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SOME PROBLEMS OF STATIC ANALYSIS OF FOLDED PLATE STRUCTURES

Lajos KOLLÁR

Department of Building Construction
Faculty of Civil Engineering
Technical University of Budapest
H–1521 Budapest, Hungary

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Abstract

Folded plate structures are extensively dealt with in the literature, but there are some neglected problems the solution of which the design engineer needs. In this paper are treated: the static behaviour of folded plate structures under partial vertical and under horizontal loads; static analysis of the extreme elements, of triangular plates loaded perpendicularly to their planes, and of plates subjected to in-plane concentrated forces; stability analysis of the plate elements.

Keywords: folded plate structures, plates subjected to perpendicular loads, plates subjected to in-plane loads, stability of plates.

1. Introduction

The classic analysis of folded plates can be found in the literature, see e.g. (BORN, 1954, 1965). There are, however, some special problems which are not treated there, although the design engineer would need their solutions. In this paper we want to deal with some of these problems.

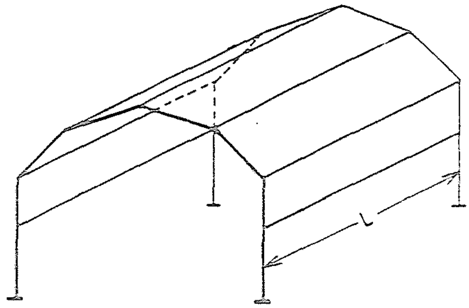
First of all, we have to clarify some notions. We can distinguish basically two kinds of folded plate structures: those consisting of long and of short plate elements. An element is called long if its length is several times larger than its width (*Fig. 1.1*); it is called short if these two dimensions are close to each other (*Fig. 1.2*). The folded plate structures with long elements can be subdivided into barrel-vault-like (*Fig. 1.1a*) and periodic ones (*Fig. 1.1b*).

We shall treat the following problems:

In the structures with long elements:

- to what extent a partially loaded fold is supported by the unloaded neighbouring folds?
- how the folds can be analysed when subjected to horizontal loads?
- what are the internal forces in the extreme plate element due to concentrated supports?
- how the plate elements can be analysed for buckling?

1.1a



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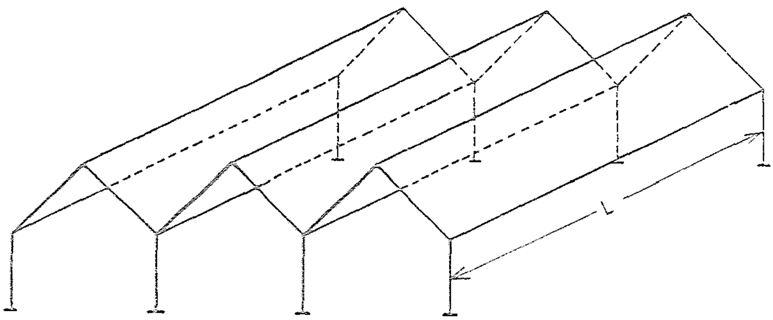


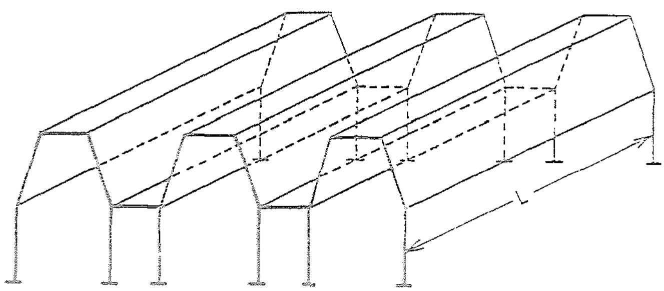


Fig. 1.1.

In the structures with short elements:

- how large are the bending and twisting moments in the triangular plates under a load perpendicular to their planes?
- what stresses arise in the triangular or quadrangular element due to concentrated forces acting in the plane of the plate?
- how the triangular plates can be analysed for buckling?

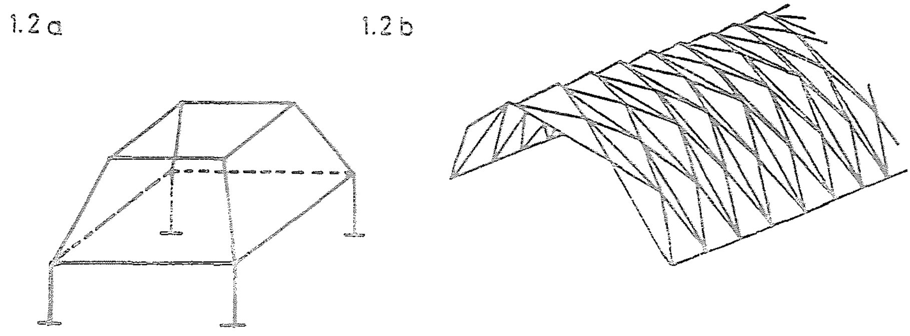


Fig. 1.2.

2. Problems of Analysis of Folded Plate Structures Consisting of Long Elements

Before dealing with the problems of analysis, we have to survey the static behaviour of folded plate structures with long elements (*Fig. 1.1*).

2.1. Static Behaviour of Folded Plate Structures with Long Elements

The plate element carries the component perpendicular to its plane of the (in most cases distributed) load by bending, and transmits it to the edges forming the border of the element. Here the supporting forces have to be resolved into components acting in the planes of the adjacent two plates, which carry them by bending (causing tension and compression) and shear, both in their own planes, to their supports. The components of the load which lie in the planes of the plate elements also cause internal forces acting in the planes of the elements.

In this 'longitudinal load-bearing' the individual plates do not act alone, they rather form, together with the adjacent elements, great thin-walled beams which are the load-bearing elements proper in the longitudinal direction. This acting together is ensured by the so-called *edge forces* which are actually shearing forces transmitted from one plate to the other along the edges (*Fig. 2.1*), and whose distribution can be considered, on a structure simply supported at both ends, with close approximation as a cosine function. Since these edge forces cause elongation also along the opposite edge of the element, we can ensure the identical elongation of the

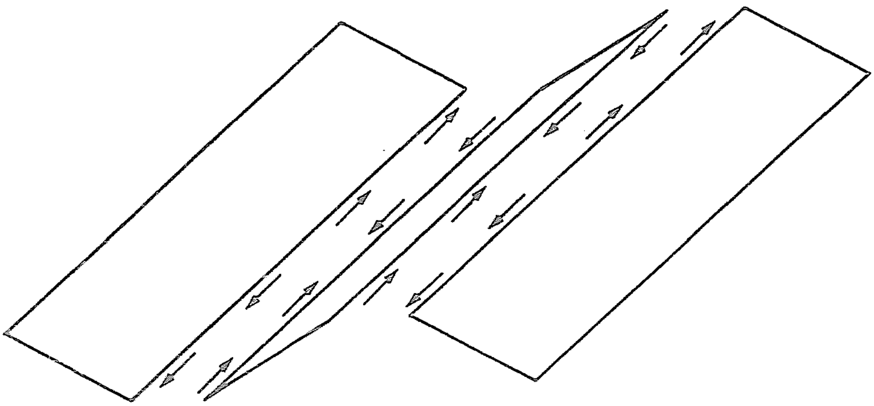


Fig. 2.1

common edge of two adjacent elements only if we consider, besides the edge force acting along the edge investigated, also the edge forces acting along the two neighbouring edges. We thus obtain the so-called three-edge-forces equations, analogous to the three-moment equations of continuous beams.

The structures with long elements can be further characterized by the fact that the edges of the elements may considerably deflect due to bending in the plane of the element (above all if the length of the elements is manifold of their width), and these deflections interact with the primary bending of the plates (their load-bearing in the transverse direction), because they appear as deflections of the supports of the continuous plate. Hence a separate investigation is needed to decide when it is permissible to consider the edges as immovable with respect to the transverse bending of the plate.

If the deflections of the edges (i.e. of the supports of the plate) cannot be neglected with respect to the transverse bending of the plate, then due to these deflections the support forces of the plate change, which also modify the loading of the longitudinal beams. Hence, the deflections of the longitudinal beams and the transverse bending moments of the plate interact.

Summing up, the internal forces of the folded plate structure are described by a system of equations in which the edge forces and the transverse bending moments of the plate (at the supporting edges) appear as unknowns (BORN, 1954).

The static behaviour of the folded plate structure becomes so complicated, as a rule, only in the case of barrel-vault-like structures, because the long plate elements cannot form a (more or less) unique cross-section without edge forces. Furthermore, the planes of the adjacent plates mostly subtend a small angle, so that they cannot hold the edge immovable. Hence, for simplifying the analysis, criteria have been established many years ago, whose fulfilment ensures that the edges can be considered as immovable, i.e. the plates can be analysed as continuous beams on fixed supports, and in the three-edge-forces equations only the edge forces appear as unknowns. Thus, no interaction takes places between plate bending moments and edge forces.

The simplest condition has been established by Gruber (BORN, 1954), according to which the edges can be considered as immovable if the angle subtended by two adjacent plates (*Fig. 2.2a*) is at least 40° .

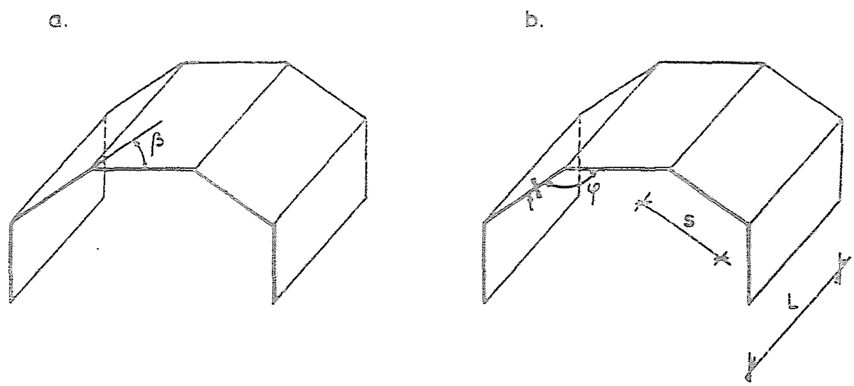


Fig. 2.2

More detailed conditions originate with Grüning (BORN, 1954), which state that (*Fig. 2.2b*) if

$$\frac{t^2}{s^2} \leq 1.6 \sin\varphi \tan\varphi \frac{s^4}{L^4},$$

then the edges can be considered as immovable; and if

$$\frac{t^2}{s^2} \geq 120 \sin\varphi \tan\varphi \frac{s^4}{L^4},$$

then the structure can be treated as an ordinary beam with a unique cross-section which does not deform, i.e. in which the distribution of the bending stresses is linear.

Finally, if t/s lies between the two above limits, the deflection of the edges has to be taken into account when computing the transverse bending moments of the plate.

