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A POSSIBLE SOLUTION OF SPATIAL TRANSFORMATION WITH METHOD OF PLANE SECTION

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Abstract

This paper presents a new solution of spatial similarity transformation using plane transformation. The transformation creates a relationship between the coordinates of different points of two coordinate systems.

The mathematical background of the transformation it discussed in detail, and a mathematical solution it presented to solve the transformation task in case of larger angles too. Finally, two examples of practical application of the transformation procedure are presented and calculated.

Keywords: Transformation, rotation, matrix.

Introduction

In geodesy and photogrammetry, it is often necessary to solve transformation between two different coordinate-systems. Helmert or similar transformation is that kind of transformation where the values of transformation in both coordinate systems are related by mathematical similarity. By use of computer systems utilization of more general and effecient methods is possible to solve the transformation task (KOVÁCS, 1984). In this study an iteration method for solving the equations of both planar and spatial transformation is presented. First, we present the plane similar transformation in detail, then the spatial transformation is discussed and finally the solution of practical examples is given.

Plane Similar Transformation

The plane similar transformation (KREILING, 1972) can be expressed for a specifie number of points n $(n \geq 2)$ by the following equation:

$$\begin{bmatrix} x \\ y \end{bmatrix}_i = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + m \begin{bmatrix} u & -v \\ v & u \end{bmatrix}_i \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \qquad (i = 1, 2, \ldots, n)$$

where:

$(x, y)_i^*$ and $(u, v)_i^*$	are the coordinates of the two systems,
$(x_0, y_0)^*$	is the shift vector,
(m)	is the scale factor,
(α)	is the rotation angle.

The solution of the previous equations can be divided into the following steps:

A. Determination the center points of the subsets which contain common points of the two coordinate values:

$$x_s = \frac{\sum x_i}{n}, \qquad y_s = \frac{\sum y_i}{n}, \qquad u_s = \frac{\sum u_i}{n}, \qquad v_s = \frac{\sum v_i}{n};$$

B. The new shift-coordinate values:

$$\overline{x}_i = x_i - x_s, \qquad \overline{y}_i = y_i - y_s,$$

$$\overline{u}_i = u_i - u_s, \qquad \overline{v}_i = v_i - v_s.$$

C. Then the following reduced equation can be derived:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix}_i + \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}_i = \begin{bmatrix} \overline{u} & -\overline{v} \\ \overline{v} & \overline{u} \end{bmatrix}_i \begin{bmatrix} m\cos\alpha \\ m\sin\alpha \end{bmatrix}$$

or

$$\mathbf{v} + \mathbf{l} = \mathbf{A}\mathbf{x}, \qquad (i = 1, 2, \ldots, n)$$

D. The normal equation $\mathbf{0} = \mathbf{A}^*\mathbf{A}\mathbf{x} + \mathbf{A}^*\mathbf{l}$ can be written as:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sum(\overline{u}_i^2 + \overline{v}_i^2) & 0 \\ 0 & \sum(\overline{u}_i^2 + \overline{v}_i^2) \end{bmatrix} \begin{bmatrix} m\cos\alpha \\ m\sin\alpha \end{bmatrix} - \begin{bmatrix} \sum(\overline{u}_i\overline{x}_i + \overline{v}_i\overline{y}_i) \\ \sum(\overline{u}_i\overline{y}_i + \overline{v}_i\overline{x}_i) \end{bmatrix}$$

from this equation the unknown values are:

$$m\cos\alpha = \frac{\sum(\overline{u}_i\overline{x}_i + \overline{v}_i\overline{y}_i)}{\sum(\overline{u}_i^2 + \overline{v}_i^2)}$$

$$m\sin\alpha = \frac{\sum(\overline{u}_i\overline{y}_i - \overline{v}_i\overline{x}_i)}{\sum(\overline{u}_i^2 + \overline{v}_i^2)}$$

The relationship between the scale and rotation angle is:

$$m = \sqrt{(m\cos\alpha)^2 + (m\sin\alpha)^2};$$

$$\cos\alpha = \frac{m\cos\alpha}{m}; \qquad \sin\alpha = \frac{m\sin\alpha}{m}$$

E. The shift-vector can be calculated according to the following equation:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_s \\ y_s \end{bmatrix} - m \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_s \\ v_s \end{bmatrix}$$

The advantage of plane tranformation procedure is not only its ability to precisely determine the rotation angle of small angles ($\alpha < 5°$), but also that of bigger ones, that is calculating the angle of is always possible due to the defined value of the trigonometric twin $\cos\alpha$ and $\sin\alpha$.

