilon_n \partial \varepsilon_l}\,\frac{\partial \varepsilon_l(\mathbf{u})}{\partial u_k}\,\frac{\partial u_k(\mathbf{q})}{\partial q_j}+$$

$$+\frac{\partial u_i(\mathbf{q})}{\partial q_m}\left(\frac{\partial \pi_{in}(\varepsilon)}{\partial \varepsilon_l}\,\frac{\partial^2 \varepsilon_l(\mathbf{u})}{\partial u_m \partial u_k}\right)\frac{\partial u_k(\mathbf{q})}{\partial q_j} + \left(\frac{\partial \pi_{in}(\varepsilon)}{\partial \varepsilon_l}\,\frac{\partial \varepsilon_l(\mathbf{u})}{\partial u_k}\right)\frac{\partial^2 u_k(\mathbf{q})}{\partial q_i \partial q_j}\Bigg)\delta q_j =$$

$$= \frac{1}{2}\delta\mathbf{f}^T(\mathbf{q},\delta\mathbf{q})\,\delta\mathbf{q}\,,$$

which can be expressed in matrix form

$$\delta^2\pi_{in} = \frac{1}{2}\delta\mathbf{q}^T\Big(\mathbf{M}^T(\mathbf{q})\;\mathbf{K}_t(\varepsilon\,(\mathbf{u}(\mathbf{q})))\;\mathbf{M}(\mathbf{q})+$$

$$+\mathbf{B}^T(\mathbf{q})\left(\sigma^T(\varepsilon(\mathbf{u}(\mathbf{q})))\,\mathcal{B}(\mathbf{u}(\mathbf{q}))\right)\mathbf{B}(\mathbf{q})+$$

$$+\left(\sigma^T(\varepsilon(\mathbf{u}(\mathbf{q})))\;\mathbf{A}(\mathbf{u}(\mathbf{q}))\right)\mathcal{U}(\mathbf{q}))\delta\mathbf{q} = \frac{1}{2}\delta\mathbf{f}^T(\mathbf{q},\delta\mathbf{q})\,\delta\mathbf{q}.$$

So we obtain

$$\delta\mathbf{f}(\mathbf{q},\delta\mathbf{q}) = \{\delta f_i(\mathbf{q},\delta\mathbf{q})\} = \frac{\partial f_i(\mathbf{q})}{\partial q_j}\delta q_j$$

which are the first variations of the reduced stresses $\mathbf{f}(\mathbf{q})$, with respect of the generalized coordinates $\mathbf{q}$.

The matrix $\mathbf{K}_t(\varepsilon)$ in $\delta\mathbf{f}$ is the structural tangent modulus

$$\mathbf{K}_t(\varepsilon) =< K_{nl}(\varepsilon) >= \frac{\partial^2\pi_{in}(\varepsilon)}{\partial\varepsilon_n\partial\varepsilon_l}$$

forming a diagonal matrix by assuming uniaxial behaviour of each material points of the structure.

The third order matrices $\mathcal{B}(\mathbf{u}(\mathbf{q}))$ and $\mathcal{U}(\mathbf{q})$ are the second derivatives of the functions $\varepsilon(\mathbf{u})$ in term of $\mathbf{u}$, and $\mathbf{u}(\mathbf{q})$ in term of $\mathbf{q}$

$$\mathcal{B}(\mathbf{u}) = \{\mathcal{B}_{lmk}(\mathbf{u})\} = \frac{\partial^2\varepsilon_l(\mathbf{u})}{\partial u_m\partial u_k} = \mathcal{B}(\mathbf{u}(\mathbf{q})),$$

$$\mathcal{U}(\mathbf{q}) = \{\mathcal{U}_{kij}(\mathbf{q})\} = \frac{\partial^2 u_k(\mathbf{q})}{\partial q_i\partial q_j}.$$

Consider now the external potential energy $\pi_{ex}(\mathbf{u}) = -\mathbf{F}^T\mathbf{u}$, where $\mathbf{F}$ is a one parameter force type dead load $\mathbf{F} = \lambda\mathbf{F}_0$, so $\pi_{ex}(\mathbf{u}) = -\lambda F_0^T\mathbf{u}$. Here the displacements $\mathbf{u}$ are work-compatible with the applied load $\mathbf{F}$.