In the latter case we may solve the problem by iteration: in the first step we determine the bending moments and support forces of the plate, assuming immovable edges; from these we compute the in-plane forces and edge forces of the plates and their edge deflections; then the influence of these deflections on the transverse bending moments of the plate and the pertinent support forces, etc. Since the transverse bending moments of the plate do not vary in the longitudinal direction (except for the vicinity of the end diaphragms), but the edge deflections do, for simplicity it is advisable to assume 2/3 of the maximum edge deflection as a constant value along the whole edge.

This iteration has not only the advantage to avoid the direct solution of the large equation system, but it is also visual: we can follow the magnitudes of the different effects, and we can also see when the iteration procedure can be stopped. So in certain cases we may allow that the edge deflections cause nonnegligible bending moments in the plate, but we require that the support forces due to these moments only slightly alter the edge forces.

We may also mention the so-called *membrane theory* of folded plates. This stipulates hinges along the edges, so that the plates carry the load as simply supported ones. Consequently, the edge forces do not interfere with the transverse bending of the plate, furthermore the load components acting in the planes of the plates can be determined from equilibrium equations only. Hence we only have to solve the three-edge-forces equations. The membrane theory is thus a simple, consistent system of assumptions. It has, however, a serious disadvantage: it does not reliably describe the static behaviour of any folded plate structure, since the r.c. plates are in every case continuous in the transverse direction. Hence this theory is suitable for theoretical investigations only; for this purpose it is, however, very useful, since it is the edge forces which ensure the connection between the plate elements in longitudinal bending. In the frame of the membrane theory these edge forces can be determined by an iteration procedure also (BECKER, 1966, 1969), analogous to the well-known moment distribution method of frames. The 'effective width', replacing the unloaded folds, will also be determined in Sect. 2.2 by using the membrane theory.

The static behaviour of the periodic folded plate structures (*Fig. 1.1b*) is much simpler than that of the barrel-vault-like ones, since in the main load-bearing (under a load uniformly distributed in the transverse direction) no such edge forces arise which should be determined in the above described way, and their edges deflect identically. (In the structure with a triangular cross-section no edge forces develop at all, and in that with a trapezoidal cross-section the arising edge forces can be determined by the elementary strength of materials). This very simple state of stresses will be disturbed only by the fact that the structure is finite in the transverse direction: here the homogeneity of the structure ceases to exist, and the edge forces caused by this fact should be computed as described above. Practically, however, we can content ourselves with guessing the internal forces (KOLLÁR, 1993). These 'edge disturbances' rapidly die out with increasing distance from the edges, see in Sect. 2.2.

KOVÁCS (1978) investigated in detail the propagation of edge disturbances on the folded plate structure with triangular cross-section, and presented the results in diagrams.

In the foregoing we assumed that the folded plate structures undergo only bending in the longitudinal direction. There are, however, structures which undergo bending and compression (*Fig. 2.3*). In these cases the aforementioned principles have to be completed by the dimensioning for compression.

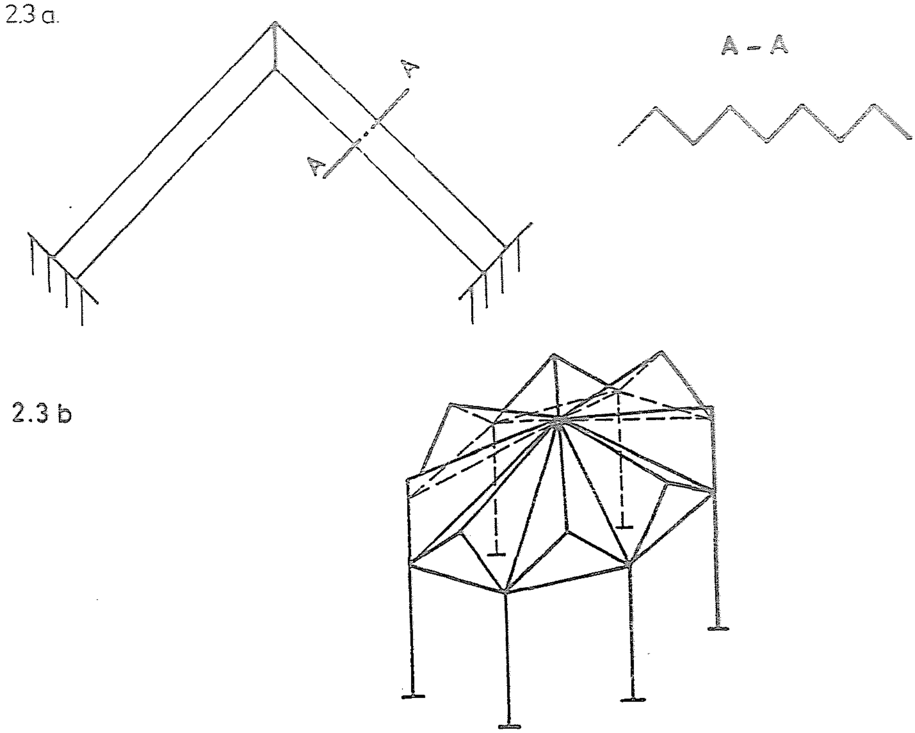
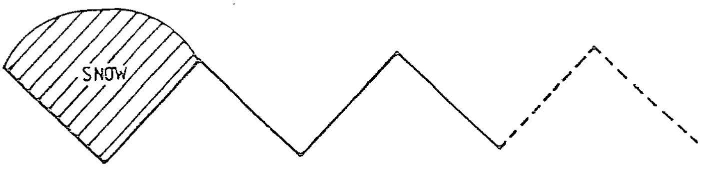


Fig. 2.3.

2.2. The 'Effective Width' Represented by the Unloaded Folds

If only one fold will be filled by snow or industrial dust (*Fig. 2.4a*), then the neighbouring, unloaded folds help the loaded one in carrying the load, mainly by the edge forces arising between them. Essentially the same phenomenon takes place if one (e.g. the extreme) fold is supported (*Fig. 2.4b*): we then investigate how this disturbance propagates.

2.4 a



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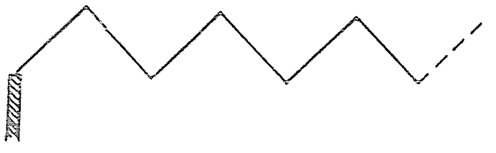


Fig. 2.4.

In order to clarify this phenomenon we will study the magnitude of the edge forces on the unloaded part of the structure.

As mentioned earlier, the three-edge-forces equations are analogous to the three-moment equations of the continuous beams. Hence let us consider such a beam with equal spans on fixed supports, infinitely long in one direction, and let us investigate the magnitudes of the bending moments at the supports due to an external moment M_1 applied at the left support; i.e. let us clarify how the influence of M_1 dies out.

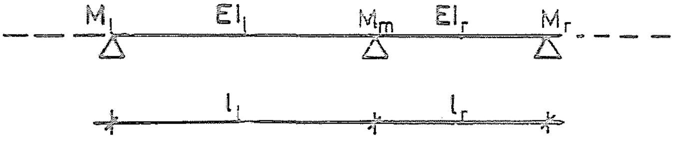


Fig. 2.5.

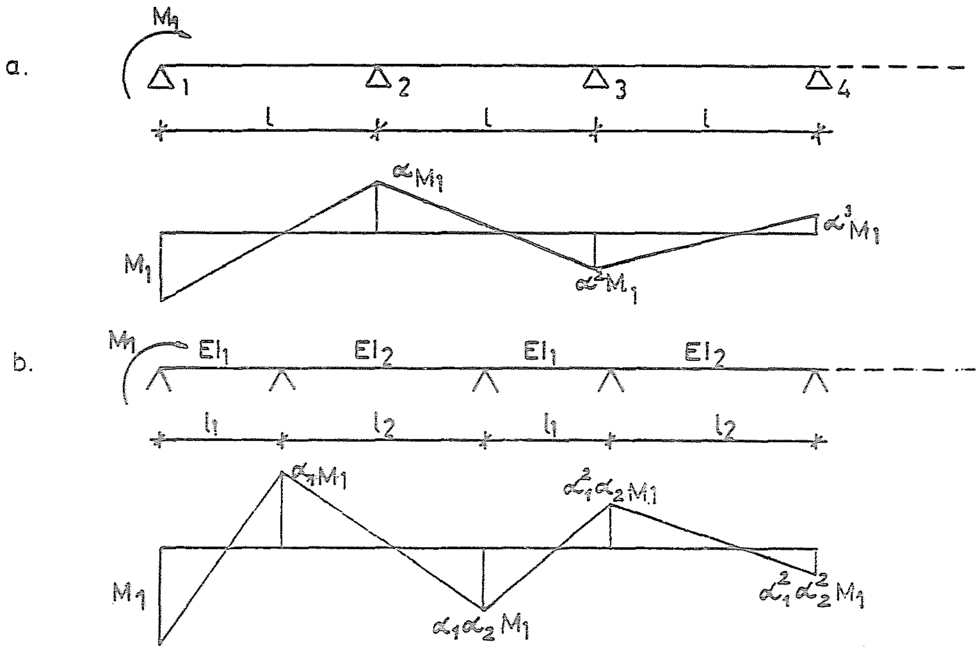


Fig. 2.6.

The three-moment equation of an unloaded beam runs (*Fig. 2.5*):

$$\frac{l_l}{EI_l}M_l + 2\Big(\frac{l_l}{EI_l} + \frac{l_r}{EI_r}\Big)M_m + \frac{l_r}{EI_r}M_r = 0. \tag{1}$$

Here the subscripts l, m and r denote left, middle and right, respectively.

First let the bending stiffness EI of the beam be constant, its spans be of equal length l, see *Fig. 2.6a.* In this case the moment M_1 decreases identically in each span to the α-fold of its value ($|\alpha| < 1$). Thus the three-moment equation becomes:

$$\frac{l}{EI}\Big(M_1 + 4\alpha M_1 + \alpha^2 M_1\Big) = 0, \tag{2a}$$

i.e.

$$1 + 4\alpha + \alpha^2 = 0, \tag{2b}$$

which yields, taking into account the requirement that $|\alpha| < 1$:

$$\alpha = -0.27. \tag{3}$$

Let us now assume that only every second span together with the bending rigidity is identical (*Fig. 2.6b*). The decrease along the spans l_1 is then α_1, along the spans l_2 it is α_2. The three-moment equations can be written for the supports 1–2–3 and 2–3–4:

$$\frac{l_1}{EI_1}M_1 + 2\Big(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\Big)M_2 + \frac{l_2}{EI_2}M_3 = 0, \tag{4a}$$

$$\frac{l_2}{EI_2}M_2 + 2\Big(\frac{l_2}{EI_2} + \frac{l_1}{EI_1}\Big)M_3 + \frac{l_1}{EI_1}M_4 = 0. \tag{4b}$$

Introducing the notation

$$\eta = \frac{l_2}{EI_2} : \frac{l_1}{EI_1} \tag{5}$$

and making use of the fact that

$$M_2 = \alpha_1 M_1, \tag{6a}$$

$$M_3 = \alpha_2 M_2, \tag{6b}$$

$$M_4 = \alpha_1 M_3, \tag{6c}$$

the two *Eqs* (4a,b) become:

$$1 + 2(1+\eta)\alpha_1 + \eta\alpha_1\alpha_2 = 0, \tag{7a}$$

$$\eta\alpha_1 + 2(\eta+1)\alpha_1\alpha_2 + \alpha_1^2\alpha_2 = 0. \tag{7b}$$

Their solution is, considering the requirement $|\alpha_i| < 1$:

$$\alpha_1 = \frac{-(5+3\eta) + \sqrt{(5+3\eta)^2 - 16}}{4}, \tag{8a}$$

$$\alpha_2 = -\Big(\frac{1}{\eta\alpha_1} + 2\frac{1+\eta}{\eta}\Big). \tag{8b}$$

If e.g. $\eta = 2.0$, then $\alpha_1 = -0.19$, $\alpha_2 = -0.34$, their average value (or more exactly, the value of $\sqrt{\alpha_1\alpha_2}$) is thus very close to -0.27, see *Eq.* (3).

