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'GAUSS' THEOREMA EGREGIUM FOR TRIANGULATED SURFACES

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Abstract

The paper deals with fundamental geometric assumptions of the static-kinematic analysis of triangulated surfaces. First, intrinsic and extrinsic properties of triangulated surfaces as analogues of those of smooth surfaces are introduced, then static-kinematic analogies between triangulated surfaces and pin-jointed single-layer space grids are dealt with. It is shown that Gaussian curvature of smooth surfaces cannot be interpreted for triangulated surfaces, and space grids, however, statements of Gauss' Theorema Egregium can be replaced for statements concerning simple and useful connections between their intrinsic and extrinsic measures.

Keywords: triangulated surfaces, polyhedra of triangle facets, single-layer space grids, extrinsic and intrinsic measures of surfaces, Gaussian curvature, inextensional deformations, static-kinematic analogies.

Introduction

Static-kinematic analysis of shells of free shape often uses replacement of discrete models because the use of exact geometric relations fails on their complexity. A common way of constructing replacement of discrete models is based on the use of triangulated surfaces. A surface can be triangulated by constructing a polyhedron of small triangle facets the vertices of which lie on the surface in question. Not prismatic folded plate roofings, single layer lattice vaults and domes may also be considered physical models of triangulated surfaces (FÖPPL, 1892, DEAN, 1965, TARNAI, 1974). Chinese laterns and Japanese origamis also are models of specific triangulated surfaces (MIURA, 1989).

A general method of constructing compatibility conditions for changes in geometry of 'smooth' surfaces is obtained by Gauss' surface theory, namely, by using the corollary of Gauss' Theorema Egregium that changes in Gaussian curvature K can be parallelly expressed in terms of strains and of changes in curvatures of the middle surface (LOVE, 1927, CALLADINE, 1983, HEGEDŰS, 1990).

Gauss' analysis has been performed on smooth surfaces having no creases and vertices. At surface points lie on creases or vertices of triangulated surfaces, principal curvatures k_1, k_2, and Gaussian curvature K cannot be interpreted, because limit transitions resulting in these measures cannot be performed. On the other hand, principal curvatures k_1, k_2, and Gaussian curvature K of plane facets are zero. Hence, triangulated surfaces have only points where application of Gauss' surface theory in its original form leads to contradictions or banalities.

The subsequent sections are devoted to show a visual elementary interpretation of Gauss' surface theory ready for use in static-geometric analysis of triangulated surfaces. All the fundamental relationships presented here are analogues of those used in the analysis of membrane shells of smooth middle surface (FLÜGGE, 1973, HEGEDŰS, 1990).

Intrinsic and Extrinsic Measures of Surfaces

Surfaces may be defined as boundaries between disjoint regions of the three dimensional Euclidean space. This definition assumes curved surfaces as three dimensional objects of the three dimensional space. But surfaces also possess a two dimensional feature, namely, that the position of any point on a surface can be specified only by two coordinates. From this viewpoint surfaces may be assumed as two dimensional manifolds of the three dimensional space. The dual two and three dimensional analysis of surface properties is the base of Gauss' surface theory (CALLADINE, 1983).

Minimum length of possible ways on a surface from point P_1 to P_2 is as a rule longer than the distance between these points in the three dimensional space. However, this minimum length may have more practical interest than the distance between P_1 and P_2 measured along the chord of the surface.

Shortest lines connecting points on a surface are called geodesics, their lengths are called geodesic distances. Geodesics starting at point P_1 in different directions may intersect at point P_2 and the geodesic distances of the starting point and the point of intersection measured in either geodesics may be different. The definition for geodesic distances and geodesic lines does not imply that there exists only one geodesic which connects two arbitrarily chosen points P_1 and P_2 of the surface. However, it implies that if points P_1 and P_2 are two arbitrarily chosen points of a geodesic, the distance between P_1 and P_2 measured on the geodesic must be a 'local' minimum of lengths of surface lines connecting P_1 and P_2.

Let geometric properties of a surface be called intrinsic properties that can be adequately interpreted in a two dimensional manifold of the three dimensional space and let measures be called intrinsic measures of a surface that can be measured using only measurements of lengths and angles on the surface in question.

Geodesic distances are typical intrinsic measures of a surface, also angles of intersection of surface lines and areas of surface figures are intrinsic measures. Expressions of intrinsic measures are also intrinsic measures.

Intrinsic measures of smooth surfaces can be expressed using only the so called first fundamental Gaussian forms E, F, and G of surfaces.

Normal directions, curvatures and torsion of surfaces cannot be interpreted without the 'third' dimension and cannot be measured using only measurements of lengths and angles on the surface, hence, they are not intrinsic properties and measures of the surface. Properties and measures of this kind are called extrinsic properties and measures. Principal curvatures k_1 and k_2, principal and asymptotic directions are the most important extrinsic meas ures of smooth surfaces.

