PERIODICA POLYTECHNICA SER. CIVIL ENG. VOL. 36, NO. 2, PP. 201–205 (1992)

CONNECTION BETWEEN THE ARC TO CHORD CORRECTIONS AND THE SPHERICAL (ELLIPSOIDAL) EXCESS

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Received: May 30, 1992.

Abstract

On the basis of the simple relation between spherical excess and arc to chord correction, formulas to compute the arc to chord correction for different map projections can be derived. If we know the arc to chord correction, however, the spherical/ellipsoidal excess and thus the area of figures on the surface of the sphere or ellipsoid, which are bounded by orthodromes, can be computed simply. Therefore we do not need to know the datum surface coordinates of a figure determined by its corner points' coordinates on a conformal map projection.

Keywords: map projections, arc to chord correction, spherical excess, ellipsoidal excess.

Introduction

It is well-known from map projection relations that the angles between tangents to different shortest lines (orthodromes = geodetic lines) passing through a certain datum surface point generally differ from the angles formed by their image on the projection surface, projected pointwise (with a real projection). One may use either ellipsoid or sphere as a datum surface. When the ellipsoid is applied as a datum surface, the projection surface can be a sphere or plane. From the sphere one generally projects onto the plane.

When a certain direction forms the angle ω with respect to a reference direction on the datum surface, and their projected images on the projection surface form the angle ω', then the angle

$$(\omega' - \omega) = \overline{\Delta}$$

is called *direction correction.* The reference direction is the first principal direction of the projection where the scale distortion has the greatest value.

The angles between the two legs on the datum surface and between their pointwise projected images have to be the same if the projection is *conformal,* and thus

$$\overline{\Delta} = 0\,.$$

On the other hand, the *arc to chord correction* has to be considered on a conformal projection as well, which follows from the fact that our projected points on the image surface are connected with shortest lines on the projection surface instead of the pointwise projected images of datum surface orthodromes. By the term arc to chord correction we denote the angle which is formed by the tangent to the pointwise projected orthodrome and the tangent to the shortest line on the image surface at a certain point. The arc to chord correction is denoted by Δ. When points are projected from the ellipsoid onto the conformal sphere we may use the term *arc to arc correction.* In the general case both legs of the projected angle are distorted, i. e. the distortion of the angle

$$A = D_2 - D_1$$

will be equal to the distortion of its legs:

$$\overline{\Delta}_A = \overline{\Delta}_2 - \overline{\Delta}_1 \quad \text{(angular correction due to direction corrections)}$$
$$\Delta_A = \Delta_2 - \Delta_1 \quad \text{(angular correction due to arc to chord corrections)}$$

We have denoted here with D_2 and D_1 the direction values of right and left legs, respectively. The total angular distortion will then be (HAZAY, 1964)

$$(\overline{\Delta}_A + \Delta_A)\,.$$

Application to Datum Surface Triangle

Let us project a datum surface triangle — bounded by orthodromes — pointwise onto the image surface (*Fig. 1*).
The *spherical/ellipsoidal excesses* are

$$\varepsilon_{\text{sphere}} = (\alpha + \beta + \gamma) - 180^\circ\,,$$

or

$$\varepsilon_{\text{ell}} = (\alpha + \beta + \gamma) - 180^\circ\,,$$

according to whether the datum surface is sphere or ellipsoid.

It can easily be seen that, according to *Fig. 1*, by taking $\overline{\Delta}_A$ into account we get α' from α, β' from β and γ' from γ; and using Δ_A one gets α'' from α', β'' from β' and γ'' from γ'. So if e. g. the image surface is plane

$$\varepsilon = (\alpha'' + \beta'' + \gamma'') - 180^\circ = \sum |\overline{\Delta}_A + \Delta_A|\,.$$

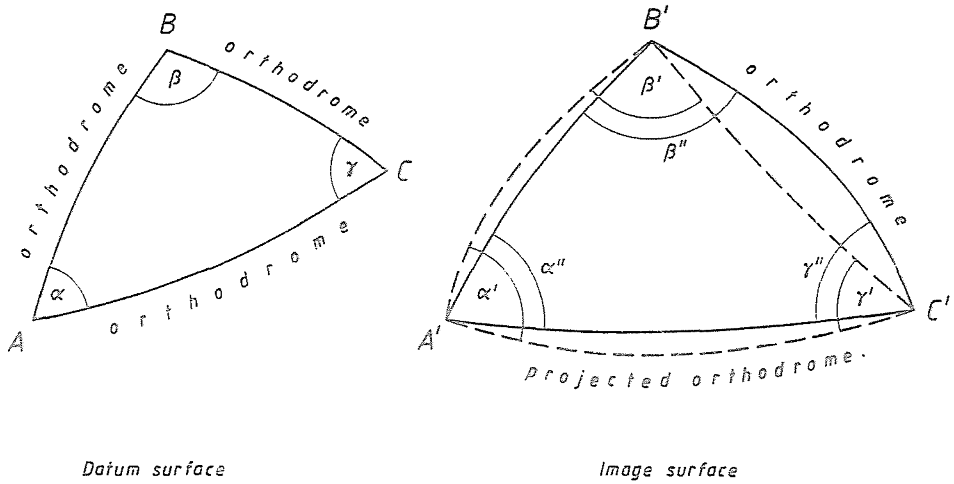

Fig. 1. Datum surface triangle and its image

When *projecting the ellipsoid onto the sphere* in the general case

$$\varepsilon_{\text{ell}} - \varepsilon_{\text{sphere}} = \sum |\overline{\Delta}_A + \Delta_A| .$$

