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CERTAIN MODIFICATIONS OF AITKEN'S ACCELERATOR FOR SLOWLY CONVERGENT SEQUENCES

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Abstract

The paper presents three modifications of the Aitken accelerator for slowly or irregularly convergent sequences. The first proposal, based on backward extension of a sequence, is particularly useful if a small number of regular sequence terms is known. The next two procedures – integral exponential and hyperbolic – can be applied to sequences with irregular or disturbed convergence. Numerical examples show essential error reduction of the limit of sequences under consideration.

Keywords: Aitken method, sequences, convergence.

1. Introduction

First proposals of procedures accelerating convergence of sequences are described to A. C. AITKEN [1] who calculated the approximate limit a_A of a sequence S_n (converging to the exact limit a) with the help of the exponential curve:

$$y(x) = a_A + b_A \exp(-\kappa_A x). \tag{1}$$

The limit a_A can here be easily calculated from three consecutive sequence terms $y_k \equiv y(k) = S_k; k = n-1, n, n+1$:

$$a_A = \frac{y_{n-1}y_{n+1} - y_n^2}{y_{n-1} - 2y_n + y_{n+1}}. \tag{2}$$

This procedure has been generalized by D. Shanks [5] to the multiple non-linear transformation:

$$y_{n,1} = S_n; \qquad y_{n,m+1} = \frac{y_{n-1,m}y_{n+1,m} + y_{n,m}^2}{y_{n-1,m} - 2y_{n,m} + y_{n+1,m}}. \tag{3}$$

In the system (3) the sequences for greater m converge for most cases faster than the basic sequence S_n.

Recently greater interest on this problem has been focused, because the majority of numerical results is given in the form of certain sequences. Many various procedures accelerating convergence of sequences have been formulated. They are gathered and discussed in detail by C. BREZINSKI [2–4] and J. WIMP [6].

The present paper proposes certain general modifications of the algorithms mentioned above. They can be applied to several terms of a regular, slowly convergent or even divergent sequence (Sec. 2) and to a sequence with disturbances (Sec. 3).

2. Backward Extension of Sequences

The transformation (3) or any multiple procedure leads to a system which is generally called 'Shanks triangle' (*Fig. 1*) [5]. In each column of this triangle the finite sequences have smaller number of terms. According to D. Shanks, for N terms of the given sequence (N being an odd number) the term $y_{M,M}$ $\left(M = \frac{N+1}{2}\right)$ gives the best approximation of the sequence limit. Our first proposal consists in filling the right upper part of the system (the shaded area in *Fig. 1*) and regarding the term $y_{1,N}$ as better approximation of the limit than $y_{M,M}$. The calculation of $y_{1,N}$ (N being here not necessarily odd) is possible after hypothetical 'backward' extension of the given sequence in the following way.

Each sequence can be treated as a sequence of partial sums of a certain series (series of differences between the consecutive sequence terms). Hence, taking as an example the series:

$$S_n = \sum_{k=1}^{n} a_k, \tag{4}$$

we can calculate the terms with smaller indices by the subtraction:

$$S_{n-1} = S_n - a_n, \tag{5}$$

and extending this procedure to negative indices we obtain:

$$\begin{aligned} S_1 &= a_1, \\ S_0 &= a_1 - a_1 = 0, \\ S_{-1} &= 0 - a_0 = -a_0, \\ S_{-2} &= -a_0 - a_1. \\ &\dots\dots\dots\dots\dots \end{aligned} \tag{6}$$