Distortion of a surface gives rise to changes in intrinsic and extrinsic measures as well. Changes in intrinsic and extrinsic measures are expressed, as a rule, in terms of surface strains and of changes of curvatures and twist, respectively.

Inextensional distortion of surfaces are changes in shape which do not result in changes in length of any surface line. If a surface is subjected to inextensional distortion, only extrinsic properties and measures vary, all intrinsic measures, and also the first fundamental Gaussian forms are preserved.

'Gauss' Theorema Egregium for Smooth Surfaces

Let Γ be a loop around a point P on a curved surface and let a vector $\mathbf{n}$ be travelling along the curve Γ in a way that it always be normal to the surface. If a radius of a sphere of unit radius moves parallel with $\mathbf{n}$, it draws another loop γ on the sphere. This is a mapping of Γ onto the sphere and γ is called the spherical image of Γ_1. Area of the figure bordered by γ is called the solid angle sustained by the surface figure which is bordered by Γ (*Fig.1*).

If areas A and a of the figures bordered by Γ and by γ, respectively, are small, then their ratio is approximately the same for any pairs of loops and the limit transition for this ratio yields

$$\lim_{A \to 0} \frac{a}{A} = K, \tag{1}$$

where K is a constant which may vary from point to point. Its sign is assumed positive if the cyclic senses of loops Γ and γ are the same. Gauss'

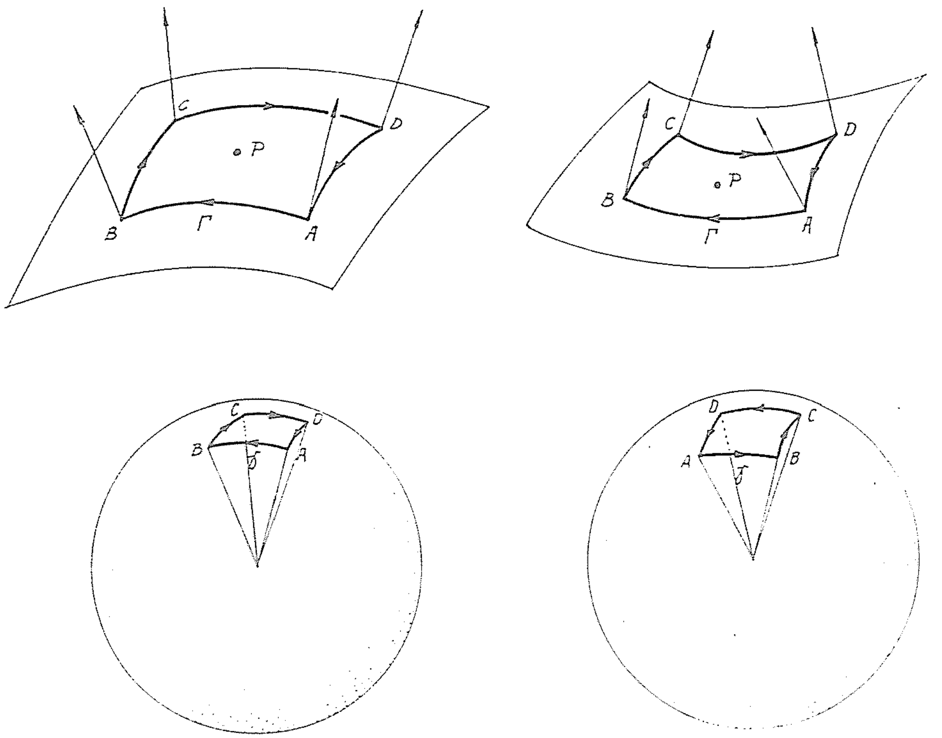

Fig. 1. Spherical images of surface figures

Theorema Egregium states that this constant is the product of principal curvatures k_1 and k_2 at point P_1 (CALLADINE, 1983):

$$K = k_1 k_2. \tag{2}$$

Points where K is positive, zero and negative are called elliptic, parabolic and hyperbolic points, respectively (*Fig. 2*).