Spatial Similar Transformation

In this procedure the calculation of the transformation equations proceeds generally within two steps. In the first one the approximate values of rotation angles are calculated. In the second step the final values of the transformation parameters are calculated through an iteration process (MINDA, 1986). This calculation concept will lead to generalization of the calculation process, so it is required to calculate the rotation matrix.

Mathematical principle of tranformation

If we consider the first modele coordinates $(U, V, W)_i$ and the second one as $(X, Y, Z)_i$ — it can be for example geodetical coordinate system, ($i = 1, 2, 3, \ldots, n$; $n \geq 3$). The transformation relation between the two coordinate systems can be written as:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + m \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + m\mathrm{R} \begin{bmatrix} U \\ V \\ W \end{bmatrix}_i$$

where: R is a 9-element ortogonal rotation matrix ($r_{11} \ldots r_{33}$),
m is the scale factor, and
$(X_0, Y_0, Z_0)^*$ is the shift coordinates.

Turning back to $\overline{X}, \overline{Y}, \overline{Z}$ and $\overline{U}, \overline{V}, \overline{W}$, the coordinates of the central point, we can obtain the following equation:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_i + \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_i = m \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \overline{U} \\ \overline{V} \\ \overline{W} \end{bmatrix}_i .$$

The rotation matrix can be broken down to a multiplication of three rotation matrices:

$$\mathbb{R} = \mathbb{R}_\omega \mathbb{R}_\phi \mathbb{R}_\kappa$$

or

$$\mathbb{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

According to this equation and considering one of the matrices as an unknown value, we can write three mediator equations systems. These are:

A.

$$(\mathbb{R}_\omega \mathbb{R}_\phi)^* \begin{bmatrix} v_x + \overline{X} \\ v_y + \overline{Y} \\ v_z + \overline{Z} \end{bmatrix}_i = m\mathbb{R}_\kappa \begin{bmatrix} \overline{U} \\ \overline{V} \\ \overline{W} \end{bmatrix} . \tag{1}$$

B.

$$(\mathbb{R}_\omega)^* \begin{bmatrix} v_x + \overline{X} \\ v_y + \overline{Y} \\ v_z + \overline{Z} \end{bmatrix}_i = m\mathbb{R}_\phi \left[\mathbb{R}_\kappa \begin{bmatrix} \overline{U} \\ \overline{V} \\ \overline{W} \end{bmatrix}_i \right] . \tag{2}$$

C.

$$\begin{bmatrix} v_x + \overline{X} \\ v_y + \overline{Y} \\ v_z + \overline{Z} \end{bmatrix}_i = m\mathbb{R}_\omega \left[\mathbb{R}_\phi \mathbb{R}_\kappa \begin{bmatrix} \overline{U} \\ \overline{V} \\ \overline{W} \end{bmatrix}_i \right] . \tag{3}$$

If $\mathbb{R}_\kappa$, $\mathbb{R}_\phi$ and $\mathbb{R}_\omega$ are considered unknown, then they can lead to the denotation of the original three plane transformation. The equations can be solved by an iteration process and this results in determining the values of κ, ϕ and ω angles. The determination of scale factor m will be discussed in the following part.

