PERIODICA POLYTECHNICA SER. CIVIL ENG. VOL. 37, NO. 1, PP. 3-11 (1993)

SOME COMPUTATIONAL ASPECTS OF THE STABILITY ANALYSIS OF NONLINEAR SPACE TRUSSES

Anna CSÉBFALVI* and Anna VÁSÁRHELYI

Kiss J. altb. u 34,
H-1126 Budapest, Hungary
*Magaslati ut 35/b, H-7625 Pécs, Hungary

Received: December 1, 1992

Abstract

A hybrid computational method is presented that seeks to combine efficiency and accuracy of the quasi-Newton method in the direct energy minimization with the robust modified homotopy method that is capable to follow post-buckling path very accurately.

Keywords: geometrically nonlinear trusses, potential energy minimization, stability, path following methods, homotopy method.

Introduction

Solution methods for geometrically nonlinear problems which have been formulated using variational principles in conjunction with a discretization technique such as the finite element method may be classified into three levels according to the method of mathematical formulation. The most widely used 'first level' formulation is the principle of stationary potential energy. The second level of formulation is obtained by expressing the condition of equilibrium directly. It can also be obtained by setting the first variation of potential energy to zero. The third level of formulation expresses the condition of incremental equilibrium. It can be obtained by setting the second variation of the energy potential to zero.

In the first scalar approach, when it is possible to establish a total potential for the system (e. g. large deformation analysis of elastic materials), the problem can be reduced to the well-known 'direct search' problem, namely the unconstrained minimization of a nonlinear function of several variables.

In the second and third vector approaches the equations of motions reduced to a system of nonlinear (linear) equations are unknown nodal point parameters of the finite element model.

Most methods used today are of incremental type combined with some schemes for equilibrium iteration (BATHE and WILSON (1976), BERGAN,

HORRIGMOE, KRAKELAND and SOREIDE (1978), WRIGGERS, WAGNER and MIEHE (1988)).

It is well known that the straightforward load incrementation-iteration tend to break down at extremum and bifurcation points of the solution path and may be quite unreliable. A large number of 'automatic' or 'adaptive' solution schemes have been proposed for dealing with stability problems with extremum points, instable branches, bifurcations, etc. However, currently no method is available that can be said to be totally reliable for every type of problems, (RIKS (1984), WATSON, KAMAT and REASER (1988)).

The scalar approach has been used for nonlinear structural analysis by several investigators. Past experiments using minimization algorithms for structural analysis reveal that at least for small scale problems, the 'direct energy minimization technique' is better suited than the most other incremental-iterative techniques for solving highly nonlinear problems. The most important result is that for comparable problems on the primary stable path the 'direct search' is superior to the standard linearized technique in terms of CPU time by a factor of 2 (KAMAT, VANDEN BRINK and WATSON (1980)) . However, such techniques are not very popular. Most of the applications of the potential energy are used for explaining the behaviour near critical points.

During the past decade significant advances have been made in the development of algorithms that solve large scale (sparse) non-linear minimization problems. Extension of the minimization methods to large problems centers on reducing the storage requirements of the second order quasi-Newton methods or improving the efficiency of the first order conjugate gradient techniques (KAMAT, VANDEN BRINK and WATSON (1980)).

In the present study for geometrically non-linear space trusses, the direct potential energy minimization technique combined with the robust homotopy algorithms (KAMAT and WATSON, (1983)) is re-examined and the effectiveness of the hybrid method for computing multiple equilibrium solutions is demonstrated.

The fundamental problem about the energy minimization is that the load response curve of the structure is a composite of stable and unstable branches rather than a single-valued function of the load. Using straightforward load incrementation with the potential energy of the system as a function to be minimized it is possible to locate only the stable equilibrium configurations. This problem is connected with the Newton method. Although this algorithm is quite efficient, it has obvious limitation in the vicinity of limit points and along unloading branches.

In a recent paper WATSON, KAMAT and REASER (1985) provide an evaluation of the globally convergent quasi-Newton method and the homo-

topy method with regard to their suitability for solving nonlinear stability problems. The algorithm extends a quasi-Newton method utilizing the model trust-region strategy to solve a system of nonlinear equations by minimizing the $F(u,l)^*F(u,l)$ least squares function. This 'quasi energy minimization' procedure used in conjunction with deflation and tunneling is theoretically useful for locating equilibrium configurations along unstable path, but in the practical size applications the use of the method is numerically very complicated. In the proposed hybrid algorithm, the standard homotopy method was used in the vicinity of limit points.