Area of a spherical triangle can be expressed as the product of its spherical excess δ and the square of radius of the sphere (KORN and KORN, 1968). Spherical excess is the difference of the sum of interior angles of the spherical triangle and that of plane triangles (*Fig. 3*):

$$\delta = \alpha + \beta + \gamma - \pi. \tag{3}$$

Spherical excess is an additive property of spherical figures. It can be interpreted for any spherical polygon by subdividing it into spherical triangles and summing up the spherical excesses of each consisting triangle,

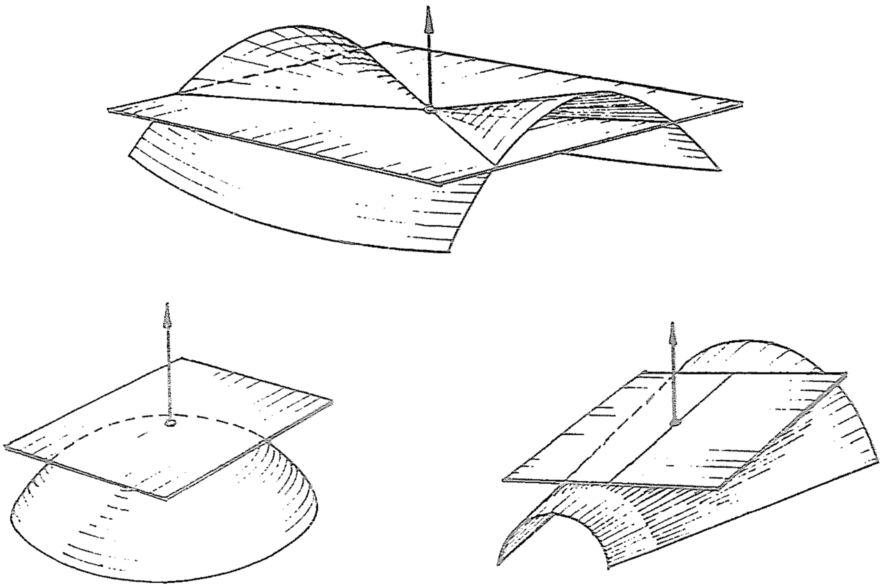

Fig. 2. Elliptic, parabolic and hyperbolic points

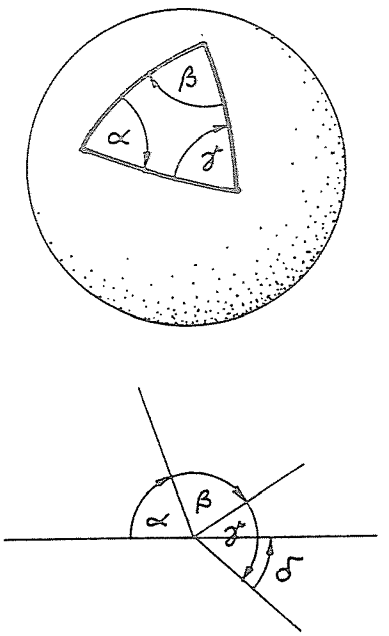

Fig. 3. Spherical excess of a spherical triangle

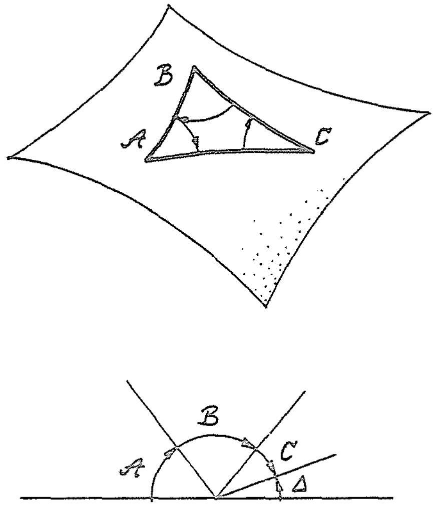

Fig. 4. Angular excess of a surface triangle

moreover, spherical excess of any spherical figure can be interpreted using a subdivision of infinitely dense triangular mesh. In this way a spherical excess δ can also be determined for spherical image γ. Its magnitude equals to a, that is, the solid angles sustained by surface figures equal with the spherical excesses of their spherical images

$$a = \delta. \tag{4}$$

Solid angles have been introduced as extrinsic measures, thus, Gaussian curvature K introduced by *Eq.* (1) and interpreted by *Eq.* (2) also proves an extrinsic measure.

Angular excess Δ of a figure on an arbitrary surface S is the analogue of spherical excess of sphere figures, that is, for surface triangles bordered by geodesics it can be expressed as,

$$\Delta = \mathcal{A} + \mathcal{B} + \mathcal{C} - \pi. \tag{5}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ stand for the interior angles of the surface triangle (*Fig. 4*). Angular excesses of surface figures also are additive measures, that is, angular excess of unions of triangles is the sum of angular excesses of the component triangles. Using a subdivision of infinitesimal triangular mesh, angular excess of any surface figure can be interpreted. Unlike spherical excess, angular excess may also take negative value, negative values of Δ are often called angular deficiencies.

Gauss' Theorema Egregium also implies that the product of principal curvatures at a point P of a surface can also be expressed as the limit transition

$$\lim_{A \to 0} \frac{\Delta}{A} = K, \tag{6}$$

where Δ and A stand for the angular excess and the area of the surface figure bordered by the loop Γ and P is always inside the loop.