By applying the displacement parameters $\mathbf{q}$, for the first variation of the compound function $\pi_{ex}(\mathbf{u}(\mathbf{q})) = -\lambda\mathbf{F}_0^T\mathbf{u}(\mathbf{q})$ we obtain

$$\delta\pi_{ex} = \frac{\partial\pi_{ex}(\mathbf{q})}{\partial q_i}\delta q_i = \frac{\partial\pi_{ex}(\mathbf{u})}{\partial u_k}\frac{\partial u_k(\mathbf{q})}{\partial q_i}\delta q_i = -\lambda F_0^k\frac{\partial u_k(\mathbf{q})}{\partial q_i}\delta q_i =$$

$$= -\lambda\mathbf{F}_0^T\,\delta\mathbf{u} = -\lambda\mathbf{F}_0^T\,\mathbf{B}(\mathbf{q})\,\delta\mathbf{q} = -\mathbf{P}^T(\mathbf{q})\,\delta\mathbf{q}$$

in which

$$\mathbf{P}^T(\mathbf{q}) = \lambda\mathbf{P}_0^T(\mathbf{q}) = \lambda\{P_0^i(\mathbf{q})\} = \lambda F_0^k\frac{\partial u_k(\mathbf{q})}{\partial q_i} = \lambda\mathbf{F}_0^T\,\mathbf{B}(\mathbf{q})$$

are the reduced external force loads.

For qualifying the stability, we need the second order increment of the external potential, too

$$\delta^2 \pi_{ex} = \frac{1}{2}\delta q_i \frac{\partial^2 \pi_{ex}(\mathbf{q})}{\partial q_i \partial q_j}\delta q_j =$$

$$= \frac{1}{2}\delta q_i \left(\frac{\partial u_m(\mathbf{q})}{\partial q_i}\frac{\partial^2 \pi_{ex}(\mathbf{u})}{\partial u_m \partial u_k}\frac{\partial u_k(\mathbf{q})}{\partial q_j} + \frac{\partial \pi_{ex}(\mathbf{u})}{\partial u_k}\frac{\partial^2 u_k(\mathbf{q})}{\partial q_i \partial q_j} \right)\delta q_j =$$

$$= -\frac{1}{2}\lambda\,\delta q_i \left(F_0^k \frac{\partial^2 u_k(\mathbf{q})}{\partial q_i \partial q_j} \right)\delta q_j = -\frac{1}{2}\delta \mathbf{P}^T(\mathbf{q}, \delta\mathbf{q})\,\delta\mathbf{q}$$

in which

$$\delta \mathbf{P}^T(\mathbf{q}, \delta\mathbf{q}) = \{\delta P_j(\mathbf{q}, \delta\mathbf{q})\} = \lambda\,\delta q_i \left(F_0^k \frac{\partial^2 u_k(\mathbf{q})}{\partial q_i \partial q_j} \right) = \lambda\,\delta\mathbf{q}^T(\mathbf{F}_0^T\,\mathcal{U}(\mathbf{q}))$$

is the first variation of the reduced load $\mathbf{P}(\mathbf{q})$.

4.2 Reduction of the Strains and Displacements. The Structural Compliance Modulus

By using the principle of the complementary energy, the statical type generalized coordinates $\mathbf{p}$ are introduced. So, the strains ε and the displacement loads $\mathbf{v}$ are reduced to be work-compatible with them. By analysing the duality of the two basic energy principles, we have to consider them in the general function space of all the state variables, namely, in the *Hu–Washizu* space (KURUTZ, 1987). By taking the saddle surface character of the *Hu–Washizu* functional into consideration, the minimum property of the potential energy and the maximum character of the complementary energy are to be used in the correct way. Thus, for the internal complementary energy the negative and for the external one the positive signs are used.

By applying the principle of the complementary energy, both the external and the internal energies have to be expressed in term of the preliminarily chosen generalized statical coordinates $\mathbf{p}$. This ensures the equilibrium required to the complementary energy as subsidiary condition. However, for the equilibrium conditions, we need the structural deflections.

