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TIME-DEPENDENT STRESS-LIMITED MECHANICAL MODELS OF ELASTO-PLASTIC PROCESSES

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Abstract

In generall, the mechanical models are described by different type of differential equations. In the approach presented here the unconstrained optimization problem that derives from the boundary problem is decomposed into a pair of constrained optimization problems. In this paper the presented models concerning the stress-limited elasto-plastic state change include the theoretical results of the earlier studies. The theoretical results are illustrated by some numerical examples.

Keywords: nonlinear programming, stress-limited problem, elasto-plastic state change.

1. Introduction

The term of plasticity is used in mechanics as a certain determination of irreversible deformation of solids produced by external actions. These deformations are caused by the change of material micro structure without cracking. Due to the plastic deformations some kinds of energy are always dissipated. Generally, the theories used in structural mechanics for describing elasto-plastic state change with time-independent material flow are formulated on the basis of macroscopic observation of the material behaviour [17]. In the phenomenological theory of plasticity, which can be tracked back to TRESCA [20] and GUEST [7], the material is assumed to be macroscopically homogeneous, isotropic and free of stress and strain in its original state. The stress and strain based approach to plasticity can be found in works DRUCKER [6] and ILYUSIN [8]. In the 50-s PRAGER [15, 16] worked hard on this topic. The mathematical work of KARUSH [10], KUHN – TUCKER [12] advanced to the development of the computation in the structural mechanics. MAIER [13] and DE DONATO [4] proposed a general and powerful matrix description of Koiter's theory [11] in plasticity. The ability of the classical theories to describe realistically the plastic deformation of metals is generally accepted, however, certain recommendations restrict their use in modelling the behaviour of the structural concrete. In the 70-s a collection of studies on elastic-plastic theories by mathematical programming is edited by MAIER – COHN [3]. The mathematical programming methods as helping tools play an important part in the structural plasticity [9, 18, 19]. The common thing is the energy dissipation it is not taken into consideration in these works.

Ordinarily the variational interpretation of a problem is created according to the type of differential equations that occur (including the initial boundary conditions), and then the approximation functions to be used have an appropriate order which follows from the type of each equation. In the approach presented here the unconstrained optimization problem that derives from the boundary problem is decomposed into a pair of constrained optimization problems.

The individual material properties are included as functions of the state characteristics and this fact connects the parts coming from the original equations into a system. Decomposition of the unconstrained optimization problem into a constrained optimization problem is not unambiguous, but the different decompositions all must satisfy the same Euler–Lagrange equations. It is executed in such a way that the primal–dual problems have physical meanings. From the mathematical point of view, the primal and the dual variables are intensive and extensive mechanically and the objective function contains different types of energies.

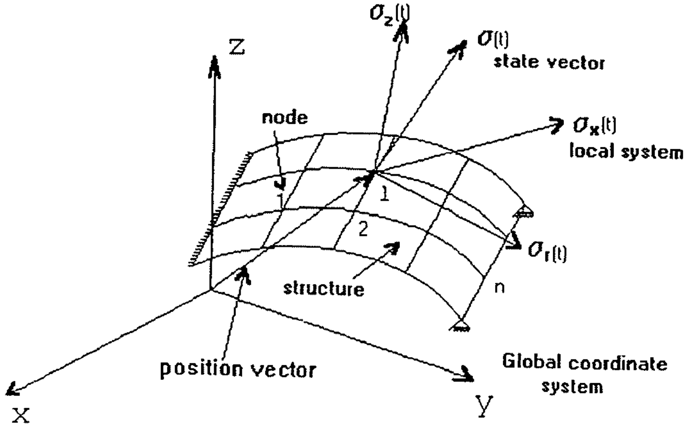


Fig. 1. Where $\sigma(t) = [\sigma_x(t), \quad \sigma_y(t), \quad \sigma_z(t), \quad \tau_{xy}(t), \quad \tau_{yz}(t), \quad \tau_{xz}(t),]$

The state variables (e. g. stresses, strains, etc.) are given in vector space by vector-scalar functions in the case of equilibrium state. The structure is discretized similarly to FEM. For a discretized structure that vector space is supposed to be a 3 dimensional space in a global co-ordinate system. Every node of the element is defined by a position vector (see *Fig. 1*). A state vector described in the local co-ordinate system is attached to each position vector. The number of the independent components depends on the freedom of the nodes (6 for the example in *Fig. 1*) and it equals n times the dimension of the state vector.