We can draw the conclusion that on continuous beams according to *Figs. 2.5* and *2.6* the moment applied at the first support decreases at the third support to less than 10% of its original value.

Let us apply these results to the edge forces of the folded plate, using the membrane theory (i.e. assuming hinges along the edges).

The three-edge-forces equations of the unloaded folded plate have the following form (cf. *Fig. 2.7*):

$$\frac{1}{s_l t_l} T_l + 2\left(\frac{1}{s_l t_l} + \frac{1}{s_r t_r}\right) T_m + \frac{1}{s_r t_r} T_r = 0. \qquad (9)$$

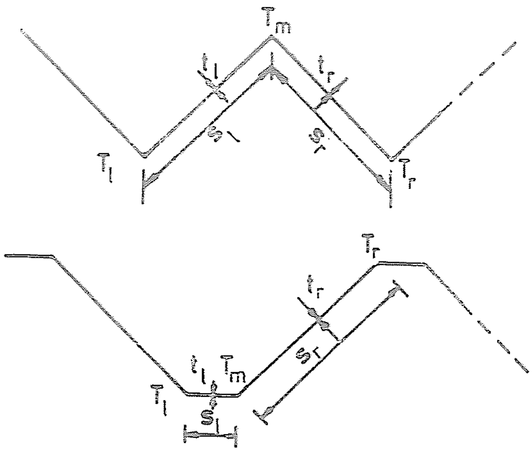


Fig. 2.7.

Hence the quantity EI/l characterizing the bending stiffness of the beam is substituted for by the area of the plate element, cf. also *Eq.* (1). Thus, on the folded plate with triangular cross-section according to *Fig. 2.7a*, the edge force decreases along one element to its −0.27-fold, while on the folded plate with trapezoidal cross-section (*Fig. 2.7b*) it decreases according to the expressions (8). It should be remarked that if we choose, on the latter type of structure, the thickness of the plates in such a way that the product ts be equal on both plates, then the role of the plates in decreasing the edge forces becomes identical, and these forces decrease to their −0.27-fold along each plate.

From all these follows that the influence of a partially loaded or a supported fold extends practically to two adjacent plate elements only.

Knowing the law of decrease of the edge forces we can determine an 'effective width' for the folded plate, which statically replaces the unloaded folds with respect to their supporting effect on the loaded one. By definition, an edge force acting on the effective width causes a constant stress σ_1 in its whole cross-section. To ensure static equivalency it is thus necessary that this stress σ_1 be equal to that caused by the edge force at its own point of application on the folded plate structure (*Fig. 2.8a,b*). Actually, the effective width should be depicted as in *Fig. 2.8c* to show that a constant stress arises along its whole width.

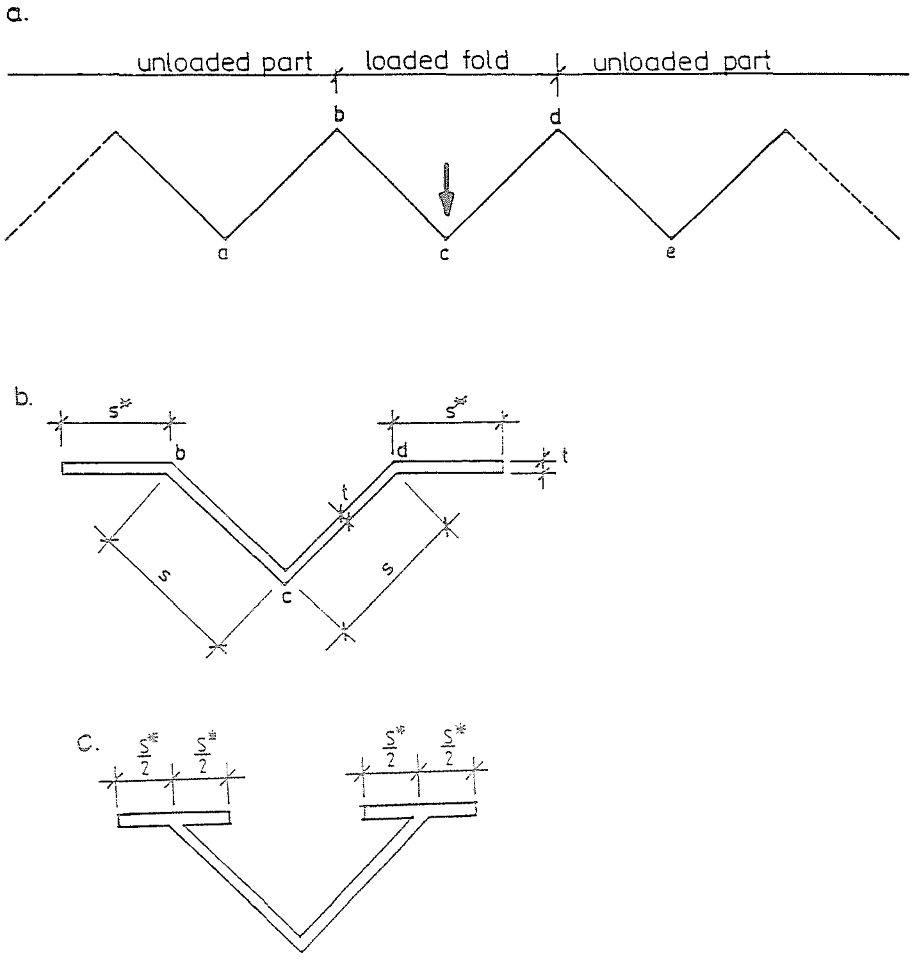


Fig. 2.8.

Let us examine a fold consisting of two plates (*Fig. 2.9a*), as a cut-out part of a large structure, acted upon along one of its edges by an edge force

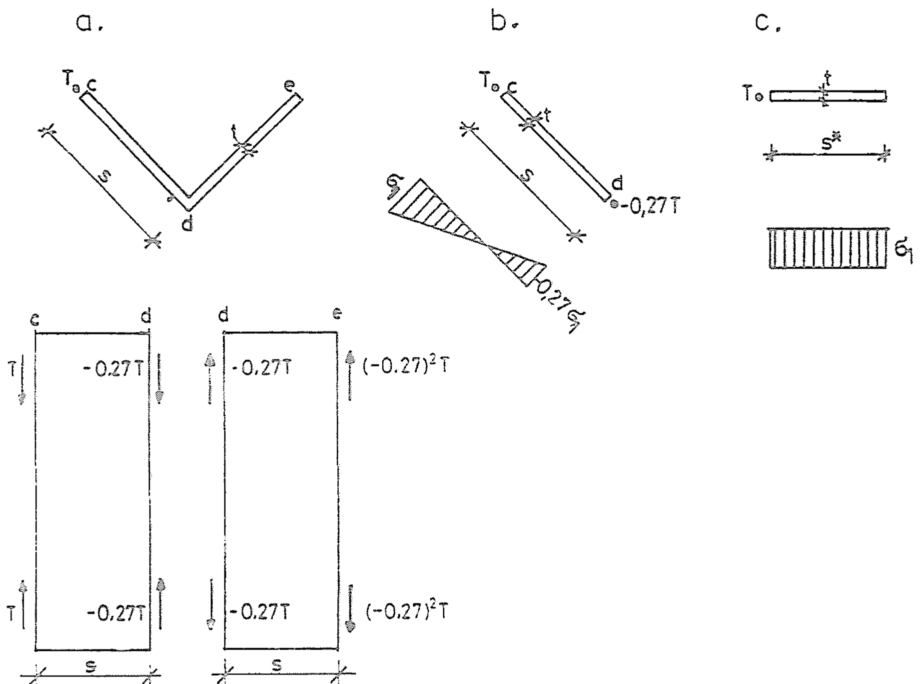


Fig. 2.9.

T, and let us determine the stress arising at the loaded edge. Since along the opposite edge of the loaded element the -0.27-fold of the edge force acts, the plate element is loaded by the two edge forces shown in *Fig. 2.9b*. These cause at the loaded left edge a stress

$$\sigma_1 = \frac{T + 0.27T}{ts} + \frac{(T - 0.27T)\frac{s}{2}}{\left(\frac{ts^2}{6}\right)} = 3.46\frac{T}{ts}, \tag{10}$$

while at the opposite edge the -0.27-fold of this value arises.

If we want that the uniform stress $T/(ts^*)$ inside the effective width s^* be equal to σ_1, the equation

$$\frac{T}{ts^*} = 3.46\frac{T}{ts} \tag{11}$$

must hold, which yields for the effective width the value

$$s^* = 0.29s, \tag{12}$$

see *Fig. 2.9c*. Substituting this effective width for the unloaded folds according to *Fig. 2.8b*, we obtain a visual picture of their supporting effect.

