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ON INVERSIONS AND CYCLES IN PERMUTATIONS

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Abstract

Permutations consisting of a single cycle are considered. EDELMAN (1987) proved that such permutations contain at least $n-1$ inversions. Moreover, he determined the number of such permutations having exactly $n-1$ inversions: 2^{n-2}. The present paper gives a new proof of the above statements and determines the number of such permutations having exactly $n+1$ inversions.

Keywords: Permutation, inversion, cycle.

Introduction

Suppose that X is a finite set whose elements are ordered. For sake of simplicity, let us choose $X = \{1, 2, \dots, n\}$ and consider them ordered in the usual way. We will consider *permutations* (rearrangements) of X. A permutation σ is the rearrangement $(\sigma(1), \sigma(2), \dots, \sigma(n))$ where this latter sequence contains each element of X exactly once, only their order is different from the usual one. A permutation is often given in its *matrix form*

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

The set of permutations of n elements are denoted by S_n. One can define a group on S_n determining the product of two permutations as the permutation obtained by the consecutive application of the rearrangements. For instance, $(1,3,2,4)(3,2,4,1) =$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}.$$

This group is called the symmetric group (of order n). Permutations have a very important role in mathematics. It is sufficient to remind the reader that the symmetric groups $S_n, (1 \le n)$ contain all finite groups. From the

practical applications, let us mention the so called *order statistics* which is a branch of mathematical statistics.
$\langle c_1, c_2, \ldots, c_k\rangle$ is a *cycle* in the permutation $\sigma \in S_n$ if $\sigma(c_1) = c_2, \sigma(c_2) = c_3, \ldots, \sigma(c_k) = c_1$. The *length* of the cycle is k. The above notation also serves to denote the permutation $\sigma \in S_n$ such that $\sigma(c_1) = c_2, \sigma(c_2) = c_3, \ldots, \sigma(c_k) = c_1$ and $\sigma(a) = a$ for all $a \in X, a \neq c_i (1 \leq i \leq k)$. It is well known (and easy to see) that any permutation $\sigma \in S_n$ can be decomposed into a product of cycles, where each element of X occurs in exactly one cycle and this decomposition is unique (up to their order). For instance,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \langle 1\rangle\langle 2,3\rangle\langle 4\rangle.$$

The number of cycles in the cycle decomposition is denoted by $c(\sigma)$. These concepts have a very clear meaning. If the permutation (arrangement) is applied repeatedly in X then any element remains in its cycle, on the other hand, if it is applied sufficiently many times, any other member of the cycle is obtained.

Another important notion is the number of inversions. We say that i and j are in *inversion* in the permutation σ if $i < j$ and $\sigma(i) > \sigma(j)$. The *number of inversions* in σ is the number of pairs i, j being in inversion. It is denoted by $I(\sigma)$. It can be considered as a measure of the 'anti-orderedness' of the permutation. Its use in the theory of determinants is well known.

It is quite natural to investigate the connection between these two parameters, $c(\sigma)$ and $I(\sigma)$. Given $c(\sigma)$, how small and how large can $I(\sigma)$ be? These questions are answered in EDELMAN (1987). A more general question is to determine $M(n,k,l)$, the number of permutations $\sigma \in S_n$ such that $c(\sigma) = k$ and $I(\sigma) = l$. In the present paper we consider only the special case $m(n,l) = M(n,1,l)$. It is known from EDELMAN (1987) that $m(n,l) = 0$ if $l < n-1$, that is, if the permutation is just one cycle then the number of inversions is at least $n-1$. Moreover, EDELMAN (1987) determines $m(n, n-1)$, as well.

In the present paper we prove the above mentioned two statements by a new method, characterize the cycles with $n+1$ inversions and determine $m(n, n+1)$. (It is easy to show that $m(n,n)$ is zero.)

Minimum number of inversions in a cycle

Our method is inductional and based on the following lemma showing the change in the number of inversions if a cycle is extended by one element into a longer one.