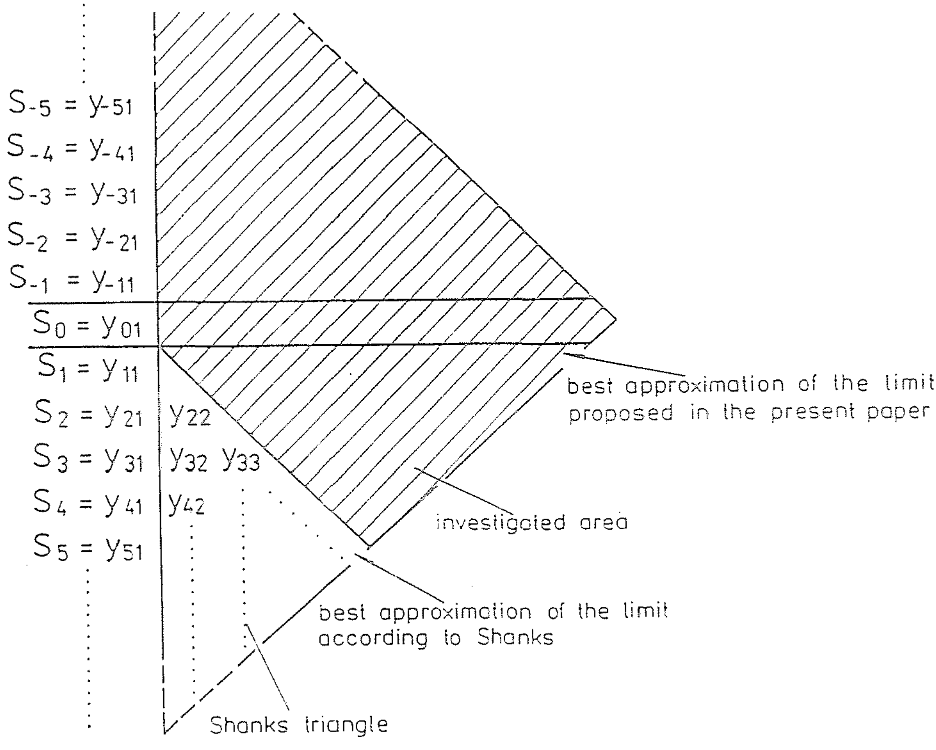

Fig. 1. Shanks triangle

Negative terms can be formed in various ways, however, observation of typical algebraic series for $n \leq 0$ suggested certain rules of their derivation. *Table 1* presents four most characteristic types of backward extensions of the series. In the first two of them we observe symmetry or antisymmetry of the sequence (with respect to the index $n = 1/2$). In the next two — the characteristic value of the zero term: $a_0 = 0$ or $a_0 = \pm\infty$ can be noticed; the remaining terms may then be formed in an arbitrary way as they do not affect the final approximation of the limit $y_{1,N}$. With the help of the Shanks formula (3) we can now build the triangles of convergence for all the types of backward extensions (*Table 2*).

Comparing character of convergence of the given sequence and the sequence of partial sums of the exemplary series (*Table 1*) we assume the particular type of the backward extension. This comparison gives as a rule not a unique answer, however, satisfactory results can be obtained by application of different schemes. Small discrepancies between $y_{1,N}$ calculated with

Table 1
Typical schemes of backward extensions

BE Schemes			Exemplary series	backward extension			classical series	
Symbol	definition (type) a_k	definition (type) S_k	$\sum$	a_{-2} / S_{-2}	a_{-1} / S_{-1}	a_0 / S_0	a_1 / S_1	a_2 / S_2
BE1	$a_0=-a_1$ $a_{-n}=-a_{n+1}$	$S_0=0$ $S_{-n}=S_n$	$\sum \frac{1}{(2k-1)^3}$	$-\frac{1}{125}$ / $\frac{28}{27}$	$-\frac{1}{27}$ / 1	-1 / 0	1 / 1	$\frac{1}{27}$ / $\frac{28}{27}$
BE2	$a_0=a_1$ $a_{-n}=a_{n+1}$	$S_0=0$ $S_{-n}=-S_n$	$\sum \frac{1}{(2k-1)^2}$	$\frac{1}{25}$ / $-\frac{10}{9}$	$\frac{1}{9}$ / -1	1 / 0	1 / 1	$\frac{1}{9}$ / $\frac{10}{9}$
BE3	$a_0=$ or $\pm\infty$ a_{-n} - arbitr.	$S_0=0$ $S_{-n}=$ or $\pm\infty$	$\sum \frac{1}{k^3}$	$-\frac{1}{8}$ / $-\infty$	-1 / $-\infty$	∞ / 0	1 / 1	$\frac{1}{8}$ / $\frac{9}{8}$
BE4	$a_0=0$ a_{-n} - arbitr.	$S_0=S_{-1}=0$ S_{-n-1} - arbitr.	$\sum \frac{k}{(2k-1)^3}$	$\frac{2}{125}$ / $-\frac{1}{27}$	$\frac{1}{27}$ / 0	0 / 0	1 / 1	$\frac{2}{27}$ / $\frac{29}{27}$

the help of various types of backward extensions make the final result more reliable. *Table 3* shows essential gains of the backward extensions proposed: BE4 gives the final error reduced 6 times, whereas BE2 — over 20 times. The structure of the triangles can have various forms depending on the scheme of the transformation (not necessarily (3) which is used in *Table 2*). For example, analogical triangles based on so called θ -transformation [2] were also successfully applied in the authors' investigations.