In case of time-dependent problems the state functions are given in both local and global co-ordinate systems and are vector-vector functions which depend on the time. Within the context of a small displacement theory, the position vectors are time-independent.

In this paper the presented models concerning the stress limited elasto-plastic state change include the theoretical results of the earlier studies [21]. The theoretical results are illustrated by some numerical examples.

2. Mathematical Background

It is supposed that the Euler and Lagrange descriptions of the structure correspond, that is the position vector does not depend on time. It means that the state characteristics can be given using function subspaces based on the local co-ordinate axes, and can be expressed via a linear combination of the (unknown) coefficients of basis functions of these spaces.

In other words, the state variables (e.g. $\sigma(x,t)$) are approximated by:

$$\sigma(x,t) \approx \sum_{i=1}^{n} \mathbf{x}(t) N(\xi), \qquad (1)$$

where $N(\xi)$ notes the shape functions depending on the position vector (ξ) and the coefficients $\mathbf{x}(t)$ depending on time.

The state variables are given in both local and global co-ordinate systems as vector-vector functions which elements depend on time. These elements are described in a function space which is determined on the local co-ordinate axes. It means that the state variables can be given as a vector with function elements on the local co-ordinate axes in a function subspace.The state variables ($\mathbf{x}(t)$) can be expressed by the generalized Fourier series according to the basis of the function subspace:

$$\mathbf{x}^{\ell}(t) = \sum_{j=1}^{s} \left[\sum_{i=1}^{\infty} \alpha_{ij} P_i(t) \right] \mathbf{e}_j^{\ell}, \quad \alpha_{ij} \in \Re, \quad P_i(t) \in L^2, \quad t \in [t_1, t_2], \quad (2)$$

where $\mathbf{e}_j^{\ell}$ ($j = 1, \ldots, s$) is j-th unit vector of the local co-ordinate system ordered to the ℓ-th node, s: number of degrees of freedom at the nodes,

$P_i(t)$: i-th element of basis of the function subspace (orthonormal polynomial system on $[t_1, t_2]$).

The state variables are described on every node in the following space: $F = L_1^2 \times L_2^2 \times \ldots \times L_s^2$ and on the whole structure

$$F^n = \left(L_2^2 \times L_3^2 \times \ldots \times L_s^2\right)^n , \tag{3}$$

where n is the number of the nodes and s is the number of freedoms.

To describe the elasto-plastic process a nonlinear mathematical programming problem is created in space F^n where the variables are the Fourier coefficients α_{ij}. It has been proved elsewhere that the Kuhn–Tucker theorem and Wolfe's duality [1] are valid in space F^n.

Then the nonlinear mathematical programming problem exists in space F^n in terms of elements $(\mathbf{x}(t))$. The general description of the nonlinear programming problem is the following in space F^n.

Primal:

$$\begin{aligned} &\min f(\sigma(x(t))) , \\ g_i(\sigma(x(t))) \le 0, \qquad & i = 1, \ldots , k; \\ h_j(\sigma(x(t))) = 0, \qquad & j = 1, \ldots , q, \\ x(t) \in F^n, \qquad & \forall t \in [t_1, t_2] . \end{aligned} \tag{4}$$

Dual:

$$\max \left\{ f(\sigma(x(t))) + \sum_{i=1}^{k} \lambda_i(t) g_i(\sigma(x(t))) + \sum_{j=1}^{q} u_j(t) h_j(\sigma(x(t))) \right\} ,$$

$$\nabla f(\sigma(x(t))) + \lambda(t) \nabla g(\sigma(x(t))) + \mathbf{u}(t) \nabla h(\sigma(x(t))) = 0 ,$$

$$g(\sigma(x(t)))\lambda(t) = 0 , \quad \lambda(t) \ge 0 , \quad x(t) , \lambda(t) , u(t) \in F^n , \quad \forall t \in [t_1, t_2] , \tag{5}$$

where the extreme is looked for every time point t, $u(t)$ and $\lambda(t)$ are the dual functions.

