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FORCES IN PRESTRESSED CONCRETE BRIDGES CONSTRUCTED BY FREE CANTILEVERING

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Abstract

There are many problems of free cantilevered prestressed concrete to be solved by more convenient methods. One of these is the determination of the internal forces during the assembly and post-tensioning of the cantilevers. This question is also called the determination of the elastic shortening loss in already anchored tendons.

A system of equations for the tendon forces as unknowns using the force method for the states of the construction of the cantilevers is written. The analytical solution of this equation is enabled by the recognition that the coefficient matrix of the system is a one-pair matrix modified by a diagonal matrix. Using the statement according to which the inverse matrix of a one-pair matrix is a symmetric tridiagonal matrix and vice versa the elements of the inverse of the one-pair matrix in the coefficient matrix can be produced. The task finally can be reduced to the inversion of the symmetric tridiagonal matrix modified by a diagonal matrix. This further problem can be solved by means of one-pair matrices formed by quantities gained by a recursive algorithm.

The importance of the result consists in the fact that the internal forces of a free cantilevered structure in an arbitrary stage of construction can be written in case of any parameters (changing cross-section, length of segments, number of tendons, etc.)

Keywords: prestressed concrete bridge, free cantilevering, force method, one-pair matrices.

Introduction

In the last decades, methods were developed and widely used for construction of prestressed concrete structures, mainly bridges, which avoid scaffolding and also moulding in its classical sense. For major spans these are the free cantilevering using precast segments or site casting using travellers. (Considering given features also the incremental launching belongs to this group.) The construction method is that at the unique phases of the construction it is to be dealt with a cantilever. This cantilever is produced step by step anchoring a prestressing tendon or tendon group at the boundary of each segment [3].

The method described in this paper allows to determine analytically the forces in the tendons by stage post-tensioning (this task is frequently called the determination of the elastic shortening loss in already anchored tendons) and the forces at different concrete cross-sections under prestress and different dead loads during construction before the closure of cantilevers at midspan or reaching the abutment. This means that the task is to calculate the unknown quantities in a statically undetermined system [1] to be solved by the force method.

The Mathematical Model, the Coefficient Matrix

In the practice, of course, many different versions of the question occur. The mathematical method which is dealt with here is fit to describe almost all cases. However, for the sake of simplicity the theoretical model will be taken with different restrictions.

The structure in *Fig. 1* is a cantilever, i.e. it is statically externally determined. At the phases of the construction the structure is elongated by segments 1, 2,..., i, ..., n, and these segments are fastened to the segment above the pier by tendons (or tendon groups) denoted by the same indices. The cantilever is statically internally as many times undetermined as many tendons (or tendon groups) are anchored. (Tendon groups contain tendons of the same profile.) In this case it is indifferent whether one clamped cantilever is constructed or balanced two cantilevers.

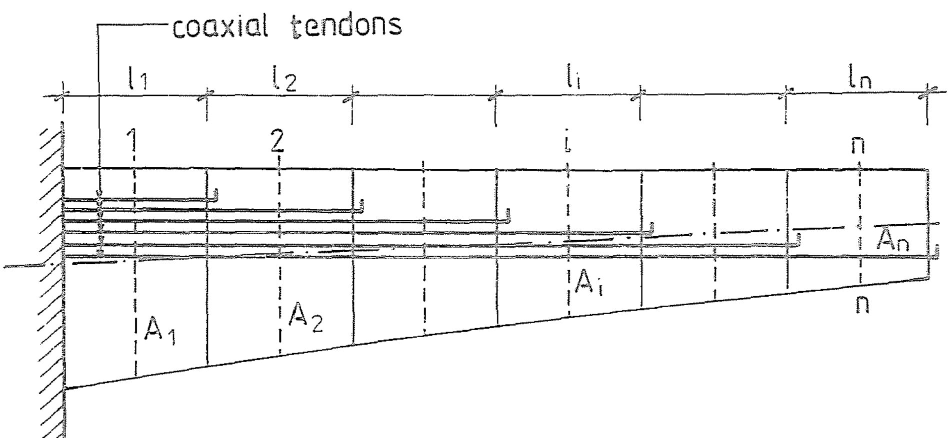

Fig. 1. Elevation of the cantilever

The force method is applied and the forces acting in tendons $1, 2, \ldots, i, \ldots, n$ are to be considered as unknowns.

If segment 1 is completed and the tendon No. 1 is prestressed, the system is statically indeterminate to the first degree and the coefficient of the equation with a single unknown is

$$a_{11} = \int \frac{M_1^2}{EI_1}\mathrm{d}x + \int \frac{N_1^2}{EA_1}\mathrm{d}x + \frac{l_1}{E_pA_{p1}}.$$

Here the first term is the relative displacement at the place where a fictitious cut is made (tendon 1) due to the unit force acting at the same place because of bending moment, the second term is that because of axial force and the third term is the same because of the elongation of the tendon; that is

$$a_{11} = \frac{e_1^2 l_1}{EI_1} + \frac{l_1}{EA_1} + \frac{l_1}{E_pA_{p1}}.$$

This formula is written supposing that the quantities having a role here are considered to be constant along one segment, but as already mentioned, there is no basic alteration in the method, if these changes are taken into account. (For the sake of a better overlook in this paper, it will be reckoned with the values at the mid length of the segments.)