LEMMA 1. *Let* $\rho = \langle c_1, c_2, \ldots, c_{n-1}\rangle$ *be a cycle in* S_{n-1} *and* $\sigma = \langle c_1, c_2, \ldots, c_{n-1}, n\rangle$. *Then*

$$I(\sigma) - I(\rho) = 1 + 2|\{i :\ 1 < i < n, c_1 < c_i, c_{n-1} < c_{i-1}\}|. \tag{1}$$

PROOF: Define the permutation $\pi = \rho\langle n\rangle$. The equality $I(\pi) = I(\rho)$ is obvious. On the other hand, σ can be obtained from π by interchanging n and c_1. To obtain (1) we have to count the increase of the inversions by this change. Since $\pi(c_{n-1}) = c_1$ and $\pi(n) = n$, thus $c_{n-1} < n$ and $c_1 < n$ imply that c_{n-1} and n are not in inversion in π. On the other hand, $\sigma(c_{n-1}) = n$ and $\sigma(n) = c_1$ imply that they are in inversion in σ. This increase is reflected by the term 1 in (1).

The pairs not containing c_{n-1} and n are not influenced by the interchange. Suppose that $x < c_{n-1}$. Then x is in inversion with c_{n-1} in π iff x is in inversion with n in σ. On the other hand, x is in inversion with n in π and it is in inversion with c_{n-1} in σ that is, the number of inversions between such an x and other elements is not changed.

We may suppose $c_{n-1} < x$. Distinguish two subcases. If $c_1 > \pi(x)$ then x is in inversion with c_{n-1} and is not in inversion with n in π, while x is in inversion with n and is not in inversion with c_{n-1} in σ. The number of these types of inversions is not changed either. In the other subcase $c_1 < \pi(x)$ is supposed. In this case x is not in inversion in π neither with c_{n-1} nor with n. However, both relations are inversions in σ. Thus the increase in the number of inversions is the double of the number of such xs. Writing x in the form c_{i-1}, (1) is obtained.

Observe that our lemma implies that the extension of the cycle increases the number of inversions by at least 1. Taking into account that a cycle of length 1 has zero inversions (a cycle of length 2 has 1 inversion), one can prove the following theorem by induction.

THEOREM 1. (EDELMAN *(1987)) If* $\sigma \in S_n, c(\sigma) = 1$ *then* $n - 1 \leq I(\sigma)$.

A cycle can be written in different forms. We call the variant having the smallest number (in our case 1) in the first place the *standard form*. A cycle $\langle c_1, c_2, \ldots, c_n\rangle \in S_n$ is called *unimodal* iff in its standard form $1 = c_1 < c_2 < \ldots < c_m > \ldots > c_{n-1} > c_n$ holds for some m (and then $c_m = n$ is obvious).

THEOREM 2. (EDELMAN *(1987)) If* $\sigma \in S_n, c(\sigma) = 1$ *then* $I(\sigma) = n - 1$ *holds iff* σ *is unimodal.*

PROOF: The previous inductional proof is used to prove the 'only if' part. The 'if' part is easier and actually contained in this proof. Suppose that the statement is true for $n - 1$ and prove it for n. Consider the following (not necessarily standard) form of σ: $\langle c_1, c_2, \ldots, c_{n-1}, n\rangle$. By Lemma 1, $I(\sigma) = n-1$ implies that $\rho = \langle c_1, c_2, \ldots, c_{n-1}\rangle$ must satisfy $I(\rho) = n-2$. By the inductional hypothesis, ρ is unimodal.

Suppose that $c_1 \neq n-1$ and $c_{n-1} \neq n-1$. Choose i satisfying $c_i = n-1$. If $c_{n-1} < c_{i-1}$ then i is in the set given in (1) therefore the increase of the number of inversions is at least 3, $I(\sigma) \geq n+1$ is a contradiction.

In the case $c_{n-1} > c_{i-1}$ take $i+1$. If $c_1 < c_{i+1}$ holds then $i+1$ is in the set given in (1) yielding a contradiction, again. Thus both $c_{n-1} > c_{i-1}$ and $c_1 > c_{i+1}$ can be supposed. As ρ is unimodal, one of the neighbours of $n-1$ must be $n-2$. That is, either c_{i-1} or c_{i+1} is equal to $n-2$. Both cases lead to contradictions.