All the numerical results obtained by the authors confirm that the application of the full triangle schemes to a small number of terms of any regular sequence considerably reduces the final error of calculations. On the other hand, for larger n the procedure can be less effective.

3. Integral Procedures

In regular sequences the relation between their terms are also regular and the approximate limit can be obtained relatively exactly with the help of several sequence terms. However, in many engineering investigations the

Table 2
Shanks transformation applied to various BE schemes.
The results regarded as the best are shown in frames

	n \ m	1	2	3	4
T(BE 1)	−3	y_{31}			
	−2	y_{21}	y_{22}		
	−1	y_{11}	y_{12}	y_{13}	
	0	0	$y_{02} = \frac{y_{11}}{2}$	y_{03}	y_{04}
	1	y_{11}	y_{12}	**y_{13}**	
	2	y_{21}	y_{22}		
	3	y_{31}			
T(BE 2)	−4	$-y_{41}$			
	−3	$-y_{31}$	$-y_{32}$		
	−2	$-y_{21}$	$-y_{22}$	$-y_{23}$	
	−1	$-y_{11}$	$-y_{12}$	$-y_{13}$	$-y_{14}$
	0	0	∞	∞	∞
	1	y_{11}	y_{12}	$y_{13} = y_{22}$	$y_{14} = y_{23}$
	2	y_{21}	y_{22}	**y_{23}**	
	3	y_{31}	y_{32}		
	4	y_{41}			
T(BE 3)	−2	∞	∞	∞	∞
	−1	∞	∞	∞	∞
	0	0	$y_{02} = y_{11}$	$y_{03} = y_{12}$	$y_{04} = y_{13}$
	1	y_{11}	y_{12}	**y_{13}**	
	2	y_{21}	y_{22}		
	3	y_{31}			
T(BE 4)	−4	y_{-41}			
	−3	y_{-31}	y_{-32}		
	−2	y_{-21}	y_{-22}	y_{-23}	
	−1	0	0	0	0
	0	0	0	0	0
	1	y_{11}	y_{12}	**y_{13}**	
	2	y_{21}	y_{22}		
	3	y_{31}			

Table 3

T(BE4) (upper results) and T(BE2) (lower results) applied to the series

$S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$ with the limit $L = \ln 2 = 0.69314718$

n \ m	1	2	3	4	5	6
0	0.00000000	0 ∞	0 ∞	0 ∞	0 ∞	0 ∞
1	1.00000000 +442.7 ‰	0.66666667 −38.20 ‰	0.70175439 +12.48 ‰ 0.70000000 +9.887 ‰	0.69271066 −0.6298 ‰ 0.69259259 −0.8001 ‰	0.69323009 +0.1196 ‰ 0.69321937 +0.1041 ‰	0.69315332 +0.0089 ‰ 0.69314902 +0.0027 ‰
2	0.50000000 −278.7 ‰	0.70000000 +9.887 ‰	0.69259259 −0.8001 ‰	0.69322970 +0.1191 ‰ 0.69321937 +0.1041 ‰	0.69315331 +0.0088 ‰ 0.69314902 +0.0027 ‰	
3	0.83333333 +202.2 ‰	0.69047619 −3.865 ‰	0.69327731 +0.1877 ‰	0.69314013 −0.0102 ‰		
4	0.58333333 −158.4 ‰	0.69444444 +1.872 ‰	0.69310576 −0.0598 ‰			
5	0.78333333 +130.1 ‰	0.69242424 −1.043 ‰				
6	0.61666667 −110.3 ‰					

regularity of the results is considerably disturbed. It is visible not only for experimental data but also in more complicated numerical algorithms where regularity vanishes because of simplifications in the calculational process.