Due to the numerical difficulties of tunneling and deflating, in our study on the stable primary path the original energy minimization approach was applied to give accurate starting information to the modified homotopy algorithm. In our approach from the first limit point the modified homotopy method was only used. We note that in the original (arc length based) homotopy method no Newton-type iterations are performed. The modified homotopy algorithm to increase the reliability of the predictions may be able to perform 'corrector type' iterations. The modification in different context was originally proposed by ABBOTT (1980).

Geometrically Nonlinear Model of Space Trusses

In this work, we consider simplified procedures for large deformation and post-buckling analysis of three-dimensional (geometrically perfect) space-truss structures, wherein each of the members is assumed to carry only one axial load. Only a conservative system of concentrated loads is assumed to act at the nodes of the trusses. The nonlinear behaviour of solids will be restricted to elastic deformations which is sufficient for most purposes concerning stability problems. The basis for this procedure is the principle of stationary potential energy for a elastic body.

In order to capture the effects of changes in global geometry of the truss a large displacement model has been adopted in our study, using a total Lagrange representation.

Let's number the nodes of the structure and the elements. In this way, two node numbers are ordered to one element number. The first one indicates the right, the second one the left side of the truss. The trusses jointed to the k-th node can be chosen by the help of the matrix [C] which contains in the k-th row the node numbers of k-th truss.

Let us denote the undeformed length of the j-th truss member by $\mathcal{L}_j$ and let $(\mathcal{L}_{ij})_{i=1,2,3,}$ be the projections of undeformed length $\mathcal{L}_j$ onto the

co-ordinate axes. The length of the member after deformation is given by,

$$L_j(u) = \left\{ \sum_{i=1}^{3} (\mathcal{L}_{j,i} + {}^{r}u_{ji} - {}^{l}u_{ji})^2 \right\}^{\frac{1}{2}}, \tag{1}$$

where $({}^{r}u_{ji})_{i=1,2,3}$ and $({}^{l}u_{ji})_{i=1,2,3}$ are the displacements at the right and left nodes of the j-th element. The displacements of the k-th node are denoted by $(u_{ki})_{i=1,2,3}$ without upper index.

The potential energy of the structure for linear elastic material is given by

$$\prod = \frac{1}{2} \sum_{j=1}^{n} E_j A_j \frac{(L_j(u) - \mathcal{L}_j)^2}{\mathcal{L}_j} - \sum_{k=1}^{m} \sum_{i=1}^{3} p_{ki} u_{ki}, \tag{2}$$

where n is the number of elements, m is the number of nodes, E_j is the modulus of elasticity and A_j is the cross sectional area of the j-th element, and $(p_{ki})_{i=1,2,3}$ denotes the components of the external load acting on the k-th node.

The vectors of displacements and applied loads are

$$\mathbb{u} = \{u_{11}, u_{12}, u_{13}, \ldots, u_{m1}, u_{m2}, u_{m3}\}, \tag{3}$$

$$\mathbb{P} = \{P_{11}, P_{12}, P_{13}, \ldots, P_{m1}, P_{m2}, P_{m3}\}. \tag{4}$$

The principle of stationary potential energy gives the following explicit equilibrium equations:

$$\frac{\partial \prod}{\partial u_{ki}} = \sum_{j}^{n} E_j A_j \frac{L_j(u) - \mathcal{L}_j}{\mathcal{L}_j} \frac{(\mathcal{L}_{j,i} + {}^{r}u_{ji} - {}^{l}u_{ji})}{L_j(u)} - P_{ki} = 0,$$

$$i = 1, 2, 3, \; k = 1, \ldots, m, \tag{5}$$

where ${}^{r}u_{ji}$ and ${}^{l}u_{ji}$ denote the i-th component of the displacements belonging to the j-th row as well as first and second columns of the matrix $[\mathbb{C}]$, respectively.

Homotopy Method

In this method we assume that the equilibrium equations (KAMAT and WATSON (1987)) have the form:

$$\mathbb{F}(\mathbb{u}, \lambda) = 0, \tag{6}$$

where $\mathbf{u}$, $\mathbf{F}$ are q vectors and λ is a scalar (load intensity factor). Assuming that there are no bifurcation points, the zero set of $\mathbf{F}(\mathbf{u},\lambda)$ is a smooth curve g which does not intersect itself or other zeros of $\mathbf{F}(\mathbf{u},\lambda)$ in the generic case, and along which

$$D\mathbf{F}(\mathbf{u},\lambda) = [D_{\mathbf{u}}\mathbf{F}(\mathbf{u},\lambda), D_{\lambda}\mathbf{F}(\mathbf{u},\lambda)] \tag{7}$$

has rank q. The exceptional cases are where γ intersects itself or has bifurcation points.