Solution of the Spatial Transformation

Considering that n points ($n \geq 3$) are given from each coordinate systems, the unknown values of transformation can be calculated through the following steps:

A. Determination the center points of subsets which contain common points of the two coordinate systems.

$$X_s = \frac{\sum X_i}{n}, \qquad Y_s = \frac{\sum Y_i}{n}, \qquad Z_s = \frac{\sum Z_i}{n},$$

$$U_s = \frac{\sum U_i}{n}, \qquad V_s = \frac{\sum V_i}{n}, \qquad W_s = \frac{\sum W_i}{n}.$$

B. The original points of coordinate systems are shifted to the center points as:

$$\overline{X}_i = X_i - X_s\,, \qquad \overline{Y}_i = Y_i - Y_s\,, \qquad \overline{Z}_i = Z_i - Z_s\,,$$

$$\overline{U}_i = U_i - U_s\,, \qquad \overline{V}_i = V_i - V_s\,, \qquad \overline{W}_i = W_i - W_s\,.$$

C. The element matrices of the rotation marix are obtained after subtition in (1), (2) and (3) and then after improvement we obtain this final form as:

I. equation group:

$$\begin{bmatrix} \overline{X}_i \cos\phi + \overline{Y}_i \sin\omega \sin\phi - \overline{Z}_i \cos\omega \sin\phi \\ \overline{Y}_i \cos\omega + \overline{Z}_i \sin\omega \end{bmatrix} = \begin{bmatrix} \overline{U} & -\overline{V} \\ \overline{V} & \overline{U} \end{bmatrix}_i \begin{bmatrix} m\cos\kappa \\ m\sin\kappa \end{bmatrix}.$$

II. equation group:

$$\text{if} \qquad \mathbb{E}_\kappa = \begin{bmatrix} \overline{W}_i & -\overline{U}_i \cos\kappa + \overline{V}_i \sin\kappa \\ \overline{U}_i \cos\kappa - \overline{V}_i \sin\kappa & \overline{W}_i \end{bmatrix} \qquad \text{then;}$$

$$\begin{bmatrix} -\overline{Y}_i \sin\omega + \overline{Z}_i \cos\omega \\ \overline{X}_i \end{bmatrix} = \mathbb{E}_\kappa \begin{bmatrix} m\cos\phi \\ m\sin\phi \end{bmatrix}.$$

III. equation group:

$$\text{if} \qquad \mathbb{E}_{\phi\kappa} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix},$$

$$\text{where} \qquad e_{11} = e_{22} = \overline{W}_i\,,$$

$$e_{12} = \overline{U}_i \sin\phi\cos\kappa - \overline{V}_i \sin\phi\sin\kappa - \overline{W}_i \cos\phi\,,$$

$$e_{21} = -\overline{U}_i \sin\phi\cos\kappa - \overline{V}_i \sin\phi\sin\kappa - \overline{W}_i \cos\phi\,,$$

$$\text{then} \qquad \begin{bmatrix} \overline{Y} \\ \overline{Z} \end{bmatrix}_i = \mathbb{E}_{\phi\kappa} \begin{bmatrix} m\cos\omega \\ m\sin\omega \end{bmatrix}.$$

The final values of the angles can be obtained by application of iteration procedure, the individual iteration steps should by performed according to I , II and III equation groups. As a begining, we consider $\omega=0$, $\phi=0$ and we replace the sinus and cosinus of the resulting angle sequencely in the equation groups. The iteration goes on until the sinus and cosinus values of the angle reach a final stable value.

D. The scale factor can be calculated according the following convention: We take the equation:

$$\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_i = m\mathbb{R} \begin{bmatrix} \overline{U} \\ \overline{V} \\ \overline{W} \end{bmatrix}_i = m \begin{bmatrix} F \\ G \\ H \end{bmatrix}_i ,$$

consider:

$$[F \quad G \quad H]_i \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_i = m_i \qquad \text{multiplications,}$$

and then the scale factor can be derived from:

$$m = \frac{\sum m_i}{n} .$$

E. The shift-vector (X_0, Y_0, Z_0) is calculated from:

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} - m\mathbb{R} \begin{bmatrix} U_s \\ V_s \\ W_s \end{bmatrix} .$$

F. We calculate the mean error using the equation:

$$v_{X_i} = X_i - X_i^s , \qquad v_{Y_i} = Y_i - Y_i^s , \qquad v_{Z_i} = Z_i - Z_I^s ,$$

where the values signed by $(^s)$ are the calculated coordinates. The mean error:

$$m_0 = \sqrt{\frac{\sum(v_{X_i}^2 + v_{Y_i}^2 + v_{Z_i}^2)}{3n-7}} = m_X = m_Y = m_Z .$$

Practical Examples

Two examples are given to demonstrate the proposed transfomation procedures accuracy and their application on larger rotation angles.