From *Eqs.* (1) and (5) it follows that both the angular excess of a surface figure and the solid angle sustained by it can be expressed as,

$$\iint_A K \mathrm{d}A. \tag{7}$$

Angles, angular excess, and area of surface figures are intrinsic measures, that is, K can also be considered intrinsic measure of the surface, despite it has been introduced as a product of extrinsic measures. Thus, Gaussian curvatures K of surfaces are preserved if the surfaces are subjected to inextensional distortion. This dual property of K is of great practical importance in the analysis of deformable surfaces.

Elements and Measures of Triangulated Surfaces

In this section elements and measures of triangulated surfaces are dealt with ones analogues of those of smooth surfaces.

Surface lines on triangulated surfaces are connected lines of planar line segments, surface figures are parts of triangulated surfaces bordered by one or more loops of surface lines. For the sake of simplicity, only simply connected surface figures bordered by one loop of positive or negative cyclic sense are dealt with. If the loop entirely lies on one triangular facet of the surface then the surface figure is a plane figure, else it is the union of plane figures. The area of a figure bordered by a loop of positive cyclic sense is the positive sum of areas of the consisting plane figures. If the boundary of the figure is a self-intersecting loop, then the area is the sum of positive and negative areas involved by closed loop segments of positive and negative cyclic senses, respectively.

Geodesic distance between points P_1 and P_2 of a triangulated surface is the length of the shortest surface line which connects P_1 and P_2. Surface lines having this property of straight lines on a plane are called geodesics of the surface.

From the last mentioned property of geodesics it follows that geodesics of a triangulated surface must be broken straight lines. It may happen that

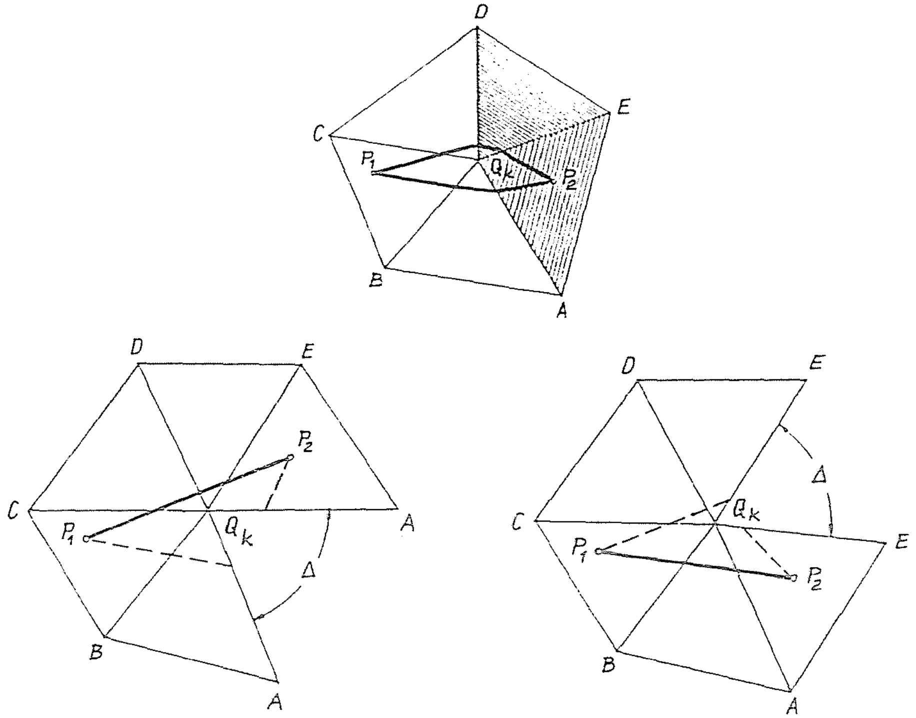

Fig. 5. Geodesics on a triangulated surface

two or more geodesics of different lengths connect surface points P_1 and P_2 (*Fig. 5*).

On a triangulated surface fragment which can be flattened into a plane using inextensional distortions only, in this flattened state, geodesics must be continuous straight lines, hence, angles of intersection of geodesics and edges are the same on adjacent triangle facets forming the edges.

Equality of angles of intersection of geodesics and edges on adjacent triangles also prevails in general. This property of geodesics of triangulated surfaces enables us to construct geodesics starting at arbitrary points in arbitrary directions of the surface unless the geodesic passes through a vertex.

Angles of intersection of surface lines can be interpreted analogously to those of lines on a plane if points of intersection lie on facets of the surface.