The smooth curve g can be parameterized by a scalar parameter t, so,

$$\mathbf{u} = \mathbf{u}(t), \qquad \lambda = \lambda(t) \tag{8}$$

along t. Then,

$$\mathbf{F}(\mathbf{u}(t),\lambda(t)) = 0 \tag{9}$$

identically in t, and the initial value problem

$$[D_{\mathbf{u}}\mathbf{F}(\mathbf{u},(t),\lambda(t))\, D_{\lambda}\mathbf{F}(\mathbf{u},(t),\lambda)(t))]\begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\lambda} \end{Bmatrix} = 0, \tag{10}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\ \lambda(0) = 0$$

has precisely γ as its trajectory (where the dot denotes differentiation with respect to t). In this approach the derivative $(\dot{\mathbf{u}}, \dot{\lambda})$ is specified only implicitly, and special techniques are required to solve the initial value problem.

Note that the initial value problem defines

$$\mathbf{y}(t) = \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\lambda} \end{Bmatrix} \tag{11}$$

up to a scaling factor, which is defined by the choice of t. For example let's choose the parameter as follows:

$$\dot{\mathbf{y}}^*(t)\dot{\mathbf{y}}(t) = 1, \tag{12}$$

and if $t = \mathbf{y}^{(k)}$ (the k-th component of $\mathbf{y}$) then

$$\mathbf{e}_k^*\dot{\mathbf{y}}(t) = 1, \tag{13}$$

where $\mathbf{e}_k$ is the kth unit vector (modified homotopy method).

The algorithm requires computation of the kernel of matrix $D\mathbf{F}$. This can be easily and efficiently done for small dense matrices, but the large

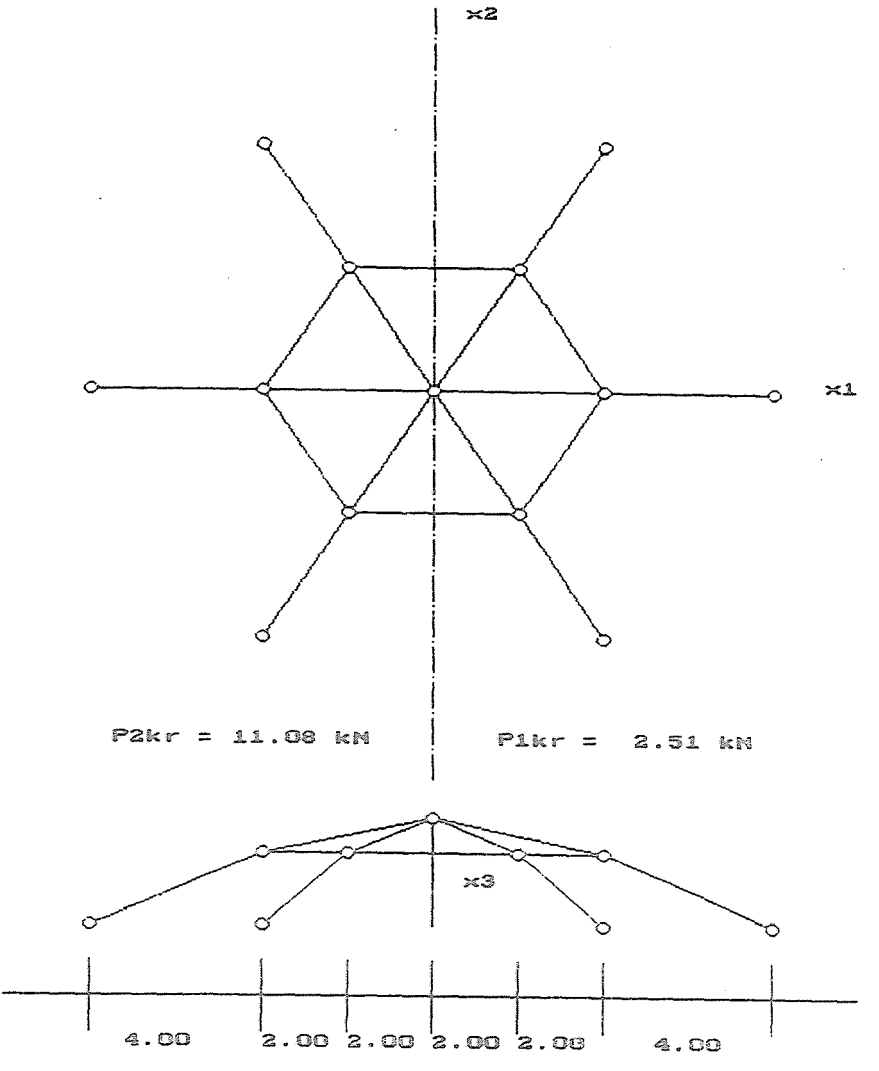

Fig. 1.