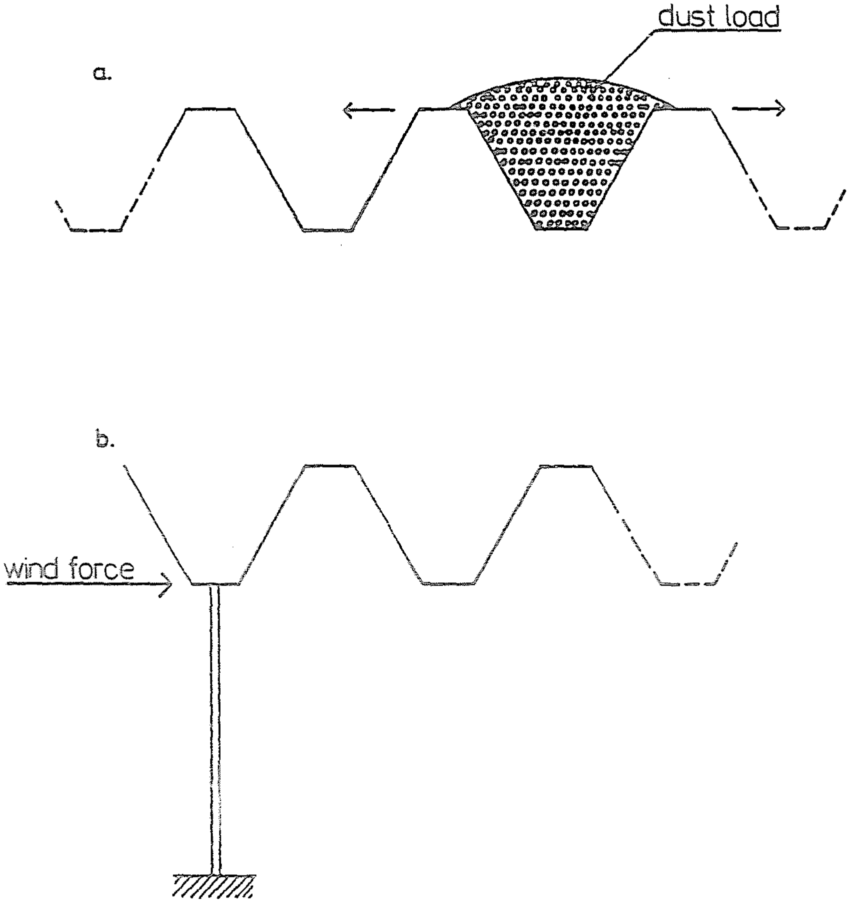


Fig. 2. 10

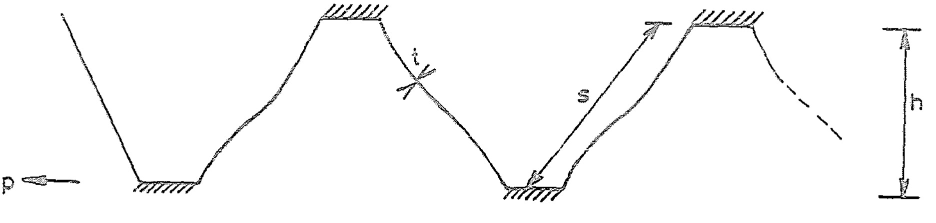


Fig. 2. 11

2.3. Analysis of the Folds for Horizontal Loads

In folded plate structures containing steep plate elements the partial load may bend the structure in the horizontal plane (*Fig. 2.10a*). The wind causes a similar effect if the folds horizontally support the side wall (*Fig. 2.10b*). The structure then acts essentially in the same way as under vertical load: the plate elements carry the load as large beams on the span L. If the horizontal deflection of the edges cannot be neglected, the plate elements will act as transverse frames and transmit the horizontal load to the other folds. In (KOLLÁR, 1974) we find an approximate method to determine the internal forces due to this type of load, the assumptions of which are the following:

On the one hand, the inclined elements of the folded plate structure can be considered steep enough to neglect the vertical deflections of their edges; on the other hand, the horizontal elements are narrow enough to prevent the rotation of the edges of the inclined elements. Hence the structure deforms as shown in *Fig. 2.11*: the horizontal plates with a section (effective width) of the inclined ones constitute U-shaped beams which are connected by springs formed by the bending stiffness of the inclined plates. So the structure can be modelled as depicted in *Fig. 2.12*: the horizontally bent beams are connected by distributed springs.

The method of analysis can be shortly summarised as follows:

The horizontal load p, uniformly distributed along the span L, is expanded into a Fourier series, of which we consider – for the time being – only the first term ($n = 1$):

$$p_1 \sin\left(\frac{\pi}{L}x\right) \tag{13a}$$

where the load amplitude p_1 is given by the expression

$$p_1 = \frac{4}{\pi}p. \tag{13b}$$

The 'spring constant' c of the inclined plates, referred to unit length, is equal to the stiffness against displacement of an inclined beam built-in at both ends:

$$c = \frac{12EI_{plate}}{h^2 s}, \tag{14a}$$

where

$$EI_{plate} = \frac{Et^3}{12(1-\nu^2)}, \tag{14b}$$

see *Fig. 2.11*, and ν is Poisson's ratio.

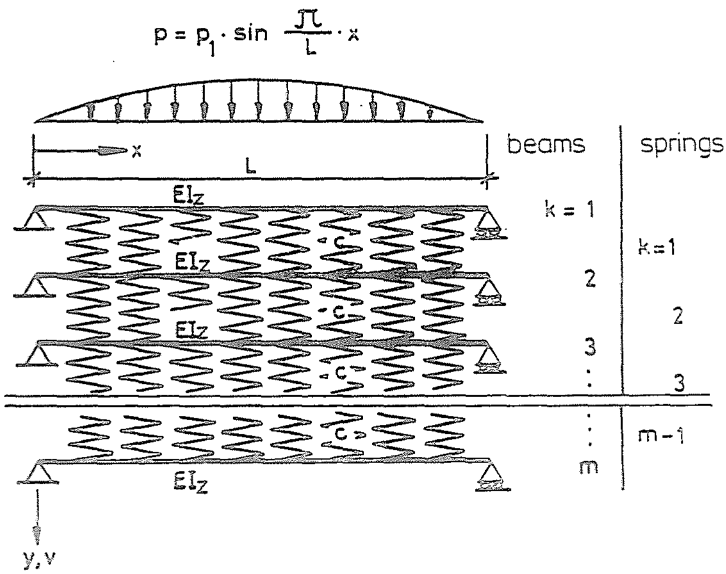


Fig. 2. 12

The effective width of the inclined elements can be assumed, taking into account what has been said in Sect. 2.2, approximately as 0.25 times the plate width s. The bending stiffness in the horizontal plane of the 'beams' with U shaped cross-section will be denoted by EI_z.

The spring force r_k between the beams k and $k+1$ is given by the difference of deflections v of the two beams. Omitting the multiplicator $\sin(\pi x/L)$ we obtain

$$r_k = c(v_k - v_{k+1}). \tag{15}$$

On the other hand, the deflection of a beam is caused by the difference of the spring forces. Introducing the parameter

$$\delta_1 = \frac{\pi^4 EI_z}{L^4}, \tag{16}$$

characterising the stiffness of a beam, the deflections of the beams k and $k+1$ become:

$$v_k = \frac{r_{k-1} - r_k}{\delta_1} \tag{17a}$$

and

$$v_{k+1} = \frac{r_k - r_{k+1}}{\delta_1}. \tag{17b}$$

Introducing (17a,b) into (15), we arrive at the following homogeneous linear difference equation for the spring force r:

$$r_{k-1} - (2 + \frac{\delta_1}{c})r_k + r_{k+1} = 0. \tag{18}$$

Assuming the solution in the form

$$r_k = c\rho^k,$$

and introducing it into (18), we obtain the characteristic equation:

$$\frac{1}{\rho} - (2 + \frac{\delta_1}{c}) + \rho = 0, \tag{19a}$$

whose roots are:

$$\rho_{1,2} = (1 + \frac{\delta_1}{2c}) \pm \sqrt{\left(1 + \frac{\delta_1}{2c}\right)^2 - 1}; \qquad (\rho_1 < \rho_2). \tag{19b}$$

Thus, the solution has the form:

$$r_k = C_1\rho_1^k + C_2\rho_2^k \tag{20}$$

This has to satisfy the boundary conditions:

$$k = 0: \qquad r_0 = p_1,$$
$$k = m: \qquad r_m = 0.$$

From these we determine the integration constants C_1 and C_2, and the solution (20) takes the final form:

$$r_k = \frac{p_1}{1 - \left(\frac{\rho_1}{\rho_2}\right)^m}\left[\rho_1^k - \left(\frac{\rho_1}{\rho_2}\right)^m \cdot \rho_2^k\right], \tag{21}$$

yielding the maximum value of the spring force k, varying according to a sine curve along the span L.

The beams are loaded by the differences of the spring forces acting at their both sides. It is the first beam which carries the greatest share of the load

$$(p_1 - r_1)\sin\left(\frac{\pi}{L}x\right), \tag{22}$$

so that the total load p_1 acting on the first beam decreases to $(p_1 - r_1)$.

All what has been said so far refers to a load varying according to a half sine wave, see *Eq.* (13a). Since the Fourier series of the uniform load is

$$p = \frac{4}{\pi} p \sum_{n=1,3,5\dots} \frac{\sin\left(n\pi x/L\right)}{n}, \tag{23}$$

the load corresponding to the n-th Fourier term has the form

$$p_n = \frac{4}{\pi}\frac{p}{n} \tag{24}$$

and thus

$$p = \sum_{n=1,3,5\dots} p_n \sin(n\pi x/L). \tag{25}$$

Hence, in the higher terms ($n = 3, 5\dots$) we have to replace L by L/n, and we have to compute the pertaining δ_n according to (16). The bending moment of the beam is obtained by differentiating (25) twice:

$$M = \frac{L^2}{\pi^2} \sum_{n=1,3,5\dots} \frac{p_n}{n^2} \sin(n\pi x/L). \tag{26}$$

Due to the action of the springs, for the first beam we have to substitute $(p_n - r_1)$ for p_n, according to (22), and for the other beams we have to write $(r_{k-1} - r_k)$ instead of p, computing every r with the n in question. The expression (26) of the bending moment shows that with increasing n the role of the corresponding terms rapidly diminishes, and in most cases it is sufficient to consider the two first terms only. Numerical examples show that the supporting effect of the springs may greatly reduce the bending moment of the unsupported first ($n = 1$) beam, in some cases even to 1/10.

2.4. The Internal Forces of the Extreme Plate Element Resting on Concentrated Supports

If we decrease to zero the horizontal dimension x of the extreme plate element reaching beyond the first support (*Fig. 2.13a,b*), then the first V-shaped beam, which would provide the continuous support of the second plate element, ceases to exist, so that this second plate becomes supported at its two corner points only. The corresponding bending moments can be taken from (STIGLAT and WIPPEL, 1983), in which the diagrams of the bending moment distribution of a cantilever plate, infinitely wide in one direction and loaded at its corner point, can be found (*Fig. 2.14*).