Since, for the complementary energy, we do not know the compatibility relations in advance, we can only use the necessary geometrical parameters being independent of each other. So we decide to consider the deflected structure with the displacement components $\tilde{\mathbf{u}}$, without knowing any relation among them.

Thus, by taking the equilibrium conditions $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{r})$ and $\mathbf{r} = \mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})$ into account, any stress functions $\sigma_i(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}}))$ are no more scalar. Consequently, for the variation of the compound function $\pi^*_{in}(\mathbf{p}, \tilde{\mathbf{u}}) = \pi^*_{in}(\boldsymbol{\sigma}(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})))$ we obtain

$$\delta\pi^*_{in} \frac{\partial \pi^*_{in}(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i} \delta p_i = \frac{\partial \pi^*_{in}(\boldsymbol{\sigma})}{\partial \sigma_l} \frac{\partial \sigma_l(\mathbf{r})}{\partial r_k} \frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i} \delta p_i =$$

$$= -\boldsymbol{\varepsilon}(\boldsymbol{\sigma})\, \delta\boldsymbol{\sigma} = -\mathbf{e}^T(\mathbf{p}, \tilde{\mathbf{u}})\, \delta\mathbf{p}$$

in which

$$\mathbf{e}^T(\mathbf{p}, \tilde{\mathbf{u}}) = \{e_i(\mathbf{p}, \tilde{\mathbf{u}})\} = \frac{\partial \pi^*_{in}(\boldsymbol{\sigma})}{\partial \sigma_l} \frac{\partial \sigma_l(\mathbf{r})}{\partial r_k} \frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i} = \varepsilon_l(\boldsymbol{\sigma}) \frac{\partial \sigma_l(\mathbf{r})}{\partial r_k} \frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i}$$

are the reduced strains, which in matrix form are

$$\mathbf{e}^T(\mathbf{p}, \tilde{\mathbf{u}}) = \boldsymbol{\varepsilon}^T(\boldsymbol{\sigma})\, \mathbf{G}(\mathbf{r})\, \mathbf{H}(\mathbf{p}, \tilde{\mathbf{u}}) = \boldsymbol{\varepsilon}^T(\boldsymbol{\sigma})\, \mathbf{N}(\mathbf{p}, \tilde{\mathbf{u}}) = \boldsymbol{\varepsilon}^T(\boldsymbol{\sigma}(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})))\, \mathbf{N}(\mathbf{p}, \tilde{\mathbf{u}})$$

in which the matrices are the following

$$\mathbf{G}(\mathbf{r}) = \{g_{ij}(\mathbf{r})\} = \frac{\partial \sigma_i(\mathbf{r})}{\partial r_j} = \mathbf{G}\,,$$

$$\mathbf{H}(\mathbf{p}, \tilde{\mathbf{u}}) = \{h_{ij}(\mathbf{p}, \tilde{\mathbf{u}})\} = \frac{\partial r_i(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_j} = \mathbf{H}(\tilde{\mathbf{u}}),$$

thus, $\mathbf{G} = \text{const}$, since $\boldsymbol{\sigma}(\mathbf{r})$ is always linear in $\mathbf{r}$, that is $\boldsymbol{\sigma}(\mathbf{r}) = \mathbf{G}\mathbf{r}$, moreover, $\mathbf{H}(\tilde{\mathbf{u}}) = \text{const}$ since $\mathbf{r}$ is always linear in $\mathbf{p}$, that is $\mathbf{r} = \mathbf{H}(\tilde{\mathbf{u}})\mathbf{p}$. Finally

$$\mathbf{N}(\tilde{\mathbf{u}}) = \{n_{ik}(\tilde{\mathbf{u}})\} = \mathbf{G}\ \mathbf{H}(\tilde{\mathbf{u}}) = \frac{\partial \sigma_i(\mathbf{r})}{\partial r_j} \frac{\partial r_j(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_k} = \frac{\partial \sigma_i(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}}))}{\partial p_k}\,.$$