For the solution it is necessary to transform the problem into the ℓ^2 space by the help of Riesz–Fischer theorem. In the ℓ^2 space the Kuhn–Tucker theorem and the duality theorems are valid [2]. The inequalities cannot be transformed into the ℓ^2 space because for the inequalities the isomorphism is not valid. The inequalities are checked at some given time point ($P_i(t_v)$, $i = 1, \ldots, \infty$, $v = 1, \ldots, w$). The nonlinear programming problem is in ℓ^2 space after the mapping (4):

Primal:

$$\min \tilde{f}(\alpha_{ij}),$$

$$\tilde{g}_{\ell v}(\alpha_{ij}) \le 0, \qquad \ell = 1, \dots, k, \qquad v = 1, \dots, w, \qquad (6)$$
$$\tilde{h}_r(\alpha_{ij}) = 0, \qquad r = 1, \dots, q, \qquad \alpha_{ij} \in \Re.$$

Dual:

$$\max\left\{\tilde{f}(\alpha_{ij}) + \sum_{\ell=1}^{k} \tilde{\lambda}_\ell \tilde{g}_\ell(\alpha_{ij}) + \sum_{r=1}^{q} \tilde{u}_r \tilde{h}_r(\alpha_{ij})\right\},$$

$$\nabla\tilde{f}(\alpha_{ij}) + \tilde{\lambda}\nabla\tilde{g}(\alpha_{ij}) + \tilde{\mathbf{u}}\nabla\tilde{h}(\alpha_{ij}) = 0,$$

$$\tilde{g}(\alpha_{ij})\tilde{\lambda} = 0\,, \qquad \tilde{\lambda} \ge 0\,, \qquad \alpha_{ij} \in \Re\,, \qquad (7)$$

where $\tilde{f}(\alpha_{ij})$, $\tilde{g}(\alpha_{ij})$ and $\tilde{h}(\alpha_{ij})$ are the pictures of the functions $f(\sigma(x(t)))$, $g(\sigma(x(t)))$ and $h(\sigma(x(t)))$ after the mapping into the ℓ^2 space, respectively. The dimensions of the mathematical programming problem increasus.

In the practice the infinite vector space is truncated to a finite one, this means a finite number of basis functions are taken into consideration in the function space.

The solution of the nonlinear programming problem is a stationer curve. The limitation of the presented model is: at least one continuous component has to be assumed, the small displacement theory is valid, stability problems are neglected.

For the numerical solution it is necessary to transform the problem into the ℓ^2 space where the Kuhn–Tucker theorem is valid [2] and the results are mapped back to the space F^n.

3. Linear Elastic Case

3.1 Time-Independent Case

In time-independent case the complementary strain energy can be written in the form of unconstrained optimization form [22] that expresses the derived variational principle. The loads and stresses depend on the place vector only.

$$\min\left\{\iiint_v \left\{B(\sigma_x, \sigma_y, \dots, \tau_{xy}) + \left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + \overline{X}\right)u + \right.\right.$$

$$+\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + \overline{Y}\right)v + \left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + \overline{Z}\right)w\Big\}\mathrm{d}V -$$

$$-\iint_{S_1}\left[(X_v - \overline{X}_v)u + (Y_v - \overline{Y}_v)v + (Z_v - \overline{Z}_v)w\right]\mathrm{d}S -$$

$$-\iint_{S_2} \left[X_v\overline{u} + Y_v\overline{v} + Z_v\overline{w}\right] \mathrm{d}S \Bigg\}, \tag{8}$$

where V, S: volume and boundary of the structure, respectively, $B(s)$: complementer strain energy function, (in linear elastic case: $B(s) = 1/2\sigma\mathbf{F}\sigma$, $\mathbf{F}$: flexibility matrix), $\mathbf{p}^* = \left[\overline{X}, \overline{Y}, \overline{Z}\right]$: vector of external loads, $\overline{\mathbf{p}}_v^* = \left[\overline{X}_v, \overline{Y}_v, \overline{Z}_v\right]$ and $\overline{\mathbf{u}}^* = [\overline{u}, \overline{v}, \overline{w}]$: prescribed value of internal loads and displacements, respectively.