If a surface line passes through a vertex, then the question, whether a kneeing point of the line takes place at the vertex or not, may be a problem which cannot be uniquely decided. The same ambiguity may also rise when surface lines intersect at vertices of the triangulated surface. For the sake of avoiding complications caused by ambiguities of this kind, vertices have to be supposed either in one or in the other side of the lines in question.

Deformations of triangulated surfaces are modelled by assuming relative rotations of facets at the edges and assuming a homogeneous state of strain inside each facet. These states of strain cannot be assumed independent of each other because changes in length of connecting triangle sides of adjacent facets must be the same.

This compatibility condition for surface strains can be automatically met using the method of characterizing the state of strain of a pin-jointed single layer space grid.

Strains in three different directions at a point of a plane adequately define an inplane state of strain, that is, homogeneous states of strain of facet triangles are adequately defined by the changes in length of the triangle sides. These shortenings or lengthenings specify compatible states of strain of triangulated surfaces if they are given as changes in length of bars of an analogue space grid having the same network as the triangulated surface.

In non-degenerate cases change in length of each bar of a pin-jointed single layer space grid can be assumed independent of the others, hence, changes in length of edges of a triangulated surface can also be assumed independent of the others. Conditions stated and met in this way are analogues of compatibility conditions for surface strains of smooth surfaces (KOLLÁR and HEGEDŰS, 1985).

A distortion of a triangulated surface is inextensional if bar lengths of the analogue space grid do not change. Joints of the analogue space grid do not prevent relative rotations of bars, however, since bar lengths do not vary, apex angles at all joints are preserved. This interpretation for inextensional distortion is in accordance with that for smooth surfaces.

Surface strains are incident to surface stresses in deformable surfaces. In the case of triangulated surfaces stresses cannot be directly attached to surface strains because equilibrium conditions cannot be met along the edges without the action of bending moments and shearing forces normal to the facets. This problem can be simply overridden by assuming replacement edge forces acting as if they were bar forces of the analogue pin-jointed space grid (HEGEDŰS, 1984, KOLLÁR and HEGEDŰS, 1985).

In case of infinitesimally small inextensional distortion, vectors of relative rotations of adjacent facets of triangulated surfaces show a statickinematic analogy with self equilibrating bar forces of the analogue space grid. This analogy resembles very much of the static-kinematic analogy of internal forces and displacements of shells (CALLADINE, 1983):

- At vertices where only three facets and three edges intersect, no relative rotation of the facets can take place; at joints connecting only three (not coplanar) bars self equilibrating nonzero bar forces cannot act.
- Convex closed triangulated surfaces cannot develop inextensional distortion, the analogue pin-jointed space grid cannot develop self stresses (FÖPPL, 1892, CALLADINE, 1983).
- States of stress of pin-jointed space grids constructed on curved surfaces can be described analogously to those of membrane shells by a stress function (HEGEDŰS, 1984). A stress function of a space grid is geometrically interpreted as the equation of a triangulated surface having the same projected network as the space grid. In nondegenerate cases, the number of linearly independent states of self-stress of grids with perfectly fixed boundary joints does not depend on the number of internal joints. If the number of boundary joints is m_b, arbitrary boundary values and boundary slopes of the stress function may define $2m_b$ linearly independent functions of self stresses of the unloaded grid, but the number of linearly independent states of sempstress is less by three:

  $$f = 2m_b - 3.$$

  The number of linearly independent states of self-stress characterized by non-zero internal bar forces and irrelevant boundary bar forces is

  $$f_i = f - m_b = m_b - 3.$$

  This number equals to the degree of freedom of compatible relative rotations at the edges due to inextensional distortions of a triangulated surface having the same network and a perfectly free boundary. The bar forces can be replaced for compatible vectors of relative rotations of the adjacent facets provided the displacements are infinitesimally small.

This static-kinematic analogy between triangulated surfaces and pin-jointed space grids may again prove much deeper than a simple coincidence of networks (HEGEDŰS, 1984, KOLLÁR and HEGEDŰS, 1985).

Angular Excesses and Solid Angles on Triangulated Surfaces

Let points $P_1, P_2, \ldots, P_i, \ldots, P_m$ be the vertices of an m-gon on a triangulated surface and let $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_m$ be interior angles at these