Further, we need the second order increment of the internal complementary energy, too

$$\delta^2 \pi^*_{in} = \frac{1}{2} \delta p_i \frac{\partial^2 \pi^*_{in}(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i \partial p_j} \delta p_j =$$

$$= \frac{1}{2} \delta p_j \left(\frac{\partial r_m(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i} \frac{\partial \sigma_n(\mathbf{r})}{\partial r_m} \frac{\partial^2 \pi^*_{in}(\boldsymbol{\sigma})}{\partial \sigma_n \partial \sigma_l} \frac{\partial \sigma_l(\mathbf{r})}{\partial r_k} \frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_j} \right) \delta p_j =$$

$$= -\frac{1}{2}\delta \mathbf{e}^T(\mathbf{p}, \delta\mathbf{p}, \tilde{\mathbf{u}})\,\delta\mathbf{p}$$

since the second derivatives of the linear functions $\boldsymbol{\sigma}(\mathbf{r})$ and $\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})$ are zero. In matrix form

$$\delta^2\pi^*_{in} = \frac{1}{2}\delta\mathbf{p}^T\,\mathbf{N}^T(\tilde{\mathbf{u}})\;\mathbf{C}_t(\sigma(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})))\;\mathbf{N}(\tilde{\mathbf{u}})\,\delta\mathbf{p} = -\frac{1}{2}\delta\mathbf{e}^T(\mathbf{p}, \delta\mathbf{p}, \tilde{\mathbf{u}})\,\delta\mathbf{p}$$

from which the *first variations of the reduced strains* $\mathbf{e}(\mathbf{p}, \tilde{\mathbf{u}})$ can be obtained

$$\delta\mathbf{e}^T(\mathbf{p}, \delta p, \tilde{\mathbf{u}}) = \{\delta e_i(\mathbf{p}, \delta\mathbf{p}, \tilde{\mathbf{u}})\} = \frac{\partial e_i(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_j}\delta p_j$$

in which we can discover the structural compliance modulus

$$\mathbf{C}_t(\sigma) =< C_{nl}(\sigma) >= \frac{\partial^2\pi^*_{in}(\sigma)}{\partial\sigma_n\partial\sigma_l}.$$

In contrast to the second variation of the internal potential which contains the second variation of the strains, we can observe that the second variation of the internal complementary energy does not contain the effect of the second variation of the stresses, since it is always equal to zero. This fact concerns the *statical linearity,* the lack of symmetry of the two basic energy principles.

Let us consider now the external complementary energy $\pi^*_{ex}(\mathbf{r}) = \mu\mathbf{v}^T\mathbf{r}$ related to a single displacement type rigid load $\mathbf{v} = \mu\mathbf{v}_0$, where μ is the load parameter.

By applying now the chosen force coordinates $\mathbf{p}$ and the equilibrium transformation $\mathbf{r} = \mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})$, for the first variation of $\pi^*_{ex}(\mathbf{r}, \tilde{\mathbf{u}}) = \pi^*_{ex}(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})) = \mu\mathbf{v}_0^T\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})$ we obtain

$$\delta\pi^*_{ex} = \frac{\partial\pi^*_{ex}(\mathbf{r})}{\partial r_k}\frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\delta p_i}\delta p_i = \mu\mathbf{v}_0^T\,\delta\mathbf{r} = \mathbf{Q}^T(\mathbf{p}, \tilde{\mathbf{u}})\,\delta\mathbf{p}$$

in which

$$\mathbf{Q}^T(\mathbf{p}, \tilde{\mathbf{u}}) = \mu\mathbf{Q}_0^T(\mathbf{p}, \tilde{\mathbf{u}}) = \mu\{Q_0^i(\mathbf{p}, \tilde{\mathbf{u}})\} = \frac{\partial\pi^*_{ex}(\mathbf{r})}{\delta r_k}\frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i} =$$

$$= \mu\mathbf{v}_0^k\frac{\partial r_k(\mathbf{p}, \tilde{\mathbf{u}})}{\partial p_i} = \mathbf{Q}^T$$

are the *reduced external displacement loads* being independent of the force parameters $\mathbf{p}$, due to the statical linearity.

For qualifying the stability, we generally need the second order increment of the external energy, too. However, in the case of the complementary energy, due to the always valid statical linearity, we find the second variation of π^*_{ex} to be zero, not only in term of $\mathbf{r}$ but in term of the generalized force coordinates $\mathbf{p}$, too:

$$\delta^2\pi^*_{ex} = \frac{1}{2!}\,\delta\mathbf{Q}^T(\mathbf{p},\delta\mathbf{p},\tilde{\mathbf{u}})\,\delta\mathbf{p} = 0$$

since $\mathbf{Q}$ is independent of $\mathbf{p}$, so $\delta\mathbf{Q} = \mathbf{0}$. Namely, the first variation of the reduced displacement loads $\mathbf{Q}$ is always zero.