The variational problem (8) can be written as a constrained variational problem using matrix notations:

$$\min \iiint_V \{B(\sigma)\} \, \mathrm{d}V, \qquad \text{complementer strain energy,}$$

$$\iiint_V (\mathbf{B}^*\sigma + \mathbf{p}) \, \mathrm{d}V = 0, \qquad \text{equilibrium equations,}$$

$$\iint_{S_1} (\mathbf{p}_v - \overline{\mathbf{p}}_v) \, \mathrm{d}S = 0, \qquad \text{stress boundary conditions,}$$

$$\iint_{S_2} \overline{\mathbf{u}} \mathrm{d}S = 0, \qquad \text{displacements boundary conditions,} \tag{9}$$

where V, S: volume and boundary of the structure, respectively, $B(s)$: complementer strain energy function, (in linear elastic case: $B(s) = 1/2\sigma\mathbf{F}\sigma$, $\mathbf{F}$: flexibility matrix), $\sigma_x = \sum_{i=1}^{\infty} \sigma_i N_i(x)$, $N_i(x)$: shape functions, $\sigma = [\sigma_i]$: vector of stresses, $\mathbf{B} = [L(N_i(x))]$: transfer matrix (in given cases it can be found in the well-known finite element books), L: differential operator, $*$: transpose, $\mathbf{p}$: vector of external loads, $\overline{\mathbf{p}}_v$ and $\overline{\mathbf{u}}$: prescribed value of internal loads and displacements, respectively.

The displacements (Lagrangian multipliers) concerning the nodes are independent of the volume integrals so we can write their problem (9).

Gaussian numerical integral calculus has been used to compute all of integrals that is the values of the stresses have been determined on the Gaussian points.

i. The unknowns of the problem are the stresses, acting on the Gaussian points. The unknown vector is partitioned according to the members, the Gaussian points, the stress freedoms.

ii. In linear elastic case the minimal value of *the complementer strain energy function* is searched for. After the integration the complementer strain energy function is:

$$\min \frac{1}{2}\langle\rho\rangle\sigma^*(G)\mathbf{F}\sigma(G)\,. \tag{10}$$

iii. *The equilibrium equations* can be written by the stresses acting on the Gaussian points on the following form:

$$\sum_{j=1}^{mGz} \mathbf{B}^*_{\mathrm{k,\,j}}\rho_j\sigma(G_j) + p_k = 0\,, \qquad k = 1, \dots, ns\,. \tag{11}$$

The columns of the matrix **B** are partitioned according to the elements of the structure (m), one hyper-block contains blocks according to the Gaussian points (G) and the dimension of one block is determined by the stress freedom of a Gaussian point (z).

A row of the matrix **B** contains the coefficients of the equilibrium equation system. The rows of the matrix **B** are partitioned according to the nodes (n) and the dimensions of the blocks refer to the displacements freedoms (s) of a node.

There are ns equations and mGz unknowns before taking into consideration the boundary conditions.

iv. In case of *displacements-type boundary conditions* — if $\mathbf{u}_k = 0$ — the k-th row is deleted from the equilibrium equation system, because the supported node at the given freedom has to be in equilibrium. These equations can be used to determine the reaction forces.

v. If the *stress-type boundary conditions* concern a given Gaussian point and stress freedom, the corresponding element of the unknown vector is zero, that is the corresponding column of the matrix **B** is multiplied by zero — this column is deleted from the matrix. The corresponding column and row are deleted from the matrix **F** as well.

If the stress-type boundary conditions concern a given point of the structure it is necessary to express the stresses at this point from the stresses acting on the Gaussian points by the help of shape functions ($N_i(x)$). These conditions give new equations to the equilibrium system. It is necessary to subtract the effect of boundaries from the complementer strain energy function, too.