Thus, the symbols are the following:

- E the Young's modulus of concrete
- I_1 the moment of inertia of the cross-section at segment 1
- A_1 the cross-section area at segment 1
- l_1 length of segment 1
- e_1 eccentricity of prestress at segment 1
- E_p Young's modulus of the prestressing tendon
- A_{p1} cross-section area of the tendon (or tendon group) No 1

Let us suppose that the eccentricity of all tendons is the same.(In *Fig. 1* they are only drawn as if they had not the same eccentricity for the sake of possibility of presentation).

After segment 2 is completed, a_{11} is unchanged according to the feature of the structure and logically

$$a_{22} = \frac{e_1^2 l_1}{EI_1} + \frac{e_2^2 l_2}{EI_2} + \frac{l_1}{EA_1} + \frac{l_2}{EA_2} + \frac{l_1 + l_2}{E_pA_{p2}}.$$

In the case of completion of segment i, all unit coefficients in the main diagonal will be according to what mentioned above, and

$$a_{ii} = \frac{e_1^2 l_1}{EI_1} + \frac{e_2^2 l_2}{EI_2} + \cdots + \frac{e_i^2 l_i}{EI_i} +$$

$$+\frac{l_1}{EA_1} + \frac{l_2}{EA_2} + \cdots + \frac{l_i}{EA_i} + \frac{l_1 + l_2 + \cdots + l_i}{E_pA_{pi}},$$

$$a_{ii} = \sum_{k=1}^{i} \left\{ \frac{e_k^2 l_k}{EI_k} + \frac{l_k}{EA_k} \right\} + \frac{\sum_{k=1}^{i} l_k}{E_p A_{pi}}.$$

Let us introduce the following notation

$$a_k = \frac{e_k^2 l_k}{EI_k} + \frac{l_k}{EA_k}; \qquad c_i = \sum_{k=1}^{i} a_k ,$$

$$d_i = \frac{1}{E_p A_{pi}} \sum_{k=1}^{i} l_k; \qquad \mathbb{D} =< d_i >,$$

i.e.

$$a_{ii} = c_i + d_i, \qquad i = 1, 2, \ldots, n.$$

Making use of the definition of the unit coefficients, the arrangement of the structure implies that the off-diagonal elements $a_{ij} (i \neq j)$ of the coefficient matrix $\mathbb{A}=[a_{ij}]$ of the system of equations are

$$a_{ij} = \begin{cases} c_i & \text{if} \quad i < j \\ c_j & \text{if} \quad i > j \end{cases}.$$

Thus the elements a_{ij} can be written as

$$a_{ij} = c_{ij} + \delta_{ij} d_i, \tag{1}$$

where

$$c_{ij} = \begin{cases} c_{\min(ij)} & \text{if} \quad i \neq j \\ c_i & \text{if} \quad i = j \end{cases}$$

and δ_{ij} is the Kronecker delta. Introducing the notation

$$\mathbb{C} = [c_{ij}], \tag{2}$$

obviously

$$\mathbb{A}=\mathbb{C}+\mathbb{D}.$$

The problem leads to the system of equations

$$\mathbb{A}\mathbf{x} = \mathbf{a}_0, \tag{3}$$

where $\mathbf{a}_0$ is the load vector. The solution of the system (3) is known if the elements of the inverse $\mathbb{A}^{-1}$ are given in an explicit form or if a convenient recursive algorithm can be formulated for calculating them.

The Inverse of the Coefficient Matrix Solution of the System of Equations

Let us give the *definition*: A matrix $\mathbb{T}=[t_{ij}]$ is called a one-pair matrix (see e.g. [2], p. 72) if its elements can be expressed in the following form:

$$t_{ij} = \begin{cases} p_i q_j & \text{if} \quad i \le j \\ q_i p_j & \text{if} \quad i \ge j \end{cases}.$$

It is to be seen that the matrix $\mathbb{C}$ defined by (2) is a one-pair matrix with $p_i = c_i, q_i = 1; (i = 1, 2, \ldots, n)$. According to a well-known theorem, the elements of the inverse of a non singular one-pair matrix can be obtained by means of certain recursions [see [2] pp. 300, 301 (5.68)...(5.71)].