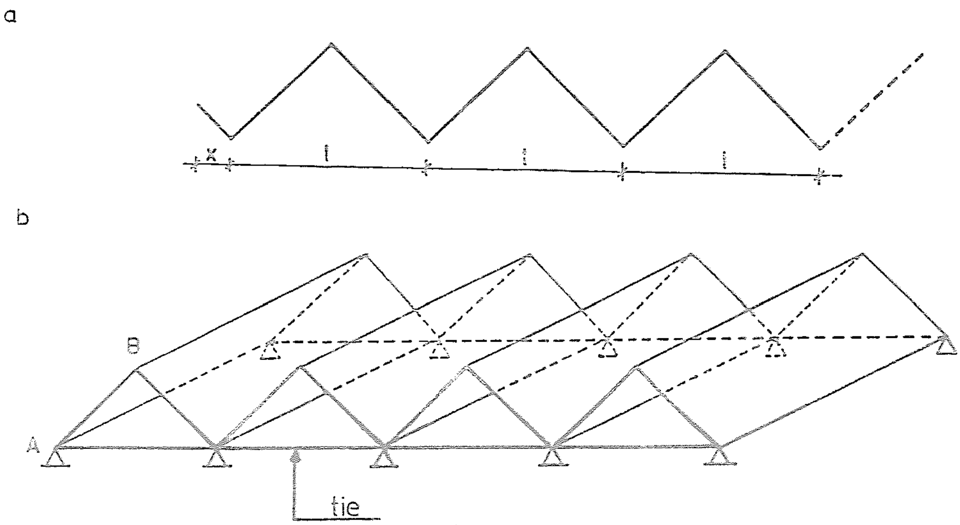


Fig. 2. 13

We thus obtain the bending moments of the extreme plate element in the following way: In the first step we assume a continuous support along the edge at the first support, and determine the moments due to the load. In the second step we add to these the moments caused by the missing continuous support and by the concentrated support forces acting at the two corner points. The missing continuous support causes the bending moments shown in *Fig. 2.15a*, while the concentrated forces give rise to those in *Fig. 2.15b*. We can see that the transverse moments along the crest B, caused by these two latter effects, give a zero sum along L. These transverse moments decrease towards the other supports according to the rules given in *Fig. 2.6*.

2.5. *Stability Problems*

Folded plates – as all composite structures – have two kinds of stability problems: instability of the individual plate elements (*local buckling*), and instability of the whole structure (*overall buckling*). The *intermediate* plates may undergo compression and/or bending (see *Figs. 1.1* and *2.3*). We assume, for the benefit of safety, hinged supports along all edges. In *Fig. 2.16a,b* we present the critical load intensity for bending according to (TIMOSHENKO and GERE, 1961).

The *extreme* plate elements also undergo compression and/or bending, but due to their free edge their critical load intensity is lower than that of the intermediate ones (*Fig. 2.16c,d,e*). The numerical values have

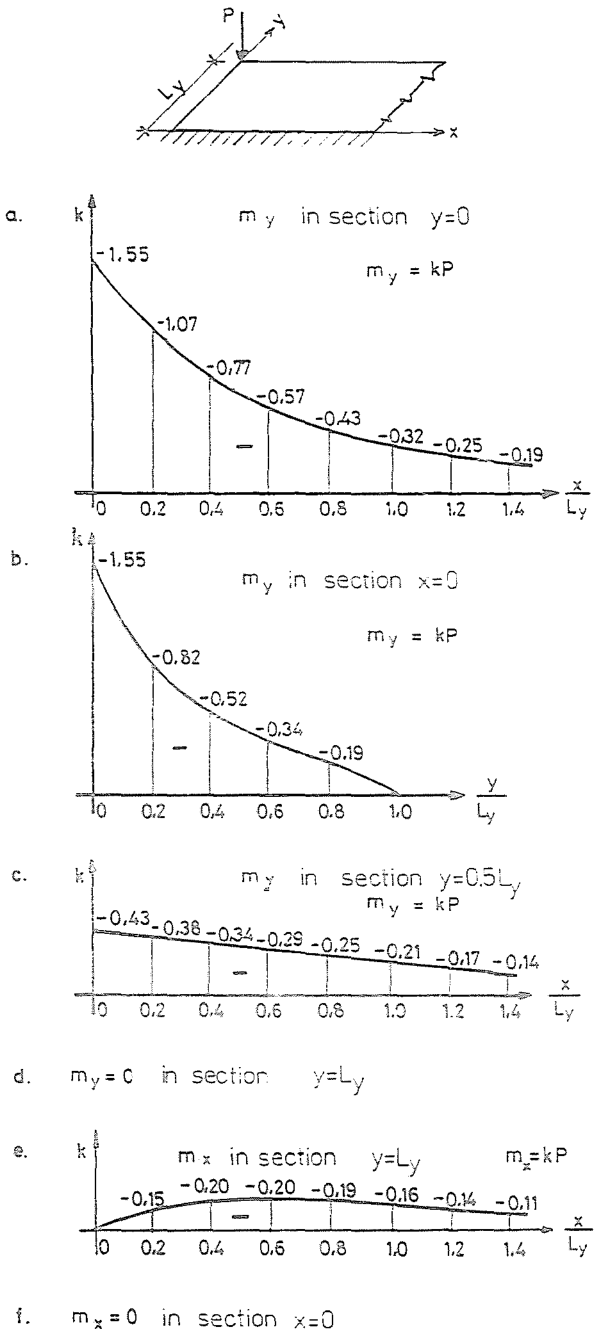

P
y
Ly
x
a.
k
m y in section y=0
m y = kP
-1.55
-1.07
-0.77
-0.57
-0.43
-0.32
-0.25
-0.19
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
x/Ly
b.
k
-1.55
m y in section x=0
m y = kP
-0.82
-0.52
-0.34
-0.19
0 0.2 0.4 0.6 0.8 1.0
y/Ly
c.
k
m y in section y=0.5Ly
m y = kP
-0.43 -0.38 -0.34 -0.29 -0.25 -0.21 -0.17 -0.14
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
x/Ly
d. m y = 0 in section y=Ly
e.
k
m x in section y=Ly
m x = kP
-0.15 -0.20 -0.20 -0.19 -0.16 -0.14 -0.11
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
x/Ly
f. m x = 0 in section x=0

Fig. 2. 14

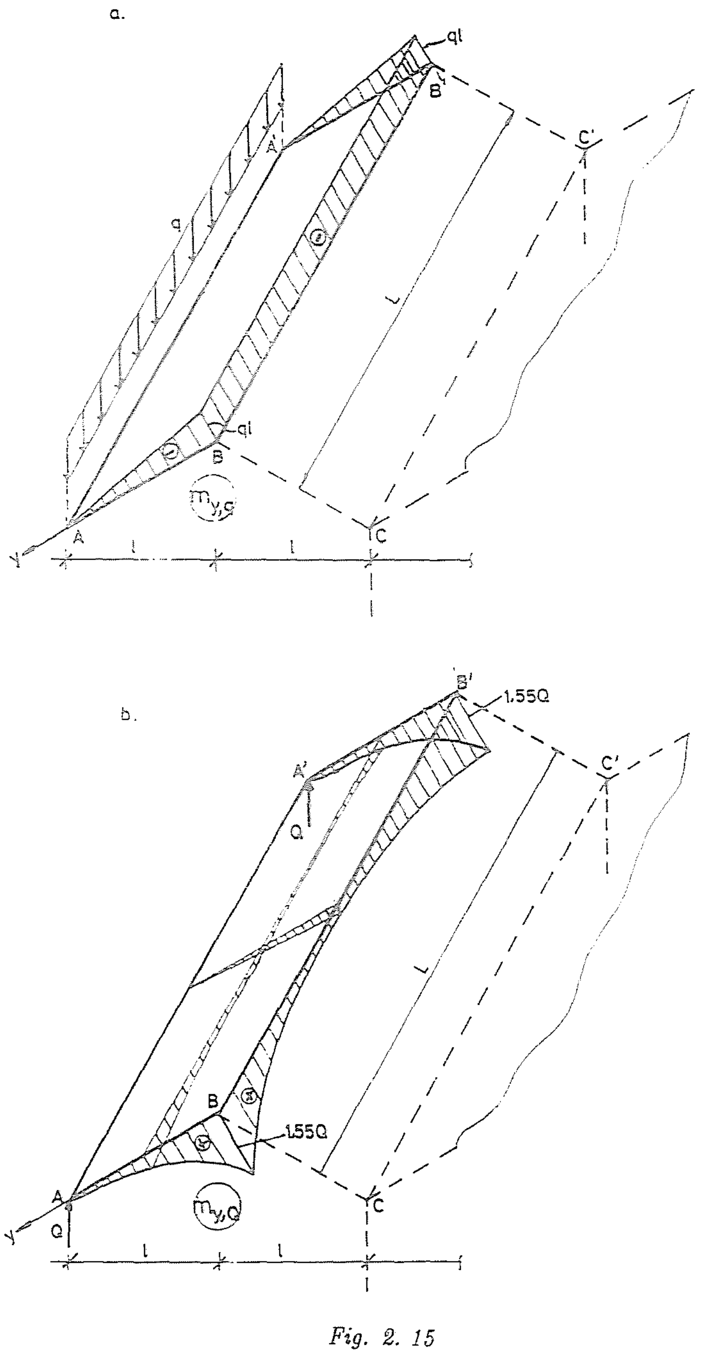


Fig. 2. 15

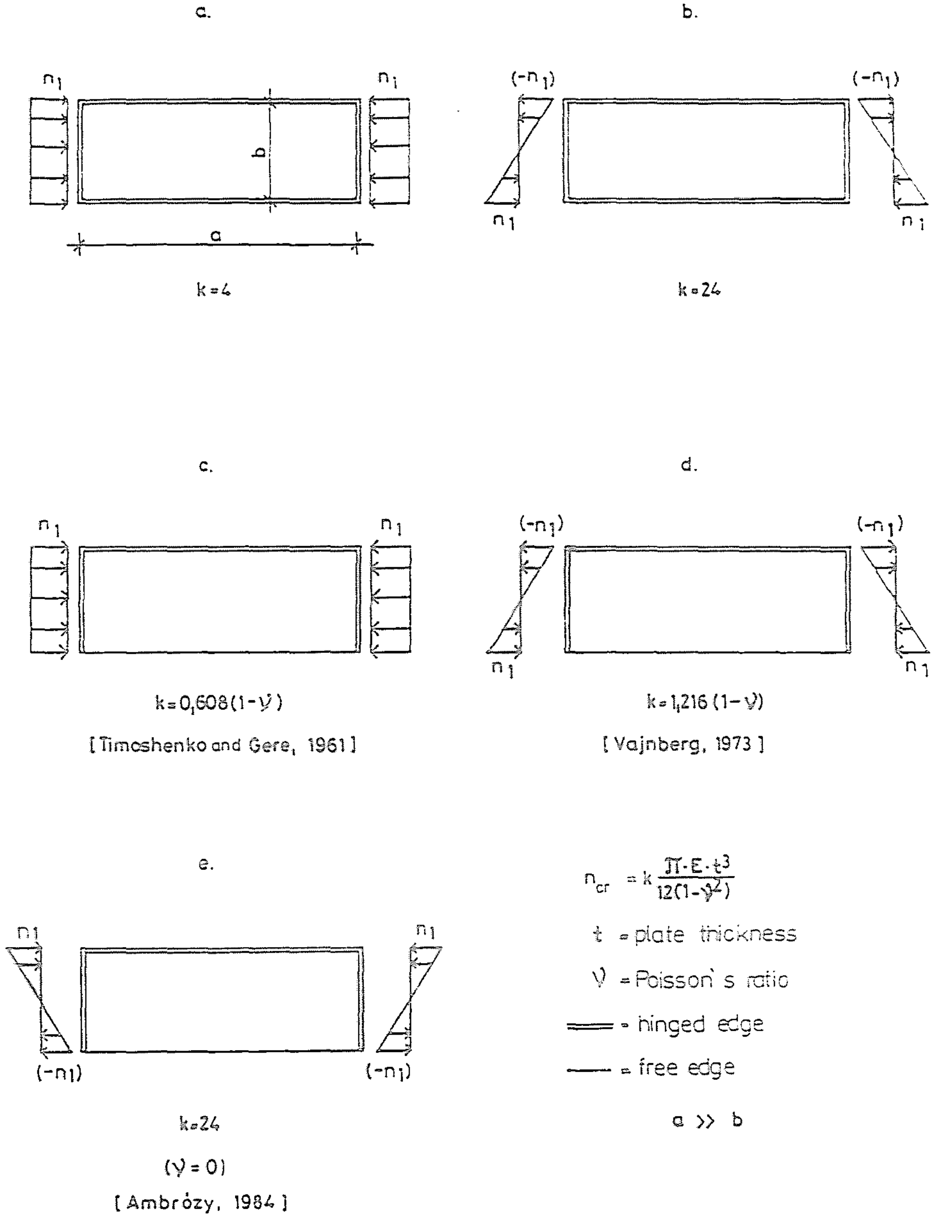

a.
b.
n_1
a
b
$(-n_1)$
$k=4$
$k=24$
c.
d.
$k=0{,}608(1-\nu)$
[Timoshenko and Gere, 1961]
$k=1{,}216(1-\nu)$
[Vajnberg, 1973]
e.
$k=24$
$(\nu=0)$
[Ambrózy, 1984]
$n_{cr} = k\frac{\pi \cdot E \cdot t^3}{12(1-\nu^2)}$
t = plate thickness
ν = Poisson's ratio
= hinged edge
= free edge
a >> b