sparse Jacobian matrix of structural mechanics presents special difficulties. For more details on the solution methods see WATSON, KAMAT and REASER (1985), KAMAT and WATSON (1983). Reliable routines for this problem are developed by KUBICEK (1976) and WATSON – FENNER (1980). A routine for modified homotopy approach was presented by ABBOTT (1980).

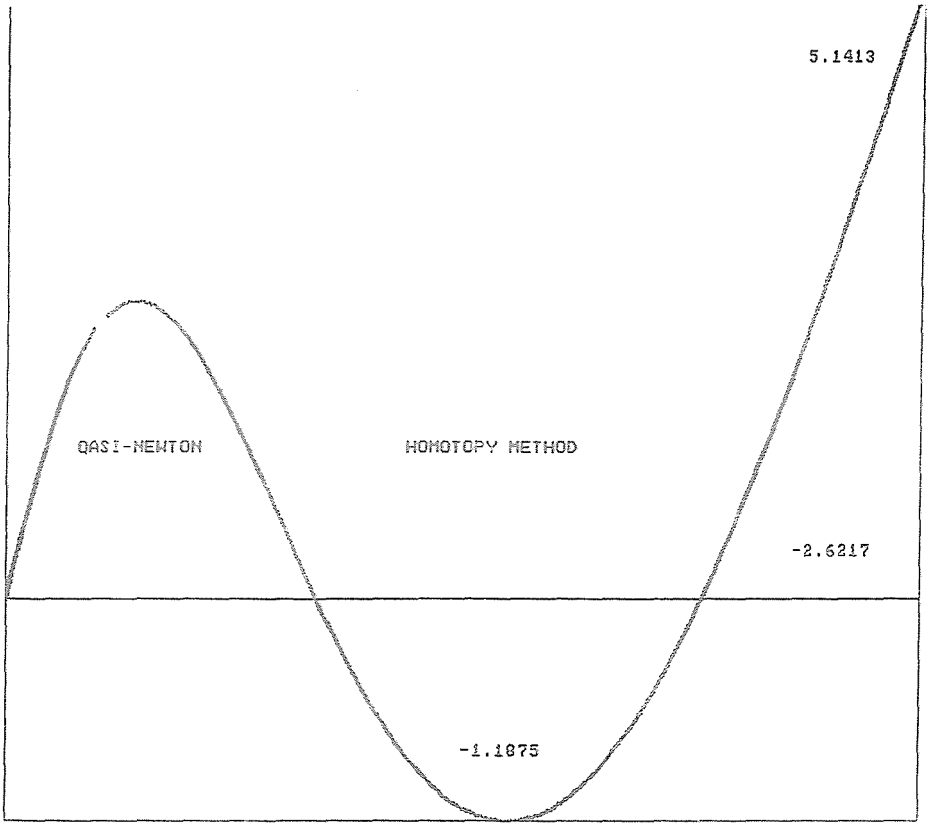

Fig. 2.

The most important characteristic of the Abbott method is that following a prediction used by HALL and WATT (1978), the equations

$$\mathbb{F}(\mathbf{y}(t_{i+1})) = 0 \tag{14}$$

can be solved with the predicted $\mathbf{y}_{i+1}^{(p)}$ as initial estimate.

We choose t to be one of the components of $\mathbf{y}$, $\mathbf{y}_k$ say.

Having set $t_{i+1} = (\mathbf{y}_k)_{i+1}$, the Jacobian of (14) is made up of q of the column of $D\mathbb{F}$ (the kth being omitted). The index k is chosen so as to make the resulting equations well conditioned. With this choice of t, (14) can be solved to arbitrary accuracy and subsequent prediction can be accurate to the order of the predictor formula employed. With this change a reduction in the number of functions and Jacobian evaluation of about 20 % and often better can be expected.