Therefore, either $c_1 = n-1$ or $c_{n-1} = n-1$ holds. The unimodality implies $c_1 > c_2 > \ldots > c_r < \ldots < c_{n-1}$ for some r in both cases. Consequently, σ is also unimodal.

THEOREM 3. *(Edelman (1987))* $m(n, n-1) = 2^{n-2}$.

PROOF: The unimodal cycles in their standard forms will be enumerated. The place of 1 is fixed, it is the first one in the ordering. We can choose the subset A of elements between 1 and n in the ordering. They are ordered by the natural order and they are followed by the rest of the set ordered backwards. The number of possible subsets A is 2^{n-2}.

The number of n-cycles with $n+1$ inversions

LEMMA 2. *(Edelman (1987)) If $\sigma \in S_n$ satisfies $c(\sigma) = 1$ then $n + I(\sigma)$ is odd.*

PROOF. Use induction and Lemma 1.

This lemma explains $m(n,n) = 0$.

A cycle is *error-unimodal* if its standard form satisfies one of the following conditions:

$$1 = c_1 < c_2 < \ldots < c_p,\ c_{p+1} = c_p - 1 < c_{p+2} < \ldots < c_m > \ldots > c_{n-1} > c_n \tag{3}$$

for some $1 < p < m \leq n$,

$$1 = c_1 < c_2 < \ldots < c_m > \ldots > c_p,\ c_{p+1} = c_p + 1 > c_{p+2} > \ldots > c_{n-1} > c_n \tag{4}$$

for some $1 < m < p < n$,

$$1 = c_1 < c_2 < \ldots < c_p,\ c_{p+1} = c_p - 2 < c_{p+2} < \ldots < c_m > \ldots > c_{n-1} > c_n \tag{5}$$

where $1 < p < m < n$ and $c_p - 1$ is in the descending part, that is, equal to some c_r, $(m < r \leq n)$,

$$1 = c_1 < c_2 < \ldots < c_m > \ldots > c_p,\ c_{p+1} = c_p + 2 > c_{p+2} > \ldots > c_{n-1} > c_n \tag{6}$$

where $1 < m < p < n$ and $c_p + 1$ is in the increasing part, that is, equal to some $c_r (1 < r < m)$.

In the proof of the following theorem we need Lemma 1 for standard forms, that is, when n is deleted from somewhere the middle. The proof of this variant is obvious.

LEMMA 3. *Let $\rho = \langle c_1, c_2, \dots, c_{m-1}, c_{m+1}, \dots, c_n \rangle$ be a cycle in S_{n-1} in standard form (that is, $c_1 = 1$) and $\sigma = \langle c_1, c_2, \dots, c_{m-1}, c_m, c_{m+1}, \dots, c_n \rangle$ where $c_m = n$. Then*

$$I(\sigma) - I(\rho) = 1 + 2|\{i : \ 1 < i < n, c_{m+1} < c_i, c_{m-1} < c_{i-1}\}|. \tag{7}$$

THEOREM 4. *If $\sigma \in S_n$ satisfies $c(\sigma) = 1$ then $I(\sigma) = n + 1$ iff σ is error-unimodal.*

PROOF: Use induction on n to prove the 'only if' part. The 'if' part is easier and actually contained in this proof. Suppose that the statement is true for $n - 1$ and prove it for n. (The first reasonable value is $n = 4$.) Delete n from $\sigma = \langle c_1, c_2, \dots, c_{m-1}, c_m, c_{m+1}, \dots, c_n \rangle$. The reduced cycle $\rho = \langle c_1, c_2, \dots, c_{m-1}, c_{m+1}, \dots, c_n \rangle$ either satisfies $I(\rho) = n$ or $I(\rho) = n - 2$. We treat these two cases separately.

1. $I(\rho) = n$. ρ has to be of the form (3)-(6). On the other hand, the set in (7) has to be empty by Lemma 3.

Suppose that (3) is valid:

$$1 = c_1 < c_2 < \dots < c_p, c_{p+1} = c_p - 1 < c_{p+2} < \dots < c_m > \dots > c_{n-1}$$

for some $1 < p < m \leq n - 1$. Let n be between c_j and c_{j+1} in σ.