Fig. 2. 16

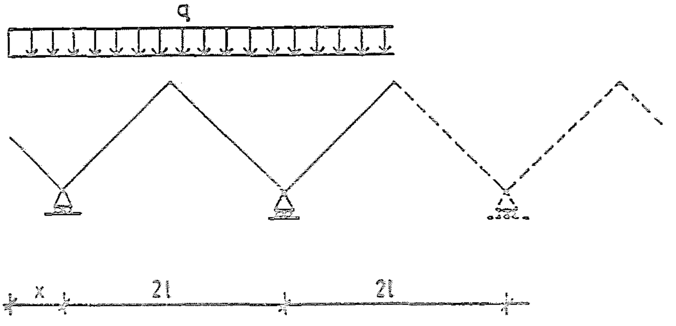


Fig. 2. 17

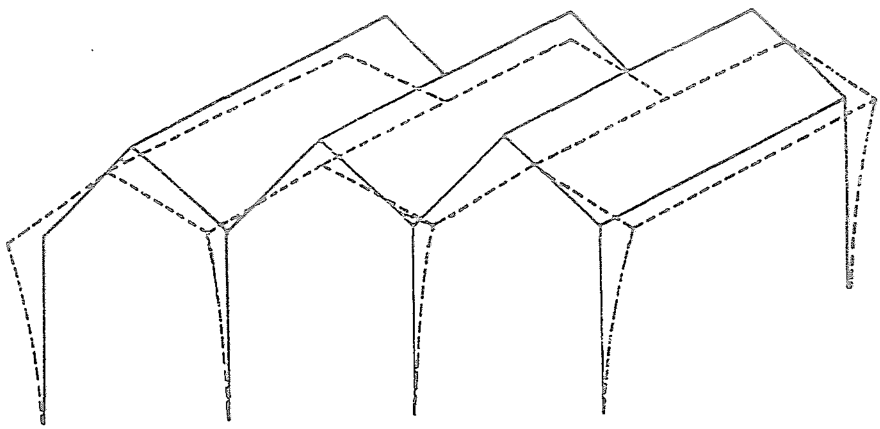


Fig. 2. 18

been taken from (TIMOSHENKO and GERE, 1961), (VAJNBERG, 1967) and (AMBRÓZY, 1984).

All there results refer to plates under bending or compression constant along their length. However, folded plate structures undergo, as a rule, a (parabolically) variable bending moment. Thus if we consider the maximum bending moment as constant along the entire length, we commit an error to the benefit of safety.

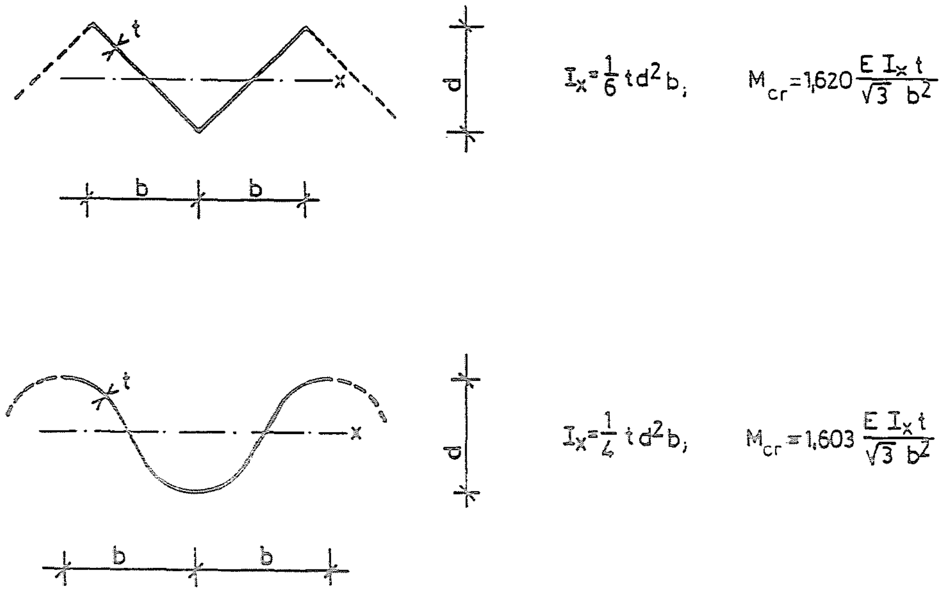


Fig. 2. 19

If the plate element undergoes at the same time bending and compression, we can use the Dunkerley formula, see e.g. in (KOLLÁR, 1991), the result of which lies on the safe side:

$$\frac{n_1^{bend}}{n_{1cr}^{bend}} + \frac{n_1^{compr}}{n_{1cr}^{compr}} = \frac{1}{\gamma}, \tag{27}$$

with γ as the safety factor.

After buckling the plate structure has an increasing load-bearing capacity in the elastic range, see e.g. in (KOLLÁR, 1991). However, the plasticizing of the material and the cracking of the concrete cause a reduction of the stiffnesses, resulting sooner or later in a decreasing load-bearing capacity.

Overall buckling of the whole structure occurs if its cross-sections deform (flatten). This comes about first of all if no diaphragms (ties) are provided at the supports. In the case of barrel-vault-like structures this generally does not happen, since diaphragms are needed to keep the shape of the cross-section anyway. Periodic folded plate structures, however, can be built without diaphragms (or ties). The folds can thus 'slide apart' (*Fig. 2.17*), the structure flattens due to bending and snaps through

(*Fig. 2.18*). The critical bending moment causing snapping has been calculated by KEREK (1981), assuming that the cross-sections are 'flat'. We present the value of the critical bending moment referred to the width of one fold, for V and sine wave formed cross-sections and constant wall thickness, in *Fig. 2.19*. The numerical values of the sine wave formed cross-section can be considered as valid for the trapezoidal cross-section as well.

After snapping the structure has a decreasing post-buckling load-bearing capacity even in the elastic range, see (KOLLÁR, 1973, 1991). Consequently, we should choose a higher safety factor than for plate buckling with increasing load-bearing capacity.

We have to mention two further modes of buckling, the critical loads of which have not yet been determined. The first is the snapping of the structure by flattening, as described above, but with supports not sliding apart (*Fig. 2.20*). This may occur only in structures with very long plate elements, where the middle part of the structure slides apart and flattens. This deformation is made possible by the shearing deformation of the plate elements in their own planes.

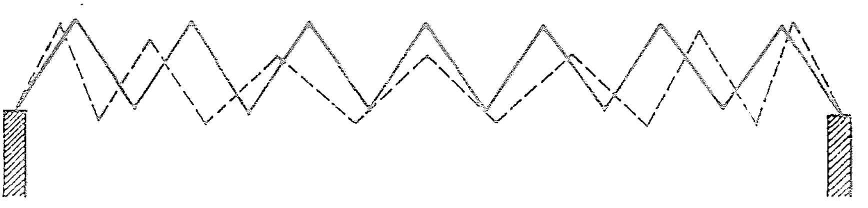


Fig. 2. 20

The second mode of instability is the buckling in the horizontal plane of the upper, U-shaped part of the trapezoidal folded plate structure (*Fig. 2.21*). This is possible only if the width of the 'U sections' is small, and the inclined plate elements provide only a weak lateral support.

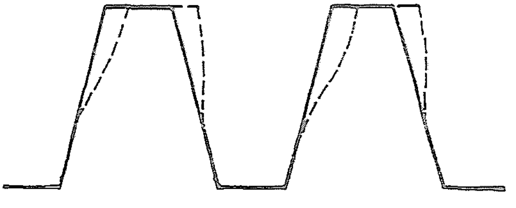


Fig. 2. 21

All we can say about these two buckling modes is that they may occur only in structures of extreme geometry.

3. Problems of Analysis of Folded Plate Structures Consisting of Short Elements

3.1. Static Behaviour of Folded Plate Structures with Short Elements

Folded plate structures consisting of short elements (*Fig. 1.2*) differ from those of long elements in the following:

- their plate elements are 'short', and mostly triangular;
- due to the shortness of the elements, the edge forces do not cause elongation in their opposite edge (since in a deep 'wall-beam' the stresses of the loaded edge die out until the opposite edge);
- the plate elements are supported mostly by each other, and not by fixed supports. Due to the shortness of the elements their edges can always be considered as unmovable with respect to the bending of the plates. The three-edge-forces equations do not exist, according to what has been said of the 'wall-beams'. Finally, due to the fact that most elements are supported by each other, the structure carries the load as a space grid whose bars are formed by the edges, as is shown by the following reasoning.

The load-bearing behaviour of the folded plate structure with short elements does not essentially differ from that of long elements: the (mostly triangular) plates transmit also here the component of the load normal to the plate by bending to the edges, where these must be resolved into components lying in the planes of the two joining plates. These components, together with the in-plane load components, cause in-plane bending and shear in the plates. The deviation from the structures with long elements is only that here the plates subjected to in-plane forces are supported only by the vertices, since it is only here where several edges meet, which can support the vertex. Thus a 'space grid' is formed by the edges (together with some 'effective width' of the plates) as bars (*Fig. 3.1*), and this grid carries the load to the supports. The analysis of space grids can be found in (KOLLÁR and HEGEDŰS, 1985).