3.2 Time-Dependent Case

The loads and stresses depend on the place vector and time (t). The parameter t is changing in a time interval. The time-dependent case of the functional minimum problem (8) can be written in the following form:

$$\min\Bigg\{ \iiint_V \Big\{ B\left(\sigma_x(t), \sigma_y(t), \dots, \tau(t)_{xy}\right) +$$

$$+ \left(\frac{\partial \sigma_x(t)}{\partial x} + \frac{\partial \tau_{xy}(t)}{\partial y} + \frac{\partial \tau_{zx}(t)}{\partial z} + \overline{X}(t) \right) u(t) +$$

$$+ \left(\frac{\partial \tau_{xy}(t)}{\partial x} + \frac{\partial \sigma_y(t)}{\partial y} + \frac{\partial \tau_{yz}(t)}{\partial z} + \overline{Y}(t) \right) v(t) +$$

$$+ \left(\frac{\partial \tau_{xz}(t)}{\partial x} + \frac{\partial \tau_{yz}(t)}{\partial y} + \frac{\partial \sigma_z(t)}{\partial z} + \overline{Z}(t) \right) w(t) \Big\} \mathrm{d}V -$$

$$- \iint_{S_1} \left[\left(X_v(t) - \overline{X}_v(t) \right) u(t) + \left(Y_v(t) - \overline{Y}_v(t) \right) v(t) + \left(Z_v(t) - \overline{Z}_v(t) \right) w(t) \right] \mathrm{d}S -$$

$$- \iint_{S_2} \left[\overline{u}(t) + \overline{v}(t) + \overline{w}(t) \right] \mathrm{d}S \Bigg\}$$

$$\forall t, t \in [t_1, t_2], \qquad (12)$$

where the vectors have function elements and the meanings of the letters are the same as ttey were in *Eq.* (8).

The constrained problem is:

$$\min \iiint_V B(\sigma(t)) \mathrm{d}V, \qquad \text{complementer strain energy,}$$

$$\iiint_V (\mathbf{B}^*(\sigma(t)) + \mathbf{p}(t)) \mathrm{d}V = 0, \qquad \text{equilibrium equations,}$$

$$\iint_{S_1} (\mathbf{p}_v(t) - \overline{\mathbf{p}}_v(t)) \, \mathrm{d}S = 0, \qquad \text{stress boundary conditions,}$$

$$\iint_{S_2} \overline{\mathbf{u}}(t) \mathrm{d}S = 0, \quad \forall t, t \in [t_1, t_2], \quad \text{displacements boundary conditions.}$$

(13)

Taking into consideration the approximation of the unknown $\sigma(x,t)$ according to the place vectors we get:

$$\sigma_x(x,t) = \sum_{i=1}^{\infty} \sigma_i(t) N_i(x).$$

The time-dependent coefficients ($\sigma_i(t)$) can be written in the function space by the help of its basic system. The basic system is chosen as an orthogonal polynom system:

$$\sigma_x(x,t) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \alpha_{ij} P_j(t) N_i(x) . \tag{14}$$

i. The unknowns of the problem are coefficients α_{ij}. The infinite series of the orthogonal polynom system ($P_j(t)$) are truncated to a finite one ($j = 1, \ldots, v$). The vector of unknown coefficients is partitioned according to the members (m), the Gaussian points (G), the stress freedoms (z) and the considered part of the orthogonal polynom system (v). The number of unknown is: $mGzv$.

ii. *The complementer strain energy function* is in linear elastic case after the integration:

$$\frac{1}{2}\langle\rho\rangle\alpha^{*}\overline{\mathbf{F}}\alpha , \tag{15}$$

where the matrix $\overline{\mathbf{F}}$ refers to the matrix $\mathbf{F}$ in a way that it contains a hyper-diagonal matrix with the same values instead of each element of $\mathbf{F}$. The dimension of $\overline{\mathbf{F}}$ is: $mGzv$, $mGzv$. $\langle\rho\rangle$ is a diagonal matrix containing the Gaussian weights. The indexes of a show are the partitions of the vector of unknowns.