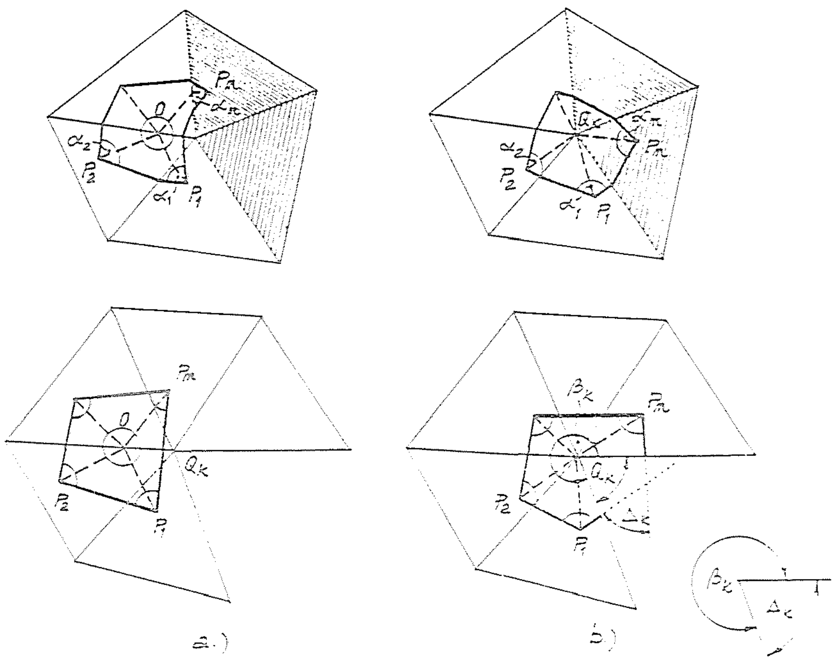

Fig. 6. Sum of interior angles of polygons on a triangulated surface

vertices. Side lines of the m-gon are geodesics of the surface, that is, straight lines between vertices lying on common facets and broken straight lines between vertices lying on different facets.

If each vertex of the m-gon lies on the same facet of a triangulated surface, then the m-gon is a plane figure. It can be assumed as the union of m plane triangles of one common vertex inside the polygon (*Fig. 6.a*) and the sum of α_i is that of angles of these triangles reduced by 2π:

$$\sum_{i=1}^{m} \alpha_i = m\pi - 2\pi. \tag{8}$$

Angular excess of an m-gon on a triangulated surface is defined analogously to that of m-gons on smooth surfaces:

$$\Delta = \sum_{i=1}^{m} \alpha_i - (m-2)\pi. \tag{9}$$

If vertices of an m-gon lie on different facets, but no vertices of the surface lie inside the m-gon, zero value for Δ is preserved, because the m-gon cut out of the surface can be flattened into a plane figure having zero angular excess.

Let one vertex Q_k of the triangulated surface lie inside the m-gon $P_1, P_2, \ldots, P_i, \ldots P_m$. One can visualize this case as the m-gon is drawn onto a roof around its apex point Q_k (*Fig. 6.b*). Let again the angular excess of the m-gon be analyzed. If the m-gon is subdivided into surface triangles in a way that the common vertex of the triangles lies at Q_k, then the sum of angles of the m-gon can be determined analogously to the formerly used method. The only difference is that the sum of angles at the common vertices of the subdivision triangles is not automatically 2π. It equals to the sum of apex angles of the roof at Q_k. Let β_k stand for this sum. Then, the sum of interior angles of the m-gon is

$$\sum_{i=1}^{m} \alpha_i = m\pi - \beta_k, \tag{10}$$

angular excess Δ of the m-gon can be expressed as

$$\Delta = \sum_{i=1}^{m} \alpha_i - (m-2)\pi = 2\pi - \beta_k. \tag{11}$$

Using the same subdivision method for a polygon involving vertices $Q_1, \ldots, Q_k, \ldots, Q_n$ with sums of apex angles $\beta_1, \ldots, \beta_k, \ldots, \beta_n$, angular excess of the polygon can be expressed as

$$\Delta = \sum_{k=1}^{n} (2\pi - \beta_k). \tag{12}$$

This result shows that Δ depends only on the sums of apex angles β_k at the involved vertices of the triangulated surface. Since Δ does not vary if the shape of the polygon varies in a way that vertices $Q_1, \ldots, Q_k, \ldots, Q_n$ remain, and no other vertices get inside it, angular excess can be attached rather to vertices $Q_1, \ldots, Q_k, \ldots, Q_n$ than to surface figures involving them.

Let angular excesses be attached to the vertices by

$$\Delta_k = (2\pi - \beta_k), \tag{13}$$

then the angular excess of any surface polygon involving vertices $Q_1, \ldots, Q_k, \ldots, Q_n$ can be calculated as

$$\Delta = \sum_{k=1}^{n} \Delta_k. \tag{14}$$

Surface figures bordered by arbitrary surface lines can be assumed as unions of infinitesimal triangles, in this way their angular excesses can also interpreted and expressed by the above formula.

Angular excesses attached to the vertices are intrinsic measures of the triangulated surface. Their values can be positive, zero or negative. Summits attached to angular excesses of positive, zero and negative value are called elliptic, parabolic and hyperbolic vertices, respectively (*Fig. 7*).