In such cases the application in *Eq.* (2) of the sequence terms relatively distant one from another is one of possible ways of estimation of the sequence limit [8, 9] (the formula (2) is valid also for any equidistant sequence terms). However, the limit obtained in this way considerably depends on the choice of these terms, especially in the cases of significant single disturbances. Such random choice of the control points (nodes) involves also unavoidable loss of information contained in the remaining terms of

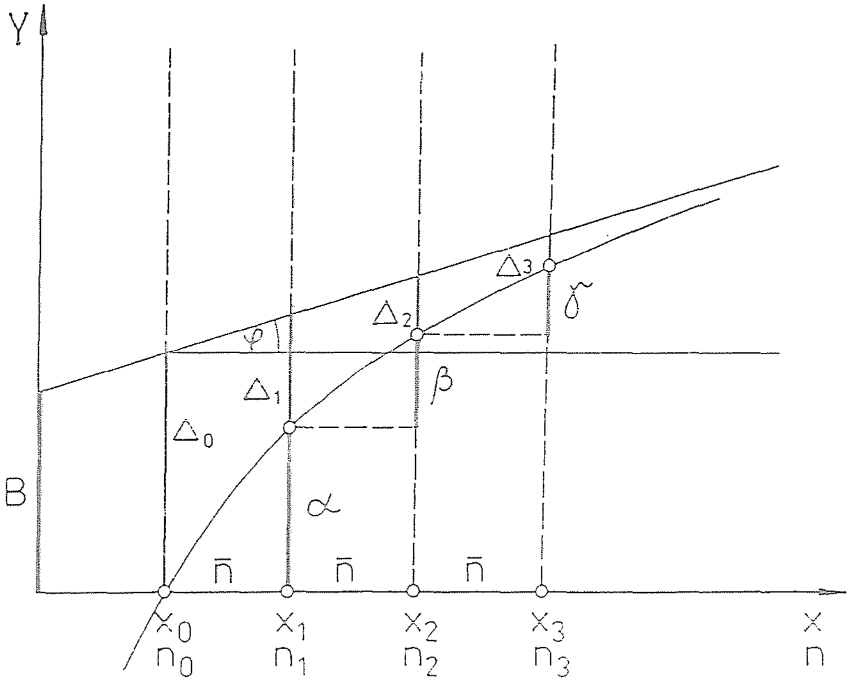

Fig. 2. Increments of the exponential function $Y_A = A_A x + B_A + C_A \exp(-\kappa_A x)$

the sequence. The present section proposes certain integral procedures using groups of the sequence terms instead of the single terms. The process of integration (summation) applied here smooths the disturbances mentioned above.

Any sequence S_n can be treated as a discrete function $y(n_i)$ with an integer argument. Extension of the argument domain to real numbers $(y(n_i) \to y(x))$ makes possible integration of this function and approximation of the integral by

$$Y_A = A_A x + B_A + C_A \exp(-\kappa_A x), \tag{7}$$

resulting from integration of the function (1).

The constants A_A, B_A, C_A and κ_A are to be found by leading the curve (7) through four equidistant points $Y(n_i); i = 0, 1, 2, 3$. The sought estimation A_A of the sequence limit a is here the tangent of the angle ϕ of the function asymptote (*Fig. 2*).

The proportions

$$\frac{\Delta_0}{\Delta_1}=\frac{\Delta_1}{\Delta_2}=\frac{\Delta_2}{\Delta_3} \quad \text{where} \quad \Delta_i=A_A n_i+B_A+Y_A(n_i), \quad i=0,1,2,3, \tag{8}$$

resulting from the properties of the curve (7), valid for equidistant n_i lead to the quadratic equation:

$$(\alpha-2\beta+\gamma)\bar{n}^2A_A^2+(3\beta^2-\alpha\gamma-\alpha\beta-\beta\gamma)\bar{n}A_A-\beta^3+\alpha\beta\gamma=0, \tag{9}$$

where $\bar{n}=n_1-n_0=n_2-n_1=n_3-n_2$.

The constants α, β, γ are the increments of the integral of the convergence function:

$$\alpha=\int_{n_0}^{n_1} y(x)dx, \qquad \beta=\int_{n_1}^{n_2} y(x)dx, \qquad \gamma=\int_{n_2}^{n_3} y(x)dx. \tag{10}$$

In particular, they can be directly the sums of the respective groups of the sequence terms.