iii. For *the equilibrium equations* the matrix $\mathbf{B}$ is extended to a matrix $\overline{\mathbf{B}}$ which has a hyper-diagonal block with the same elements instead of every element of $\mathbf{B}$. The dimension of $\overline{\mathbf{B}}$ is: nsv, $mGzv$. The generalised Fourier coefficients of the elements of external load vector are noted by β_{jk} according to the orthogonal polynom system ($P_j(t)$). The equilibrium equations have to be satisfied in every time point of t ($\forall t, t \in [t_1, t_2]$):

$$\sum_{j=1}^{mGz} \rho_j \overline{\mathbf{B}}^{*}_{\mathrm{ljk}} \alpha_{jk} + \beta_{lk} = 0 , \quad l = 1, \ldots, ns, \quad (k = 1, \ldots, v, \forall l, \forall j) . \tag{16}$$

iv., v. The *displacements-type and stress-type boundary conditions* are taken into consideration as it happened in the time-independent case, not only the corresponding rows and/or columns are deleted now but also the corresponding hyper rows and/or columns.

Transforming the problem into the ℓ^2 space it is enough to compute with the generalised Fourier coefficients (Riesz–Fisher theorem).

Summarising in case of time-dependent linear elastic case one has to solve infinite numbers of quadratic programming problems. The equality equations are linear with the same coefficient matrix and different right hand side and are the same objective function.

4. Stress-Limited Linear Elastic, Plastic Case

4.1 Time-Independent Case

The problem — given by (10), (11) and boundaries — is extended with restrictions for the measure of stresses:

$$\min \frac{1}{2}\langle\rho\rangle\sigma^*(G)\mathbf{F}\sigma(G),$$

$$\sum_{j=1}^{mGz} \mathbf{B}^*_{\mathrm{k,j}}\rho_j\sigma(G_j) + p_k = 0, \qquad k = 1, \ldots, ns,$$

$$f_i(\sigma) \leq 0, \qquad i = 1, \ldots, mG. \tag{17}$$

4.2 Time-Dependent Case

The time-dependent problem can be formulated by the help of the problem defined by (15), (16) and boundaries and time dependent plastic yield conditions.

The time-dependent plastic yield conditions can be written as a function of stresses:

$$f_k(\sigma(t)) \approx f_k\left(\sum_{i=1}^{v}\alpha_{ij}P_i(t), \qquad j = 1\ldots z, \sigma_L\right) \leq 0,$$

$$k = 1, \ldots, mG, \qquad \forall t, t \in [t_1, t_2], \tag{18}$$

where σ_L: limit stress.

The inequalities cannot be transformed into the ℓ^2 space as it happened in case of equalities. It is necessary to check the inequalities in some given points in the interval of t.

Summarising the time-dependent stress-limited system can be formulated by the following mathematical programming problem:

$$\min \frac{1}{2}\langle\rho\rangle\alpha^*\overline{\mathbf{F}}\alpha,$$

$$\sum_{j=1}^{mGz} \rho_j\overline{\mathbf{B}}^*_{\mathrm{ljk}}\alpha_{jk} + \beta_{lk} = 0, \qquad l = 1, \ldots, ns, \qquad (k = 1, \ldots, v, \forall l, \forall j)$$

boundary conditions,

$$f_k\left(\sum_{i=1}^{v}\alpha_{ij}P_i(t_c), \qquad j=1,\dots z,\sigma_L\right)\le 0\,,$$

$$k=1,\dots,mG\,, \qquad c=1,\dots,w\,, \tag{19}$$

where w means the discretized time points where the inequalities are examined.

The problem (19) contains $mGzv$ unknowns, nsv equalities, mGw inequalities. The problem (19) cannot be separated into a series of mathematical programming problems because of the yielded conditions against of the elastic case.

5. Numerical Example

As an illustration of the proposed computational methods, let'sconsider a three-supported beam with time-dependent loading. The data of the structure can be seen in *Fig. 2*. The loads act on the nodes. The functions of the external loads are given by

$$F(t) = 9.75 + 0.25t + 0.75t^2 + 1.25t^3\,.$$

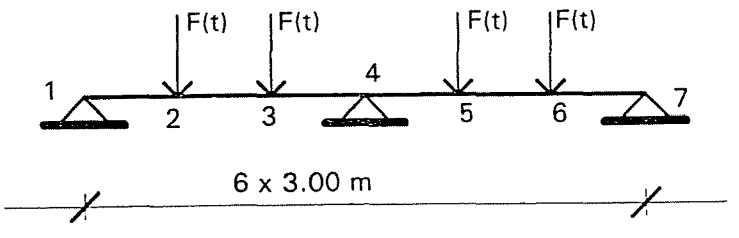


Fig. 2.