In contrast to the existing second variation of the external potential, we can state that the second variation of the external complementary energy vanishes.

5. Stability of Equilibrium by Using the Variational Principle of the Potential Energy

By introducing the tangentially equivalent elastic structure, the inelastic problems can be solved in small loading steps by a series of quasi-elastic analyses. Consequently, for the stability analysis of inelastic structures, the infinitesimal increment of the potential energy is needed (BAŽANT and CEDOLIN, 1991, Chap. 10).

5.1 Equilibrium Paths of Load Controlled Structures

Considering the related tangentially equivalent elastic structure of a nonlinearly nonelastic one, for the equilibrium at $\mathbf{q} = \mathbf{q}_0$, the first order infinitesimal increment of the total potential must vanish. By taking the previously detailed first order increments of the internal and external potential into account, for the condition of equilibrium, we have

$$\delta\pi = \mathbf{f}^T(\mathbf{q})\,\delta\mathbf{q} - \lambda\mathbf{P}_0^T(\mathbf{q})\,\delta\mathbf{q} = 0, \quad \mathbf{q} = \mathbf{q}_0\,.$$

If the variation $\delta\mathbf{q}$ of the generalized coordinates $\mathbf{q}$ are arbitrary, the scalar expression leads to a system of equations

$$\mathbf{f}(\mathbf{q}) - \lambda\mathbf{P}_0(\mathbf{q}) = \mathbf{0}$$

which reads in matrix form

$$\mathbf{M}^T(\mathbf{q})\,\sigma\left(\varepsilon\left(\mathbf{u}(\mathbf{q})\right)\right) - \lambda\mathbf{B}^T(\mathbf{q})\,\mathbf{F}_0 = \mathbf{0}, \qquad \mathbf{q} = \mathbf{q}_0$$

related to the linear comparison solid including the tangent modulus $\mathbf{K}_t$. For a global analysis, related to the total domain of possible deflections, the instantaneous tangent modulus $\mathbf{K}_t(\varepsilon)$ is needed.

The set of unknowns $(\mathbf{q}, \lambda)$ of the total potential consists of two parts: the displacements $\mathbf{q}$ and the load parameter λ. The *equilibrium state* $\mathbf{q}(\lambda)$ is the kinematical state of the structure by which the structure is in equilibrium at the load level λ. The equilibrium paths $\lambda(\mathbf{q})$ are equilibrium load-deflection functions, load functions in term of the equilibrium positions $\mathbf{q}$. The equilibrium path is the geometrical place of the equilibrium states (THOMPSON – HUNT, 1974).

In most cases of practical interests, the equilibrium conditions are highly nonlinear in $\mathbf{q}$. Matrices $\mathbf{B}(\mathbf{q})$ and $\mathbf{M}(\mathbf{q})$ generally consist of trigonometrical or other transcendent functions of $\mathbf{q}$, so the equilibrium paths can hardly be obtained in a closed form. Thus, for the numerical handling of the equilibrium conditions approximations are needed. Generally, for the material functions, polygonal approximations are needed, so the problem, by using nonsmooth analysis can be solved (see KURUTZ, 1989, 1993, 1994).

5.2 Stability of Equilibrium Paths of Load Controlled Structures. The Function of the Structural Tangential Stiffness Matrix

By using the tangentially equivalent elastic structure, the stability of equilibrium can be decided on the basis of the tangential stiffness matrix of the structure, obtained by the second order increment of the total potential (BAŽANT and CEDOLIN, 1991, Chap. 10), namely

$$\delta^2\pi = \frac{1}{2}\delta\mathbf{f}^T(\mathbf{q}, \delta\mathbf{q})\,\delta\mathbf{q} - \frac{1}{2}\delta\mathbf{P}^T(\mathbf{q}, \delta\mathbf{q}, \lambda)\,\delta\mathbf{q} = \frac{1}{2}\delta\mathbf{q}^T\mathbf{K}(\mathbf{q}, \lambda)\,\delta\mathbf{q}.$$