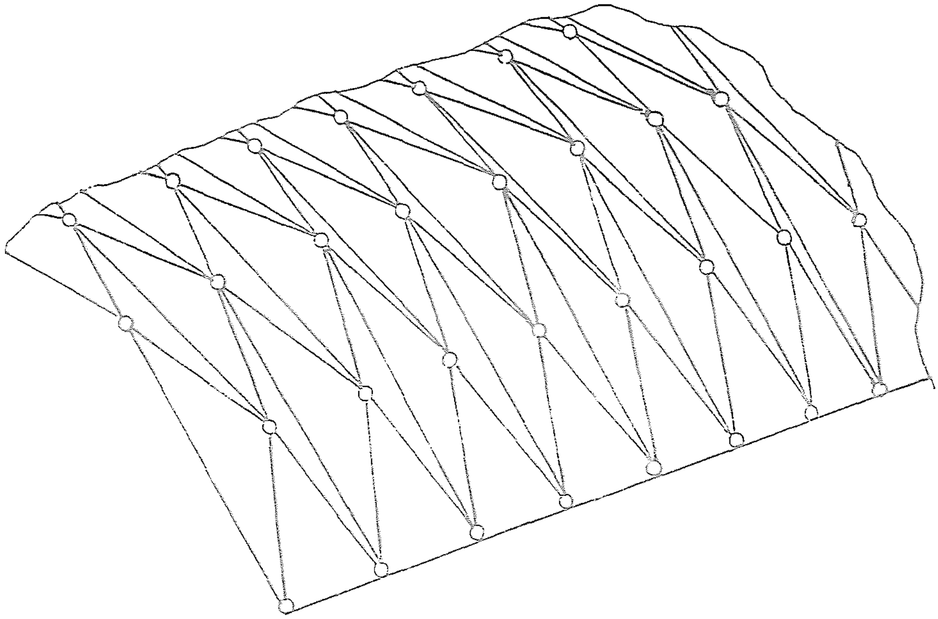


Fig. 3.1

3.2. The Static Behaviour of the Triangular Plates

For the analysis of the folded plate structures with short elements we have to know the bending moments and support forces of the triangular plates. In (BREITSCHUH, 1974) diagrams can be found for these quantities.

The support forces along the edges of the plate elements of arbitrary shape can also be determined in the following simple, approximate way.

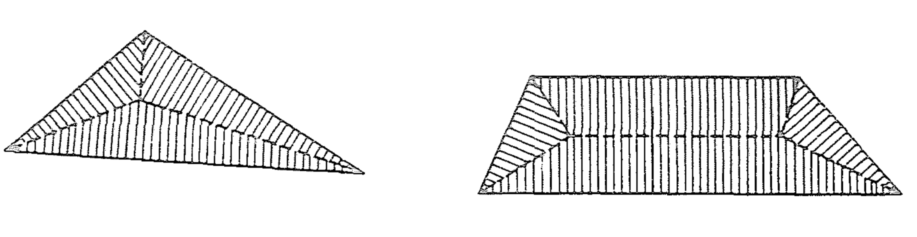
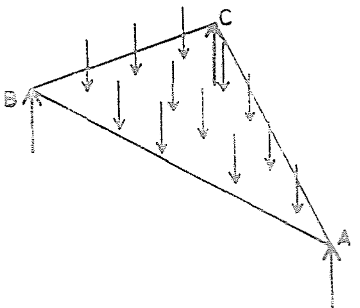


Fig. 3.2

Starting from the corners we draw the bisectrix lines and, if they do not intersect in one single point, we connect the intersection points (*Fig. 3.2a,b*). The geometrical figures thus obtained yield the forces acting on the sides, together with their distribution.

The forces transmitted from the triangular plates to the vertices, necessary to know the internal forces of the *whole structure* (as a space grid), can also be simply determined. These forces have to be statically equivalent to the load acting on the plate. The problem can be solved by considering the plate as a 'table with three legs', which can be easily solved by using equilibrium equations only (*Fig. 3.3*).



Equilibrium equations:

$\Sigma M_{\overline{AB}} = 0$

$\Sigma M_{\overline{BC}} = 0$

$\Sigma M_{\overline{AC}} = 0$

$(\Sigma P = 0)$

Fig. 3.3

As far as the forces loading the plate in its own plane are concerned, we find that they consist of three parts: the components of the load lying in the plane of the plate, the support forces resolved in the planes of the two plates, and the bar forces of the space grid, which act from the vertices along the edges. The latter ones constitute a force system being in equilibrium in itself, and the former two force groups are held in equilibrium by the in-plane components of the support forces acting at the vertices.

Applying these force systems (all being in equilibrium) to the plate element, we can calculate the in-plane forces in the plate. However, we have to make one remark.

Since the 'bar forces' of the space grid act along the edges, we have to divide them between the effective widths of the two adjacent plates. The basic principle of the division should be the compatibility of deformations: the two adjacent plates should undergo the same compression (elongation) due to the forces acting on them. For simplicity, however, we may divide these bar forces in half and half, provided the widths and thicknesses of the two adjacent plates do not differ considerably. The 'effective width' of the plate, valid for this case, will be treated in the next section.

3.3. The Stresses Arising in the Plate due to Concentrated In-plane Forces

In folded plate structures with short elements, concentrated forces act on the plates along their edges. These forces are mostly applied at the corner points of the elements which behave as 'discs' (i.e. plates loaded in their own planes). Such forces act along the inclined edges of the structure shown in *Fig. 3.4*, and a horizontal tension force acts e.g. along the edge AB of the triangular element OAB. Along the inclined and horizontal edges of the structure of *Fig. 3.5* also act concentrated forces, and vertical forces at points A and B of the quadrangular element ABCD. The 'bar forces' arising in the edges of the structure of *Fig. 1.2b* have a similar effect.

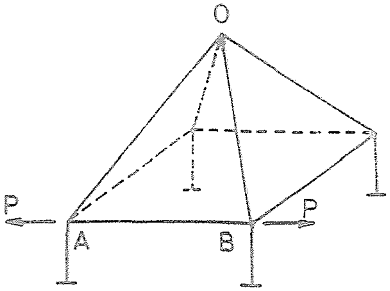


Fig. 3.4.

The problem is: what stresses are caused inside the disc by these concentrated forces. To put it in another way: what is the 'effective width' of the disc subjected to a concentrated force acting along its edges?

The problem can be solved by the theory of elasticity. Before presenting the results, however, we shall try to give a clear picture of the phenomenon by simple reasoning. (When in the following we shall speak of 'stresses', we actually mean 'stresses multiplied by the thickness of the disc': $n = t\sigma$).

Let us consider the rectangular disc subjected to two concentrated forces along one edge (*Fig. 3.6*). The vertical sections of the disc undergo eccentric compression. The 'natural' state of stress of an eccentrically compressed bar is the linear stress distribution, given by the elementary strength of materials (*Fig. 3.7*), since the pertaining strain energy becomes a minimum as compared with other (curvilinear) stress distributions. This linear stress distribution, however, can come about only at a certain distance from the cross-section where the force acts, since in this cross-section the stress distribution corresponds to the application of the concentrated force (*Fig. 3.8*). According to the principle of Saint-Venant, this distance

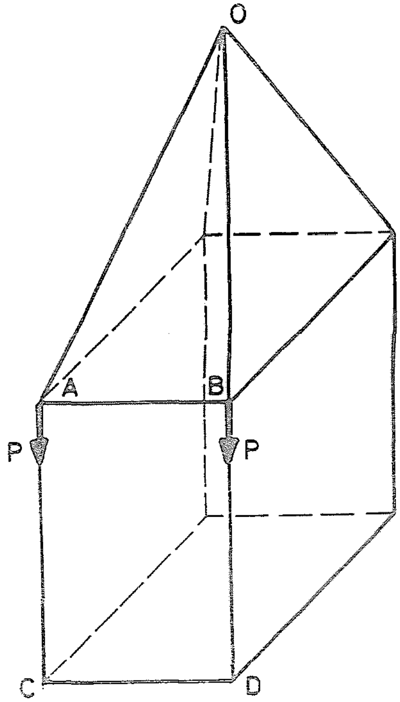


Fig. 3.5.

is equal to the height H of the cross-section, because the effect of the concentrated force can be composed from two parts, see *Fig. 3.9* : load case a) causes a linear stress distribution in every cross-section of the disc, while load case b) – being a force system in equilibrium – causes stresses only inside a length which is equal to the loaded section, i.e. H.

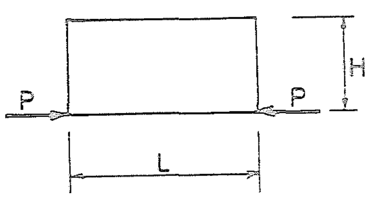


Fig. 3.6.

It follows from the foregoing that if the height of the disc is not greater than half its length ($H \leq L/2$), then the stress distribution in the middle

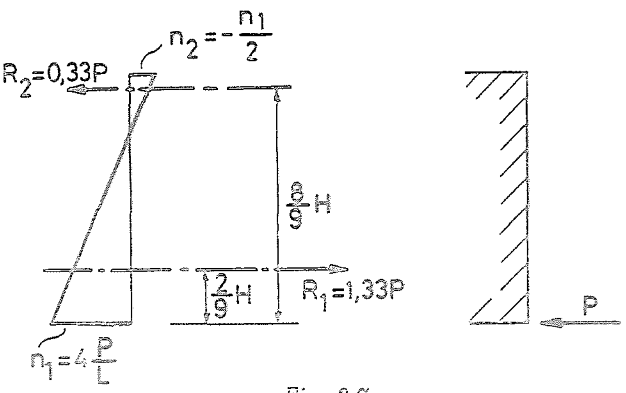


Fig. 3.7.

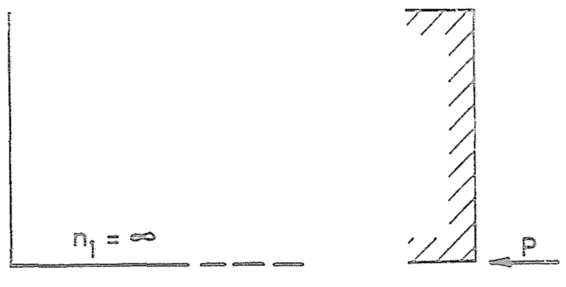


Fig. 3.8.

cross-section is linear, and the resultant R_1 of the compressive stresses acts at a distance $(2/9)H$ from the compressed edge.

If, on the other hand, the height of the disc is greater than its length ($H > L$), we can apply Saint-Venant's principle in the direction of the height: the force system in equilibrium causing disturbance consists of the two concentrated forces P, acting on the bottom 'cross-section', which cause stresses only inside a distance equal to L. Hence in the part of the disc above the height $H = L$ no stresses arise.