The loads are approximated by the Legendre polynomial system up to four members. The structure is divided into 6 members with 7 nodes. The unknowns are the moments and the shear forces at the Gaussian points of the members. The number of the unknowns is 96.

On the basis of the general form, the statically admissible inner forces are determined by the equilibrium equations and force boundary conditions. The objective function contains the complementary potential energy. The

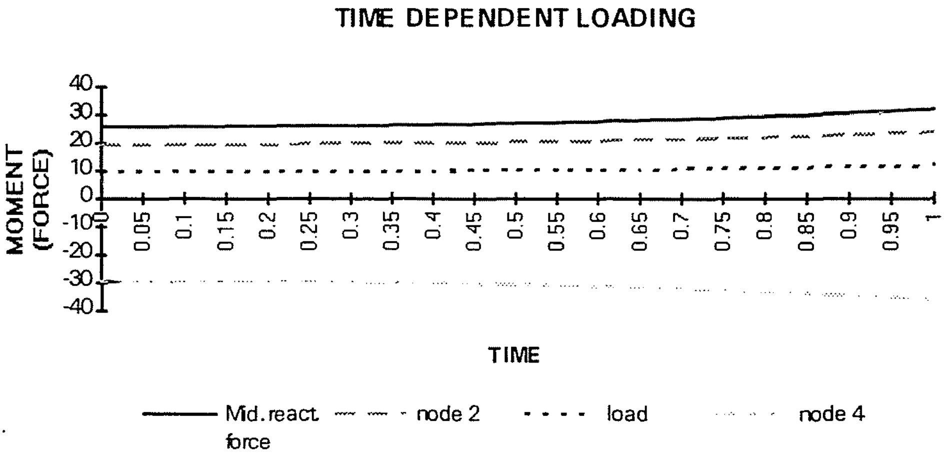
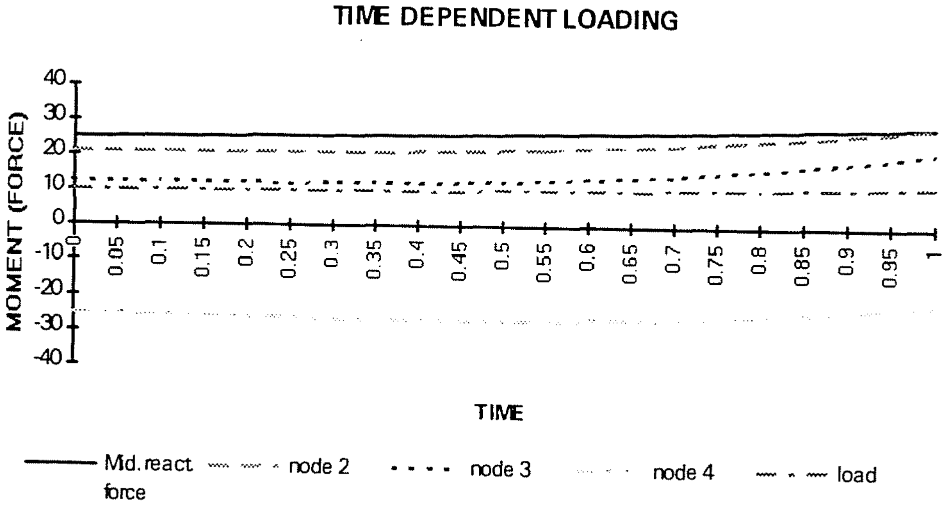

TIME DEPENDENT LOADING
40
30
20
10
0
-10
-20
-30
-40
MOMENT (FORCE)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
TIME
Mid.react force
node 2
load
node 4

Fig. 3.