Numerical Results

The proposed hybrid method was validated on the snap-through response of a hexagonal truss structure, the data for which are shown in *Fig. 1.* In this paper the computation of the primary path is stopped after the first two limit points. These points are associated with a local snap-through of the upper dome, see *Fig. 2.*

The hybrid algorithm begins as a quasi-Newton algorithm with an assumed or specified load step. The 'jump' of the minimization was taken to be an indication of the existence of a critical point in the vicinity. The modified homotopy method was therefore initiated. For the smooth transition from the direct minimization to the modified homotopy method, the last eight points were stored in every step. Because of its good stability properties we have chosen the Adams–Bashforth predictor for the prediction of $\mathbf{y}(t_{i+1})$ from the past values of $\mathbf{y}$ and $\dot{\mathbf{y}}$. Using the predicted point as initial iterate, the new points were computed with the modified Newton methods. For computing the kernel of $D\mathbf{F}$ matrix the QR transformation was used. The limit points can be calculated very accurately by the homotopy method with no difficulty. In the example exact Hessian and Jacobian were used.

References

ABBOTT, J. P. (1980): Computing Solution Arcs of Nonlinear Equation with a Parameter, *The Computer Journal* Vol. 23, pp. 85-89.

BATHE, K. J.– WILSON, E. L. (1976): Numerical Methods in Finite Element Analysis, Prentice-Hall, Englewood Cliffs, N.J.

BERGAN, P. G.– HORRIGMOE, G.– KRAKELAND, B. – SOREIDE, T. H. (1978): Solution Techniques for Nonlinear Finite Element Problems, *Int. J. Numer. Methods Eng.*, Vol. 12, pp. 1677-1696.

HALL, G. – WATT, J. M. (1978): Modern Numerical Methods for Ordinary Differential Equations, Oxford.

KAMAT, M. K. – VANDEN BRINK, D. J. – WATSON, L. T. (1980): Nonlinear Structural Analysis Using Quasi-Newton Minimization Algorithms that Exploit Sparsity, in C. Taylor, E. Hinton and D. R. J. Owen: Numerical Methods for Non-linear Problems, Pineridge Press, Swansea.

KAMAT, M. P. – WATSON, L. T. (1983): A Quasi-Newton versus a Homotopy Method for Nonlinear Structural Analysis, *Computers and Structures*, Vol. 4, pp. 579-585.

KAMAT, M. P. – WATSON, L. T. (1984): Determination of Multiple Equilibrium Solutions through a Deflation Technique with Tunelling, in C. Taylor, E. Hinton and D. R. J. Owen: Numerical Methods for Non-Linear Problems, Vol. 2, Pineridge Press, Swansea.

KAMAT, M. P.– WATSON, L. T. – VANDEN BRINK, D. J. (1984): Comparison of a Globally Convergent Quasi-Newton with an EBE Method for Nonlinear Structural

Problems, in W. K. Liu, T. Belytschko and K. C. Park: Innovative Methods for Nonlinear Problem, Pineridge Press, Swansea.

KUBICEK, M. (1976): Dependence of Solution of Nonlinear Systems on a Parameter, Algorithm 502, *CACM-TOMS*, No. 2, pp. 98-107.

RIKS, E. (1984): Some Computational Aspects of the Stability Analysis of Nonlinear Structure, *Comp. Meth. Appl. Mech. Eng.*, No. 47, pp. 219-259.

RIKS, E. (1984): Bifurcation and Stability, a Numerical Approach, in W. K. Liu, T. Belytschko and K. C. Park: Innovative Methods for Nonlinear Problems, Pineridge Press, Swansea.

SCHWEIZERHOF, K. H. – WRIGGERS, P. (1986): Consistent Linearization for Path Following Methods in Nonlinear FE Analysis, *Comp. Meth. Appl. Mech. Eng.*, No. 59, pp. 261-279.

WATSON, L. T. – FENNER, D. (1980): Algorithm 555, Chow-Yorke Algorithm for Fixed Points of C^2 Maps [C5], *ACM-TOMS*, No. 6, pp. 252-259.

WRIGGERS, P.– WAGNER, W. – MIEHE, C. (1988): A Quadratically Convergent Procedure for the Calculation of Stability Points in Finite Element Analysis, *Comp. Meth. Appl. Mech. Eng.*, No. 70, pp. 329-347.

WATSON, L. T.– KAMAT, M. P. – REASER, M. H. (1985): A Robust Hybrid Algorithm for Computing Multiple Equilibrium Solutions, *Eng. Comp.*, No. 2, pp. 30-34.

WRIGGERS, P. – SIMO, J. C. (1990): A General Procedure for the Direct Computation of Turning and Bifurcation Points, *Int. J. Numer. Methods Eng.*, No. 30, pp. 155-175.