It is not difficult to verify that

$$\mathbb{C}^{-1} = \begin{bmatrix} c_1 & c_1 & c_1 & \ldots & c_1 \\ c_1 & c_1+c_2 & c_1+c_2 & \ldots & c_1+c_2 \\ c_1 & c_1+c_2 & c_1+c_2+c_3 & \ldots & c_1+c_2+c_3 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ c_1 & c_1+c_2 & c_1+c_2+c_3 & \ldots & c_1+c_2+\ldots+c_n \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} \frac{1}{c_1}+\frac{1}{c_2} & -\frac{1}{c_2} & 0 & . & . & . & . \\ -\frac{1}{c_2} & \frac{1}{c_2}+\frac{1}{c_3} & -\frac{1}{c_3} & 0 & . & . & . \\ 0 & -\frac{1}{c_3} & \frac{1}{c_3}+\frac{1}{c_4} & -\frac{1}{c_4} & 0 & . & . \\ . & 0 & -\frac{1}{c_4} & . & . & . & . \\ . & . & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & -\frac{1}{c_n} & \frac{1}{c_n} \end{bmatrix}$$

Since the solution of the *Eq.* (3) can be written in the form

$$\mathbf{x} = (\mathbb{C}+\mathbb{D})^{-1}\mathbf{a}_0,$$

the task is to find the inverse of $\mathbb{C}+\mathbb{D}$ when the inverse $\mathbb{C}^{-1}$ is known. In order to get the solution $\mathbf{x}$, i.e. the vector formed by the unknown tendon forces, let $\mathbb{C}$ and $\mathbb{D}$ be factored out to the left and to the right, respectively:

$$\mathbf{x} = (\mathbb{C}+\mathbb{D})^{-1}\mathbf{a}_0 = \mathbb{D}^{-1}(\mathbb{D}^{-1}+\mathbb{C}^{-1})^{-1}(\mathbb{C}^{-1}\mathbf{a}_0). \tag{4}$$

Substituting the identity

$$(\mathbb{D}^{-1}+\mathbb{C}^{-1})^{-1}\mathbb{C}^{-1} =$$

$$= (\mathbb{D}^{-1}+\mathbb{C}^{-1})^{-1}(\mathbb{D}^{-1}+\mathbb{C}^{-1}-\mathbb{D}^{-1}) = \mathbb{I}-(\mathbb{D}^{-1}+\mathbb{C}^{-1})^{-1}\mathbb{D}^{-1}$$

into (4) we get

$$\mathbf{x} = \mathbb{D}^{-1}\mathbf{a}_0 - \mathbb{D}^{-1}(\mathbb{D}^{-1}+\mathbb{C}^{-1})^{-1}\mathbb{D}^{-1}\mathbf{a}_0.$$

Since the matrix $\mathbb{D}^{-1}+\mathbb{C}^{-1}$ to be inverted is a symmetric tridiagonal matrix, its inverse is a one-pair matrix the elements of which can be obtained by a simple recursive algorithm (see [2] p. 300).

Let the inverse of $\mathbb{D}^{-1}+\mathbb{C}^{-1}$ be denoted by $\mathbb{R}=[r_{ij}]$, then

$$(\mathbb{D}^{-1}+\mathbb{C}^{-1})^{-1} = \mathbb{R} = [r_{ij}],$$

where

$$r_{ij} = \begin{cases} u_i v_j & \text{if} \quad i \le j \\ v_i u_j & \text{if} \quad i \ge j. \end{cases}$$

Obviously, one factor of the parameters $u_i, v_i (i = 1,2,3,\dots n)$ can be arbitrarily chosen, so let us substitute $u_1 = 1$. The further parameters $u_i (i = 2,3,\dots,n)$ can be obtained by the recursive algorithm

$$\begin{aligned} u_1 &= 1, \\ u_2 &= c_2\Big(\frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{d_1}\Big), \\ u_{i+1} &= c_{i+1}\left\{\Big(\frac{1}{c_i}+\frac{1}{c_{i+1}}+\frac{1}{d_i}\Big)u_i - \frac{1}{c_i}u_{i-1}\right\}, \qquad i = 2,\dots,n-1. \end{aligned}$$

Introducing the parameter u_0 according to the formula,

$$u_0 = (\frac{1}{c_n}+\frac{1}{d_n})u_n - \frac{1}{c_n}u_{n-1},$$

(u_0 is proportional to the determinant of the tridiagonal matrix to be inverted), we get the recursive algorithm for the parameters v_i:

$$\begin{aligned} v_n &= \frac{1}{u_0}, \\ v_{n-1} &= c_n\Big(\frac{1}{c_n}+\frac{1}{d_n}\Big)\frac{1}{u_0}, \\ v_{n-i} &= c_{n+1-i}\left\{\Big(\frac{1}{c_{n+1-i}}+\frac{1}{c_{n+2-i}}+\frac{1}{d_{n+1-i}}\Big)v_{n+1-i} - \frac{1}{c_{n+2-i}}v_{n+2-i}\right\}. \\ i &= 2,3,\dots,n-1 \end{aligned}$$

In knowledge of u_i and v_i, the unknowns x_i can be obtained in the following form:

$$x_i = \frac{a_{i0}}{d_i} - \frac{v_i}{d_i}\sum_{j=1}^{i}\frac{u_j}{d_j}a_{j0} - \frac{u_i}{d_i}\sum_{j=0}^{n-i-1}\frac{v_{n-j}}{d_{n-j}}a_{n-j,0},$$

$$i = 1, 2, \ldots, n.$$

Conclusion

The importance of the derived result is given by the fact that the technical parameters of the task can be taken arbitrarily. Thus e.g. the cross-section can change, also the length of the segments, the number of prestressing tendons, etc. For structures of different forms of girders, different tendon layouts, different prestressing force and dead load distribution the solution can be given by substitution.

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