The state of stress of a disc with the height $H = L$, subjected to two concentrated forces, has been determined by BAY (1938) with the method of finite differences, subdividing the disc into 5+5 parts. His results are shown in *Fig. 3.10a.* The stress distribution of the section $A - A$ can also be considered valid for the middle cross-section $k - k$, so the magnitudes

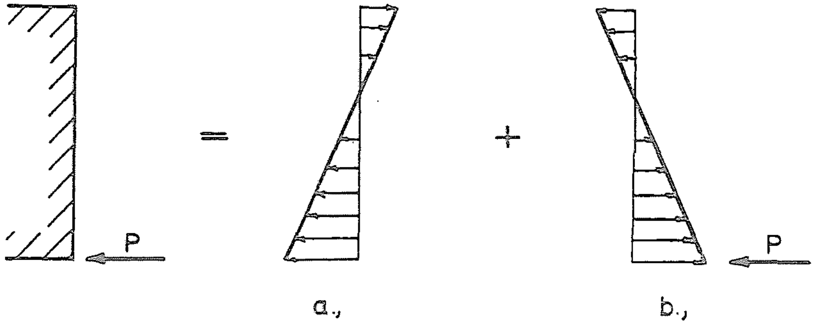


Fig. 3.9.

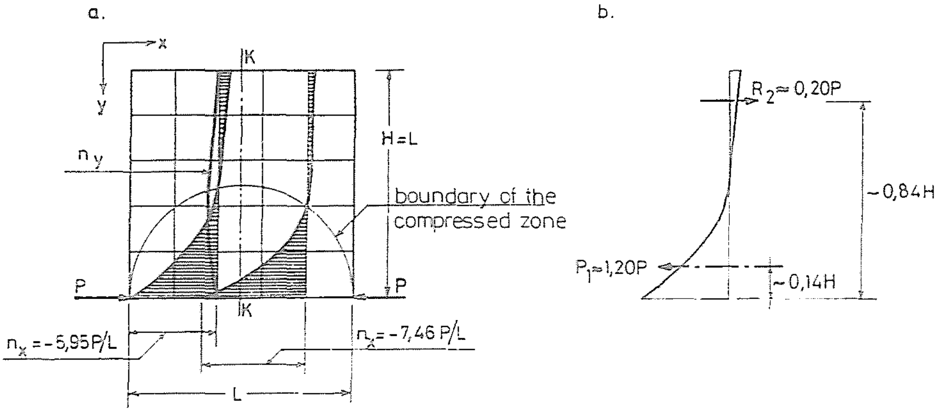


Fig. 3. 10.

and situations of the resultants of the tensile and compressive stresses are given by *Fig. 3.10b.*

It is worth while to note that the compressive stresses propagate into the inside of the disc according to the semicircle of the radius $r = H/2$. The resultant R_1 of the compressive stresses lies closer to the loaded edge than in the case of a linear stress distribution (*Fig. 3.7*).

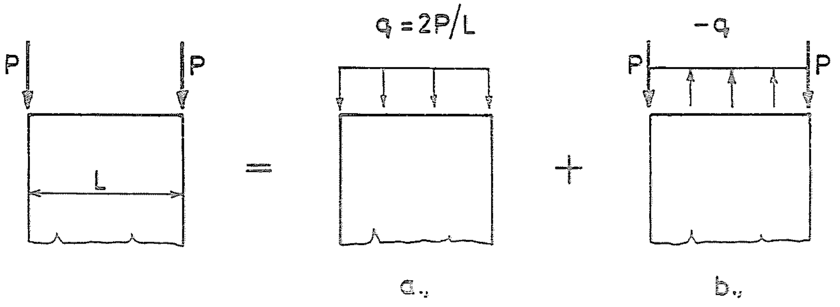


Fig. 3. 11.

For the disc loaded by two vertical forces at its two upper corner points (*Fig. 3.11*) we can apply a similar reasoning: the effect of the two forces can be replaced by a force system corresponding to the 'natural' one (a) and another being in equilibrium (b). The latter one causes stresses only inside a distance L, so that at this distance the distribution of the stresses becomes uniform.

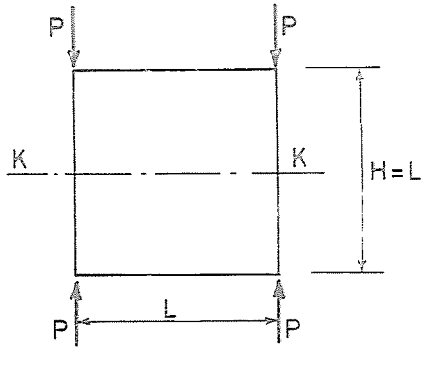


Fig. 3. 12.

If, on the other hand, a square disc is loaded by four concentrated forces (*Fig. 3.12*), then the stress distribution cannot become uniform until the middle cross-section $k-k$, but remains curved. This distribution is obtained by adding the diagram of *Fig. 3.10a* to its mirror image.

The internal forces of the triangular disc (*Fig. 3.13a*) are, according to (BAY, 1938), similar to those of the rectangular one, with the difference that the stress distribution in the middle cross-section can be considered linear only if $\tan\alpha \leq 0.5$, i.e. $\alpha \leq 26.5°$, which means that the height H of the triangle is not more than one fourth of the basis length L.

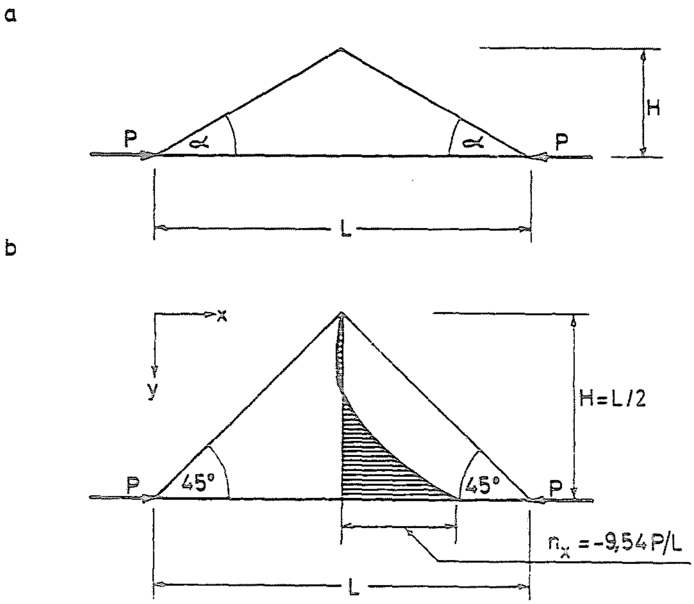


Fig. 3. 13.

The stress distribution in the middle cross-section of a triangular disc with $H = L/2$ is shown in *Fig. 3.13b*. The magnitudes and situations of the resultants of the tensile and compressive stresses correspond approximately to *Fig. 3.10b*.

3.4. Stability Problems

The buckling problems of triangular plates constituting the folded plate structures with short elements are by far not so well clarified as those of rectangular plates. We found the solution of two cases in the literature: for the equilateral and for the rectangular isosceles triangular plate (*Fig. 3.14*), see in (TIMOSHENKO and GERE, 1961) and (WITTRICK, 1954). In the figure we give the critical compressive force of the hinged plates for hydrostatic compression ($n_x = n_y = n$).

For practical applications it is advantageous to replace the triangular plate by a rectangular one exhibiting an equivalent stability behaviour. For this reason we investigated the following problem: if we want to replace the isosceles triangular plate by a rectangular one having the basis of the triangle as one side (*Fig. 3.15*), what height the rectangular plate should have in order to yield the same critical load for hydrostatic compression as the triangular one?

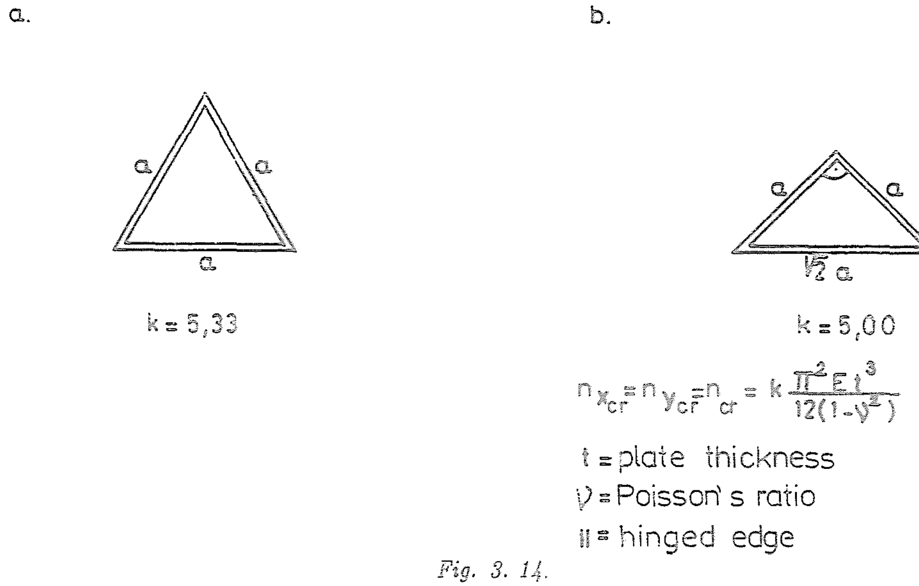


Fig. 3. 14.

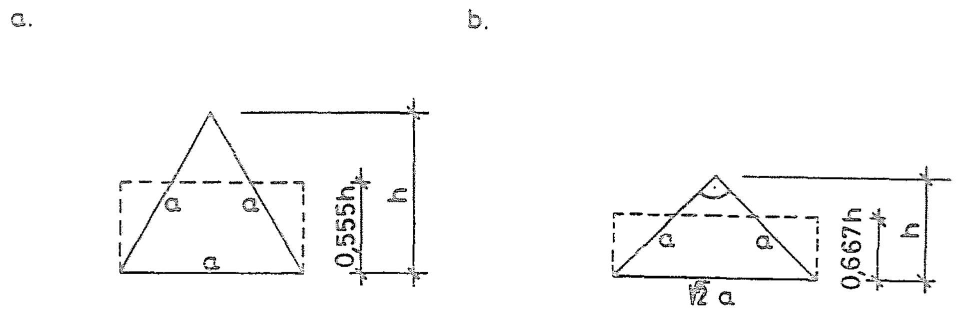


Fig. 3. 15.

Assuming hinged edges for both plates we found that for the two cases shown in *Fig. 3.15a,b* the height should be $0.555h$ and $0.667h$ respectively. For flatter triangles we did not find any result, but considering the tendency of these two cases we may assume that for flatter triangles the height of the rectangle should be somewhat greater than $0.667h$.

All what has been said so far is valid for hydrostatically compressed plates. We could not set up a comparison for other loading cases. Hence we recommend to assume a higher safety factor when substituting a rectangular plate for the triangular one. The plates exhibit an increasing post-buckling load-bearing capacity, see Sect. 2.5, so the reasonings presented there should be considered.

The overall instability of the folded plate structures with short elements is the buckling of the space grid consisting of bars along the edges. This phenomenon can be investigated by a programme based on the second-order theory, i.e. which analyses the internal forces on the deformed shape.

Acknowledgement

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