Here the matrix

$$\mathbf{K}(\mathbf{q}, \lambda) = \{K_{ij}(\mathbf{q}, \lambda)\} = \frac{\partial^2\pi_{in}(\mathbf{q})}{\partial q_i \partial q_j} + \frac{\partial^2\pi_{ex}(\mathbf{q})}{\partial q_i \partial q_j}, \qquad \mathbf{q} = \mathbf{q}_0$$

is the well-known *tangential stiffness matrix,* namely the *Hesse matrix of the potential* at the equilibrium state $\mathbf{q} = \mathbf{q}_0$. Thus, the condition of the stability can be decided on the basis of the tangential stiffness matrix: for stability it has to be positive definite.

By using the expressions detailed above, and by substituting the functions of the smooth equilibrium paths $\lambda(\mathbf{q})$ into $\mathbf{K}(\mathbf{q}, \lambda)$, the *function of the tangential stiffness matrix* for qualifying the equilibrium paths can be obtained

$$\mathbf{K}(\mathbf{q}) = \mathbf{M}^T(\mathbf{q})\,\mathbf{K}_t(\varepsilon(\mathbf{u}(\mathbf{q})))\,\mathbf{M}(\mathbf{q}) + \mathbf{B}^T(\mathbf{q})\,\sigma^T(\varepsilon(\mathbf{u}(\mathbf{q})))\,\mathcal{B}(\mathbf{u}(\mathbf{q}))\mathbf{B}(\mathbf{q}) +$$

$$+\mathbf{A}^T(\mathbf{u}(\mathbf{q}))\,\sigma^T(\varepsilon(\mathbf{u}(\mathbf{q})))\mathcal{U}(\mathbf{q}) - \lambda(\mathbf{q})\mathbf{F}_0^T\mathcal{U}(\mathbf{q})\,.$$

The stability of equilibrium paths and all the influencing effects can be studied by analysing the above terms of the tangential stiffness matrix.

6. Stability of Compatibility by Using the Variational Principle of the Complementary Energy

Similarly to the introduction of the complementary energy, which was a mathematical fiction, we introduce the stability analysis of geometrical states of structures, based on the complementary energy.

6.1 Compatibility Paths of Displacement Controlled Structures

The term 'compatibility path' was introduced by TARNAI in (TARNAI, 1990), related to finite mechanisms. Analogy between the bifurcation of equilibrium and compatibility paths was detailed in his thesis, for structures consisting of rigid elements.

In this paper, for the stability analysis of compatibility, solid structure is supposed.

For the condition of compatibility at the state $\mathbf{p} = \mathbf{p}_0$, the first order increment of the total complementary energy vanishes, namely

$$\delta\pi^* = -\mathbf{e}^T(\mathbf{p}, \tilde{\mathbf{u}})\,\delta\mathbf{p} + \mu\mathbf{Q}_0^T(\tilde{\mathbf{u}})\,\delta\mathbf{p} = 0,\ \mathbf{p} = \mathbf{p}_0$$

leading to the system of equations if $\delta\mathbf{p}$ are arbitrary

$$-\mathbf{e}(\mathbf{p}, \tilde{\mathbf{u}}) + \mu\mathbf{Q}_0(\tilde{\mathbf{u}}) = \mathbf{0}\,, \qquad \mathbf{p} = \mathbf{p}_0\,.$$

The classical problem of HILL, the linear comparison solid, can be extended to the dual problem, too. Thus, by excluding strain softening the condition of compatibility can be written in the form

$$-\mathbf{N}^T(\mathbf{p}, \tilde{\mathbf{u}})\,\varepsilon(\sigma(\mathbf{r}(\mathbf{p}\,\tilde{\mathbf{u}}))) + \mu\mathbf{H}^T(\mathbf{p}, \tilde{\mathbf{u}})\mathbf{v}_0 = \mathbf{0}\,, \qquad \mathbf{p} = \mathbf{p}_0$$

by applying the expressions detailed above. This relation is the compatibility equation, resulting the relations among the displacement components $\tilde{\mathbf{u}}$ at the statical state $\mathbf{p} = \mathbf{p}_0$.