TIME DEPENDENT LOADING
40
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MOMENT (FORCE)
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0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
TIME
Mid.react force
node 2
node 3
node 4
load

Fig. 4.

displacement boundary conditions are taken into consideration in the objective function.

To solve this problem we used a quadratic programming system. The results can be seen in *Fig. 3*. One can see that this method and the traditional algorithms give the same result.

If there are some restrictions for the inner forces they give inequalities in the mathematical programming problem. In our example, the Huber–Mises–Hencky yield condition is used. In *Fig. 4* one can follow the formation of the plastic hinges during the process.

Acknowledgements

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References

1. BAZARRA, M. S. – SHETTY, C. M.: Nonlinear Programming, Theory and Algorithms, John Wiley & Sons , New York, 1979.
2. BHAKTA, P. C. – ROYCHANDHURI, S.: Optimization in Banach Spaces, *Journal of Math. Analysis and Applications,* No. 134, pp. 460–470, 1988.
3. COHN, M. Z. – MAIER, G. (Eds.): Engineering Plasticity by Mathematical Programming, Pergamon Press, New York, 1979.
4. DE DONATO, O.: On Piecewise-linear Constitutive Laws in Plasticity, *Tech. Rept., ISTC,* 1974.
5. DRUCKER, D. C. – PRAGER, W.: Soil Mechanics and Plastic Analysis or Design, *Aust. Appl. Math.,* Vol. 10, 1952.
6. DRUCKER, D. C.: Basic Concepts in Plasticity, Handbook of Eng. Mech., W. Flügge, (Ed.) McGraw-Hill, 1962.
7. GUEST, J. J.: On the Strength of Ductile Materials under Combined Stress, *Phil. Magazine,* Vol. 50, 1900.
8. ILYUSIN, A. A.: On the Foundation of the General Mathematical Theory of Plasticity, *Voprosy Teorii Plastichnosti,* AN SSR, Moscow, 1961.
9. KALISZKY, S.: Plasticity, Theory and Engineering Application, Elsevier Science Publishers, Amsterdam, 1989.
10. KARUSH, W.: Minima of Functions of Several Variables with Inequalities as Side Conditions, M.S. Thesis, Dept. of Mathematics, University of Chicago, 1939.
11. KOITER, W. T.: Stress-strain Relations, Uniqueness and Variational Theorems for Elastic-plastic Materials with Singular Yield Surfaces, *Quart. Appl. Math.,* Vol. 11, 1953.
12. KUHN, H. W. – TUCKER, A. W.: Nonlinear Programming, *2nd Berkeley Symp. Math. Statistics and Probability,* Berkeley, 1951.
13. MAIER, G.: A Matrix Structural Theory of Piecewise Linear Elastoplasticity with Interacting Yield Planes, *Meccanica,* Vol. 5, 1970.
14. MUSCHIK, S.: Non-equilibrium Thermodynamics with Applications to Solids, Springer-Verlag, Wien, New York, 1993.

15. PRAGER, W.: Recent Developments in the Mathematical Theory of Plasticity, *J. Appl. Phys.,* Vol. 20, 1949.
16. PRAGER, W.: The Theory of Plasticity — a Survey of Recent Achievements, *Proc. Inst. Mech. Eng.,* Vol. 169, p. 41, 1955.
17. SAWCZUK, A.: Mechanics and Plasticity of Structures, Ellis Horwood Ltd., Chichester, 1989.
18. SAWCZUK, A. – BIANCHI, G. (Eds.):Plasticity Today, Elsevier Science Publishers, London, 1985.
19. LLOYD SMITH, D.: Mathematical Programming Methods in Structural Plasticity, Springer Verlag, Wien, New York, 1990.
20. TRESCA, H.: Mémoire sur l'écounlement des corps solides soumis á des fortes pressions, *Compte-Rendu,* Vol. 59, Paris, 1864.
21. VÁSÁRHELYI, A. – LÓGÓ, J.: Mathematical Models of Processes in Mechanics, *Structural Optimization,* Vol. 7. 1994.
22. WASHIZU, K.: Variational Methods in Elasticity and Plasticity, Pergamon Press, Oxford, London, 1968.