If conformal projection is assumed

$$\varepsilon_{\text{ell}} - \varepsilon_{\text{sphere}} = \sum |\Delta_A| . \tag{1}$$

When *projecting the ellipsoid onto the plane* in the general cases

$$\varepsilon_{\text{ell}} = \sum |\overline{\Delta}_A + \Delta_A| .$$

For conformal projection

$$\varepsilon_{\text{ell}} = \sum |\Delta_A| . \tag{2}$$

When *projecting the sphere onto the plane* in the general case

$$\varepsilon_{\text{sphere}} = \sum |\overline{\Delta}_A + \Delta_A| .$$

Assuming conformal projection:

$$\varepsilon_{\text{sphere}} = \sum |\Delta_A| . \tag{3}$$

Conformal projections are applied for geodetical purposes in almost every case, therefore the *Eqs.* (1, 2, 3) are important for us. The formulas of arc to chord correction of each conformal projection can be determined from the simple relation between spherical/ellipsoidal excess and the angular corrections.

After the arc to chord correction had been determined the spherical/ellipsoidal excess and thus the area of the ellipsoidal or spherical triangle could be computed simply.

$$F_{\text{ell}} = \frac{M \cdot N \varepsilon_{\text{ell}}}{\rho}, \qquad F_{\text{sphere}} = \frac{R^2 \varepsilon_{\text{sphere}}}{\rho}, \tag{4}$$

where M and N are the principal radii of curvature of the ellipsoid along the direction of meridian and prime vertical, respectively, at the centre of the triangle. In the case of a sphere: $M \cdot N = R^2$ (R is the spherical radius). We have denoted by ρ the analytical unit of angle.

Conclusions

The area of any figure, which lies on the surface of ellipsoid or sphere and bounded by orthodromes, can be calculated from the *Eq.* (4) after the ellipsoidal/spherical excess of the covering triangles had been summed up (*Fig. 2*).

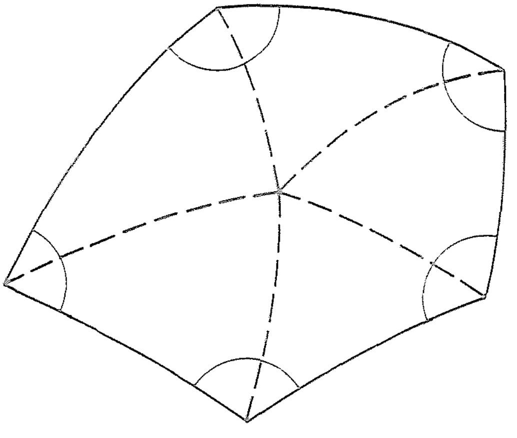

Fig. 2. Datum surface figure bounded by orthodromes

It is suitable not to compute ε from spherical trigonometrical relations, but first the plane coordinates of corner points in a conformal projection should be computed from datum surface coordinates, and then the proper expressions are applied to compute the angular corrections of inner angles at corner points of the delimiting polygon. The total angle distortion of such figures appears along sides of the delimiting polygon, because the sum of angles between the sides which meet at the inner point is 2π both on the datum and on the image surface. After this the absolute values of angular corrections are summed and the area is calculated from the *Eq.* (4).

The Gauss-Krüger or Lambert conformal conical projections can be used for example, to compute the area of an ellipsoidal surface figure. In case of a spherical surface figure the stereographic projection should be applied suitably because of its simple relations.

The area of figures bounded by parallel circles cannot be computed in this way because the parallel circle is an ellipsoidal or spherical small circle, so it is not a geodetical line. In such cases the area under question may be computed by integrating the area of infinitesimal surface element between the appropriate limits of geographic latitude and longitude:

$$F_{\text{ell}} = \int_{\varphi_1}^{\varphi_2} \int_{\lambda_1}^{\lambda_2} M \cdot N \cos\varphi \, \mathrm{d}\varphi \, \mathrm{d}\lambda$$

$$= a^2(1-e^2) \int_{\varphi_1}^{\varphi_2} \int_{\lambda_1}^{\lambda_2} \frac{\cos\varphi}{(1-e^2\sin^2\varphi)^2} \, \mathrm{d}\varphi \, \mathrm{d}\lambda ,$$

where a denotes the semi-major axis of ellipsoid and e is the first numeric eccentricity

$$F_{\text{sphere}} = R^2 \int_{\varphi_1}^{\varphi_2} \int_{\lambda_1}^{\lambda_2} \cos\varphi \, \mathrm{d}\varphi \, \mathrm{d}\lambda .$$

References

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