First Example — Normal Angle

This example depends (practically) on data given by SCHWIDEFSKY and ACKERMANN (1976).

Table 1
Photo Coordinates

Psz	*U*	*V*	*W*
11	0.018	79.931	149.872
13	9.962	79.949	147.890
31	0.022	−79.995	151.915
33	0.000	−79.948	154.922

Table 2
Geodetic Coordinates

Psz	*X*	*Y*	*Z*
11	5083.205	5852.099	527.925
13	5780.080	5906.365	571.549
31	5210.870	4258.446	461.810
33	5909.264	4314.283	455.484

The result of transformation were concluded as:

$$\omega = +1.98581\,\mathrm{g}, \qquad \phi = -2.01138\,\mathrm{g}, \qquad \kappa = +5.05327\,\mathrm{g},$$

$$m = 10.006532,$$

$$X_0 = 4999.402\,\mathrm{m}, \qquad Y_0 = 5000.462, \mathrm{m}, \qquad Z_0 = 1999.902\,\mathrm{m}.$$

Table 3
Errors in cm

Psz	ΔX	ΔY	ΔZ
11	−6.2	5.3	−2.3
13	4.9	−2.5	2.3
31	12.2	−2.4	2.0
33	−11.6	−0.4	−2.0

Table 4
Plotter Coordinates

Psz	*U*	*V*	*W*
21	7.816	71.373	144.924
22	71.254	19.797	140.650
23	83.656	−73.591	148.673
24	16.455	−65.833	151.705

Second Example — Large Angle

This exaple is taken from the DSR1 analytical plotter experements.

Table 5
Geodetic Coordinates

Psz	*X*	*Y*	*Z*
21	50641.17	49326.54	887.05
22	49540.94	49934.56	976.96
23	48138.44	49571.11	862.76
24	48636.24	48657.71	828.54

Table 6
Errors in cm

Psz	ΔX	ΔY	ΔZ
21	−3.8	0.7	5.6
22	6.3	2.7	−7.7
23	−0.7	−1.9	7.1
24	−1.8	−1.5	−5.0

The results were

$$\omega = +199.0414\,\mathrm{g}, \qquad \phi = -0.1593\,\mathrm{g}, \qquad \kappa = -124.4748\,\mathrm{g},$$

$$m = 15.370402,$$

$$X_0 = 49674.97\,\mathrm{m}, \qquad Y_0 = 48837.83\,\mathrm{m}, \qquad Z_0 = 3155.32\,\mathrm{m}.$$

Conclusions

An important advantage of the tranformation procedure is that it is possible to have a solution even in case of larger angles: this is due to the fact that the determination of trigonometric twin (sinus, cosinus) of any angle leads necessarily to the prcise definition of the angle itself. Comparing to other transformation methods; the procedures presented here can be classified among the precise ones. The method is simpler since there is no need to calculate the inverse matrix, which is also simplify the computation process and shortened its execution time.

A disadvantage of this method is that all of three coordinates of control points are required to have a defined solution to the transformation process. Special attention has to be paid when the coordinate system is rotating together, where the rotation angle can have multi-value, this is mainly due to the existence of the rotation matrix $\mathbb{R}$ and its central matrix $\mathbb{R}_\phi$ (case of larger ϕ values) where the other two ($\mathbb{R}_\omega$, $\mathbb{R}_\kappa$) multiplication factors can lie right or left of $\mathbb{R}_\phi$ without changing the final value of $\mathbb{R}$.

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