This equation seems to be formally equal to the dual expression of equilibrium condition, however, the statical linearity makes it different. In contrast to equilibrium problems, here the force parameters $\mathbf{p}$ appear only in the material functions.

In order to investigate a global analysis related to the domain of possible forces $\mathbf{p}$, the instantaneous compliance modulus $\mathbf{C}_t(\sigma)$ is needed. Thus, polygonal approximation is offered again (see KURUTZ, 1993, 1994).

The set of unknowns $(\mathbf{p}, \mu)$ of the total complementary energy consists of two parts: the force coordinates $\mathbf{p}$ and the load parameter μ. The compatibility states $\mathbf{p}, (\mu)$ are the statical states of the structure by which the structure is in compatibility with the given displacement load, at the load level μ. The compatibility paths $\mu(\mathbf{p})$ are compatible deflection-force functions, namely, the rigid load in term of the compatible reaction forces $\mathbf{p}$. The compatibility paths are the geometrical place of the compatibility states.

6.2 Stability of Compatibility Paths of Displacement Controlled Structures The Function of the Structural Tangential Flexibility Matrix

By using the concept of tangentially equivalent elastic structure, the stability of compatibility can be decided on the basis of the tangential compliance matrix of the structure, obtained by the second order increment of the total complementary energy:

$$\delta^2\pi^* = -\frac{1}{2}\delta\mathbf{e}^T(\mathbf{p}, \delta\mathbf{p})\,\delta\mathbf{p} = -\frac{1}{2}\delta\mathbf{p}^T\mathbf{C}(\mathbf{p})\,\delta\mathbf{p}$$

since $\mathbf{Q}$ is constant in $\mathbf{p}$, so $\delta\mathbf{Q} = \mathbf{0}$. Thus, due to the statical linearity, from the expression $\delta^2\pi^*$ the load parameter μ vanishes.

In the case of the variational principle of the complementary energy, the second variation of the total complementary energy consists of only the internal part, so the matrix

$$\mathbf{C}(\mathbf{p}) = \{C_{ij}(\mathbf{p})\} = -\frac{\partial^2\pi^*_{in}(\mathbf{p})}{\delta p_i \delta p_j} \qquad \mathbf{p} = \mathbf{p}_0\,.$$

Thus, the function of the tangential flexibility matrix, the function of the Hesse matrix of the complementary energy is as follows

$$\mathbf{C}(\mathbf{p}) = \mathbf{N}^T(\tilde{\mathbf{u}})\;\mathbf{C}_t(\sigma(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})))\;\mathbf{N}(\tilde{\mathbf{u}})$$

leading to the conclusion that the function of the tangential flexibility matrix contains the force parameters only in the term of the material function.

Thus, the stability of compatibility depends on the material behaviour, the actual compliance moduli $\mathbf{C}_t(\sigma)$ only.

For a global analysis, the instantaneous compliance modulus $\mathbf{C}_t(\sigma(\mathbf{r}(\mathbf{p}, \tilde{\mathbf{u}})))$ is needed. These moduli are the inverse of the tangential stiffnesses. Since from this analysis the strain softening is excluded, the tangential compliance moduli are not allowed to be negative. If the material becomes infinitely rigid (perfect locking), then the compliance modulus becomes zero. This leads to the loss of stability of compatibility.

In contrast to the stability of equilibrium, where the material softening can cause stability losing, in the case of the stability of compatibility the material hardening leads to the losts of stability.

Conclusion

Dual variational principles of stability analyses were introduced. Similarly to the stability analysis of equilibrium, stability analysis of compatibility can be investigated. Similarly to the concept of the tangent modulus, as a basis of the analysis, the concept of compliance modulus was used.

One dimensional material functions and simple discrete structures were used. Path stability analysis was investigated. Similarly to the equilibrium paths of load controlled structures, compatibility paths of displacement controlled structures can be obtained.

The stability conclusions can be decided similarly to the classical cases. While the stability of equilibrium states and paths is based on the function of the structural tangential stiffness matrix, the stability of compatibility states and paths is based on the structural tangential flexibility matrix.

Numerical illustrations can be studied in the papers of the author, in (KURUTZ 1989, 1993, 1994).

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