The equation (9) has two real roots:

$$A_{A1}=\frac{\alpha\gamma-\beta^2}{\bar{n}(\alpha-2\beta+\gamma)}, \qquad A_{A2}=\frac{\beta}{\bar{n}}. \tag{11}$$

In a general case only the first root A_{A1} represents the approximation of the sequence limit. However, if simultaneously $\alpha\to\beta$ and $\gamma\to\beta$, also $A_{A1}\to A_{A2}$ and the second formula can be practically used. If $\bar{n}=1$ the root A_{A1} reduces to the form (2), therefore this expression can be considered as a generalization of Aitken's procedure. The case $\alpha-2\beta+\gamma=0$ but $\alpha\gamma-\beta^2\neq 0$ means parabolic variation of the integral, i.e. the lack of the asymptote. This relation suggests integral linearity of the sequence which means its divergence (more accurately — the lack of its limit and antilimit [5]).

The integral procedure presented can also be applied to oscillatory (in the integral sense) converging sequences. This is visible in the proportions (8) in which Δ_i and Δ_{i+1} can have different sign.

Analogically to the exponential procedures presented above, we can introduce procedures based on a hyperbolic curve:

$$y_H=a_H+b_H x^{-\kappa_H}. \tag{12}$$

In this case the expression:

$$a_H = \frac{y_p y_r - y_q^2}{y_p - 2y_q + y_r}, \tag{13}$$

analogical to the form (2) is valid for $q^2 = pr$, where p, q, r are numbers of the sequence terms. This relation was often used by the authors for quick rough estimation of the limit in the case of hyperbolic type sequences and series.

The integral hyperbolic procedure leads also to a quadratic equation with two roots:

$$A_{H1} = \frac{\alpha\gamma - \beta^2}{\bar{n}(4\alpha - 4\beta + \gamma)}, \qquad A_{H2} = \frac{\beta}{2\bar{n}}, \tag{14}$$

from which the first one should be taken in a general case. Similarly to the roots (11) the second one is valid only for $4\alpha = 2\beta = \gamma$ (then both roots coincide) and the case $4\alpha - 4\beta + \gamma = 0$ but $\alpha\gamma - \beta^2 \neq 0$ means the lack of the limit estimate.

4. Comparison of Discrete and Integral Procedures

The sequence of 24 consecutive partial sums of the series:

$$S_n = \sum_{k=1}^{n} 0.9^{k-1} \to G = 10.0, \qquad n = 1, 2, 3, \ldots, 24, \tag{15}$$

disturbed in a random way inside the limits

a) determined in percentages ($\pm 0.05 S_n$) b) constant (± 0.5)

was chosen to compare effectiveness of the discrete and integral procedures. 300 random samples were formed and estimators of a mathematical expectation a_{300} and a standard deviation σ_{300} were calculated. *Table 4* shows results of the example for:

1. discrete exponential procedure with maximal possible distances between the control points (nodes) ($n = 2, 13, 24$) – E_d
2. integral exponential procedure ($n = 8, 16, 24$) – E_i

It contains also 10 extreme values obtained in each case mentioned above and the number of the results inside the limits ($9.5 - 10.5$) accepted as admissible. The results show evident superiority of the integral exponential procedure proposed. The hyperbolic procedures gave here worse results because of the exponential type of the series (15). On the other hand, the hyperbolic procedures gave better results for trigonometric series in which

Table 4
Investigation of the limit of the series $S_n = \sum_{k=1}^{n} 0.9^{k-1}$ with local disturbances

Limits of disturbances	$\pm 0.05\ S_n$		± 0.5	
Procedures applied	E_d	E_i	E_d	E_i
Five minimal sample values	8.9359	9.1818	8.8763	9.1126
	8.9551	9.1940	8.9502	9.1725
	8.9863	9.3246	8.9520	9.2019
	9.0010	9.3873	8.9774	9.2491
	9.0090	9.4093	8.9886	9.2841
Number of results within acceptable limits (9.5÷10.5)	166	245	130	228
Five maximal sample values	11.6246	10.8419	12.3323	11.2858
	11.6648	10.8780	12.4759	11.3092
	11.8651	10.9418	12.4856	11.3264
	12.0159	10.9754	12.6759	11.7988
	12.1871	11.0812	12.6827	12.5644
Math. expectation a_{300}	10.0440	10.0369	10.1855	10.0693
Stand. deviation σ_{300}	0.6472	0.3530	0.8198	0.4725

the coefficients have a hyperbolic form. This was also observed during investigations of the boundary series method [7] in which the solution was obtained in the form of certain combinations of trigonometric series.

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