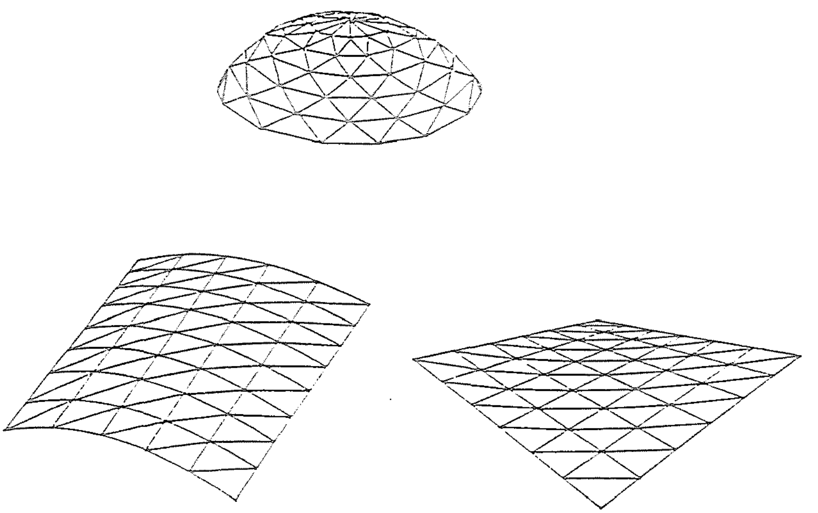

Fig. 7. Triangulated surfaces with elliptic, parabolic, and hyperbolic vertices

Spherical images of surface figures on triangulated surfaces are also defined exactly in the same way as it was done for surface figures on smooth surfaces. However, travelling normal vector $\mathbf{n}$ of the surface behaves differently: at boundary segments lying on facets of the surface its direction does not vary at all, at creases it has no uniquely defined direction on the diametral plane normal to the crease in question. Nevertheless, if all directions of the sector limited by normal vectors of the adjacent facets is considered to belong to the kneeing point of the boundary then spherical images can be uniquely constructed as spherical polygons bordered by great circle arcs.

Spherical images of surface figures which lie on one facet of triangulated surfaces are single points on the sphere, the solid angle sustained by such figures is obviously zero. Spherical images of figures lying on more than one facet in a way that no vertices are included are great circle arcs enclosing zero area. Solid angle sustained by such figures is zero as well. For sustaining a non-zero solid angle, a surface figure must involve at least one vertex of the triangulated surface.

Let a surface figure include one vertex Q_k of the triangulated surface and let the construction of its spherical image be analyzed (*Fig. 8*). If the centre of the sphere of unit radius is fixed at Q_k then the m sided roof W_k formed by planes of m facets intersecting at Q_k cuts a spherical m-gon S_k from the sphere. Let radii normal to the faces of W_k be the edges of another roof W_k'. This roof also consists of m faces, that also cut a spherical m-gon S_k' from the sphere.

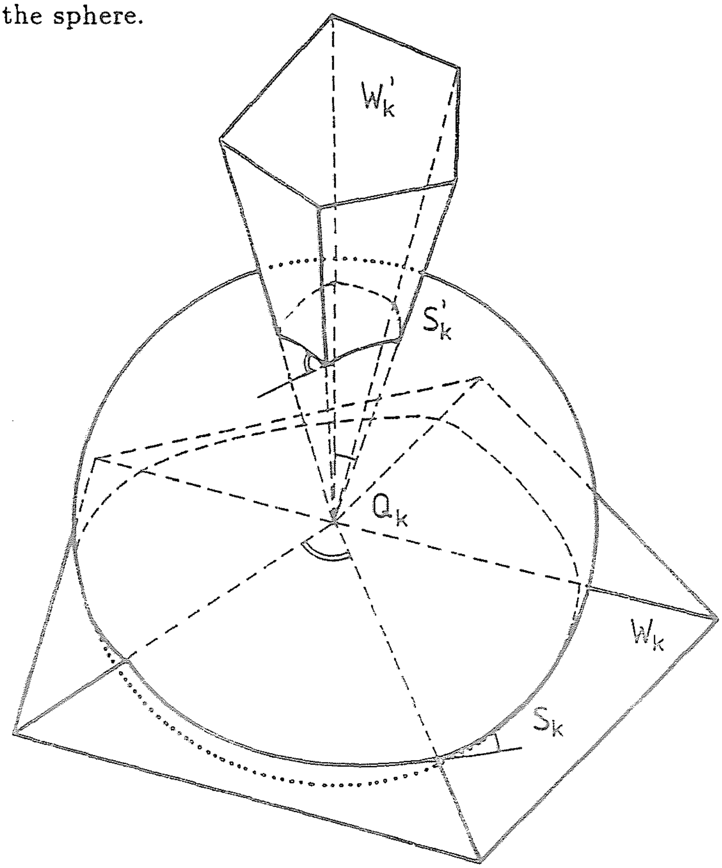

Fig. 8. Solid angle attached to a vertex of a triangulated surface

Spherical images of surface figures involving only one vertex Q_k can only differ from S_k' in having extensions of zero area, irrelevant in calculating solid angles. Hence, S_k' can be considered the spherical image of any loop drawn around the vertex Q_k on W_k. This result clearly shows that solid angle sustained by surface figures involving Q_k can be attached rather to the vertex than to the figures themselves.