If $m < j$ then $c_j < c_m$ and $c_{j+1} < c_{m+1}$ holds, m is in the set in (7), a contradiction. If $j < m - 1$ then $c_{j+1} < c_m$ holds. $c_j < c_{m-1}$ also holds, unless $j = p = m - 2$. $m - 1$ is in the set in (7), a contradiction, again. If $j = p = m - 2$ and $c_p < n - 2$ then $c_{m+1} = n - 2$ therefore we have $c_j < c_m, c_{j+1} < c_{m+1}$. Here m gives the contradiction.

It is easy to see that σ is error-unimodal in the remaining cases: 1) $j = m$, 2)$j = m - 1$, 3)$j = m - 2 = p$ and $c_{m-2} = n - 2$. In the last case, σ is of type (6).

The case of (4) is symmetric to the previous one.

Suppose that (5) is valid:

$$1 = c_1 < c_2 < \dots < c_p, c_{p+1} = c_p - 2 < c_{p+2} < \dots < c_m > \dots > c_{n-1} > c_n,$$

where $1 < p < m < n$ and $c_p - 1$ is in the descending part, that is, equal to some $c_r (m < r \leq n)$. Let n be between c_j and c_{j+1} in σ.

If $m < j$ then $c_j < c_m$ and $c_{j+1} < c_{m+1}$ hold, m is in the set in (7), a contradiction. If $j < m-1$ then $c_{j+1} < c_m$ holds. $c_j < c_{m-1}$ also holds, unless $j = p = m-2$. $m-1$ is in the set in (7), a contradiction, again. If $j = p = m-2$ then we have $c_j < c_m$ and $c_{j+1} = c_p - 2 < c_p - 1 = c_r \leq c_{m+1}$. Here m gives the contradiction.

It is easy to see that σ is error-unimodal in the remaining cases: 1. $j = m$, 2. $j = m-1$.

The case of (6) is symmetric to the previous one.

2. $I(\rho) = n-2$. In this case ρ must be unimodal. The set in (7) has to contain one element.

Let ρ be a cycle $\langle c_1, c_2, \ldots, c_{n-1}\rangle$ where $1 = c_1 < c_2 < \ldots < c_m > \ldots > c_{n-1}$. Suppose that n is between c_j and c_{j+1} in σ.

Let $m+1 < j$. Then $c_j < c_m, c_{m+1}$ and $c_{j+1} < c_{m+1}, c_{m+2}$ imply that the set in (7) contains at least two elements (m and $m+1$), this is a contradiction. The case when $j < m-2$ is analogous.

If $j = m-2$ then two subcases are distinguished. 1) $c_{m-1} < c_{m+1}$ implies that the set in (7) contains both m and $m+1$. 2) $c_{m-1} > c_{m+1}$ implies $c_{m-1} = n-2$ and then σ is error-unimodal of type (4).

If $j = m-1$ or $j = m$ then the set in (7) is empty, this is a contradiction.

The case $j = m+1$ is analogous to the case $j = m-1$.

THEOREM 5. *$m(n, n+1) = (3n-10)2^{n-4}$ if $4 \leq n$.*

PROOF: By Theorem 4, it is sufficient to enumerate the error-unimodal cycles. An error-unimodal cycle of type (3) is determined by the choice of c_p and the set of elements in the increasing part. It is obvious that $3 \leq c_p \leq n-1$, that is, c_p can have $n-3$ different values. Then the places of $1, c_p, c_p - 1, n$ are fixed, 2^{n-4} subsets can be chosen from the remaining elements. The number of error-unimodal cycles of type (3) is $(n-3)2^{n-4}$. Type (4) gives the same result. In the case of (5), c_p has only $n-4$ different values. On the other hand, in this case the places of $1, c_p, c_p-1, c_p-2, n$ are all determined, thus the number of error-unimodal cycles of type (5) is $(n-4)2^{n-5}$. The same is true for (6). Adding up the results in the four cases, the statement of the theorem is obtained.

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