For the same reason as S'_k, S_k can also be considered the spherical image of any loop drawn around vertex Q_k on W'_k. Thus, S_k and S'_k are spherical images of each other. Angles and side lengths of S_k and S'_k are connected as follows.

Side lengths of S'_k are equal to the hinge angles at the respective radial edges of W_k, that is, to the respective exterior angles of S_k. Exterior angles of S'_k are equal to the side lengths of S_k, that is, to the respective apex angles of W_k

$$\beta_{k1}, \ldots, \beta_{kj}, \ldots, \beta_{km}.$$

Solid angle a sustained by spherical m-gon S_k or by any surface figure on W_k which involves Q_k is equal to the area of spherical m-gon S'_k that can be obtained as its spherical excess multiplied by the square of the unit radius. Using the notation formerly introduced, a can be written as

$$a = \sum_{j=1}^{m}(\pi - \beta_{kj}) - (m-2)\pi = 2\pi - \sum_{j=1}^{m}\beta_{kj} = 2\pi - \beta_k. \tag{15}$$

This equation shows that solid angle a sustained by a surface figure involving vertex Q_k of the triangulated surface is the same as angular excess Δ of the figure defined by *Eq.* (12).

Spherical images of triangulated surfaces as a whole can also be interpreted. In typical cases, there is a one to one duality between elements of a triangulated surface and those of its spherical image. Spherical image of each facet is a point assigned by a radius normal to the plane of the facet in question. Spherical images of edges are arcs of great circles connecting points which are spherical images of facets. Planes of the great circles are normal to the corresponding edges. Spherical images of vertices Q_k are closed spherical polygons of the area a_k. This mapping of the triangulated surface shows that solid angles can be directly attached to the vertices, because only vertices of the surface sustain solid angles.

Sum of apex angles β_k determines angular excess and solid angle by the same formula (see *Eqs.*(13) and (15), hence,

$$a_k = \Delta_k. \tag{16}$$

The background and the statement of this equation may be considered Gauss' Theorema Egregium for triangulated surfaces. The dual character of sums of apex angles of triangulated surfaces is the analogue of the dual character of Gaussian curvature K of smooth surfaces.

Solid angle is an additive measure, that is, solid angle sustained by the union of surface figures is obtained as the sum of solid angles sustained by these figures. If a surface figure includes vertices $Q_1, \ldots, Q_k, \ldots, Q_n$ with sums of apex angles $\beta_1, \ldots, \beta_k, \ldots, \beta_n$, then the solid angle sustained by the figure can be expressed as,

$$a = \sum_{j=1}^{m} s(2\pi - \beta_k) = \sum_{j=1}^{m} \Delta_k = \Delta. \tag{17}$$

Conclusions

The results of the above analysis can be summed up as follows.

Sums of apex angles at vertices involved by a surface figure of a triangulated surface determine both the angular excess and the solid angle sustained by the figure in a way that the two measures can be expressed in the same algebraic form. These sums are intrinsic measures of the triangulated surface, hence, solid angles can also be assumed intrinsic measures, invariant against inextensional distortions of the surface.

For the same reason, changes in solid angles caused by distortions of a triangulated surface can be expressed by calculating changes in angular excesses.

Bisecting normals of facet triangles form trivalent nets of surface lines that divide triangulated surfaces into surface polygons in a way that each vertex Q_k can be associated with a polygon of the area A_k. The network formed by bisecting normals is topologically equivalent with the network formed by the spherical images of the edges of the surface, to each polygon on the sphere a polygon on the surface can be associated. One also can compute the ratio of areas:

$$K_k = \frac{a_k}{A_k} = \frac{\Delta_k}{A_k}. \tag{18}$$

Let the network of a triangulated surface be 'refined' in a way that additional vertices are put into the network e.g. by replacing superficies of shallow tetrahedra for facets or broken lines for edges. This refinement does not essentially change the values of K_k computed for the new vertices if the shape of the refined surface gets closer and closer to a smooth surface. It tends to the value of K of this surface, hence, in case of sufficiently uniform network of a triangulated surface ratii K_k are approximate values of the Gaussian curvature of the original surface.

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