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# SOME RECENT APPLICATIONS OF THE KERNEL FUNCTION OF GENERALIZED WEYL FRACTIONAL INTEGRALS

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## Abstract

Quite recently, connections of unusual type have been discussed by the author between the so-called 'fractional calculus' as a new branch of analysis and strong summation processes, furthermore, between fractional integration and certain number theoretic approximation methods. In the following, two different aspects of these inherences are considered: I. a new verification for the powerful method of $(D)$-summation in case of trigonometric series is given; II. such a generalization of the famous Franel theorem on Riemann's hypothesis (1924) is presented which shows the deeper background of the topic in the field of Diophantine approximations.

*Keywords:* fractional integration, Fourier analysis, summation methods, zeta-functions, Diophantine approximations.

## Introduction

About thirty years ago, the author published a new theory of generalized integro-differential operators (called '$W_s$-limits'), widening H. WEYL's concept of fractional integration. (See MIKOLÁS, 1959.) As it is well known, the Weyl fractional integral of order $\theta > 0$ is defined by

$$f_\theta(x) = \Gamma(\theta)^{-1} \int_{-\infty}^{x} f(t)(x-t)^{\theta-1} dt \qquad (0 < \theta < 1, \quad 0 < x < 1), \quad (1\text{-}1)$$

where $f$ denotes a Lebesgue integrable function of period 1 with $\int_0^1 f(t)dt = 0$. (Cf. WEYL, 1917). (1-1) is a pendant of the classical RIEMANN–LIOUVILLE fractional integral over $(x_0, x)$

$$_{x_0}I_x^\nu f = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} f(t)(x-t)^{\nu-1} dt \qquad (\operatorname{Re}\nu > 0), \qquad (1\text{-}2)$$

which can be regarded as a natural extension to non-integral order $\nu$ of Cauchy's integral solution for the initial-value problem

$$y^{(m)}(x) = f(x) \quad (x_0 < x < x_1); \qquad (1\text{-}3)$$
$$y(x_0) = y'(x_0) = \cdots = y^{(m-1)}(x_0) = 0,$$

and has become since the turn of this century — together with (1-1) — an essential expedient of mathematics, physics and technical sciences. (Cf. MIKOLÁS, 1975.)

In the above-mentioned theory of *generalized Weyl fractional integrals of complex order* the following fundamental facts hold: for the '$W_s$-integral' in question we have the representation

$$f_s(x) = \int_0^1 f(x-t)[\mathcal{Z}_s(t) - \mathcal{Z}_s(x)]dt \quad (\operatorname{Re} s > 1) \qquad (1\text{-}4)$$

and the so-called *kernel function*[1] $\mathcal{Z}_s(u)$ occurring here may be written in the form

$$\mathcal{Z}_s(u) = \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^s} \cos\left(2n\pi u - \frac{\pi s}{2}\right) \quad (u \text{ real, non-integer}); \qquad (1\text{-}5)$$

the corresponding fractional ('$W_s$-') derivatives are received on the basis of the holomorphy of (1-4) as function of $s$. Herewith — surprisingly — the theory is closely connected with properties of an important class of higher special functions by relation:

$$\mathcal{Z}_s(u) = \Gamma(s)^{-1}\zeta(1-s, \langle u \rangle), \qquad (1\text{-}6)$$

$\langle u \rangle = u - [u]$ denoting the 'fractional part' of $u$.

On the right of (1-6), $\zeta(s,u)$ means the *zeta-function of Hurwitz*, familiar in number theory, defined for $\operatorname{Re} s > 1$, $u$ real, $u \neq 0, -1, -2, \ldots$ as the sum $\sum_{m=0}^{\infty} (m+u)^{-s}$ and for other $s \neq 1$ by analytic continuation with respect to the complex variable $s$. (We know that $\zeta(s,1) = \zeta(s)$ is the *Riemann zeta-function.*)

Remark that the *main properties of the kernel* $\mathcal{Z}_s$ are:

[1]For the terms *'kernel function'* and *'singular integral'* we refer e.g. to HARDY-ROGOSINSKI (1944) and FEJÉR (1949).

(I) For any fixed $u \in (0,1)$, $\mathcal{Z}_s(u)$ is an entire function of $s$, i. e. it is regular everywhere on the $s$-plane; furthermore for any fixed $s$ and non-integral real $u$, the formula $\frac{\partial}{\partial u}\mathcal{Z}_{s+1}(u) = \mathcal{Z}_s(u)$ holds.

(II) Since $\mathcal{Z}_0(u) \equiv -1$, $\mathcal{Z}_p(u) = -B_p(u)$ $(p = 1, 2, \dots)$, the kernel $\mathcal{Z}_s(u)$ can be regarded as a common generalization of the Bernoulli polynomials.[2]

(III) We have the relations:

$$(-1)_s = \mathcal{Z}_s(u) \quad \text{for } u \in (0,1), \quad s \text{ arbitrary}; \tag{1-7}$$

$$\int_0^1 \mathcal{Z}_{s_1}(u)\mathcal{Z}_{s_2}(x-u)du = \mathcal{Z}_{s_1+s_2}(x) \tag{1-8}$$

for $x \in (0,1)$ and $\operatorname{Re} s_1, \operatorname{Re} s_2 > 0$.

## Application of $\mathcal{Z}_s(u)$ to New Summation Methods for Ordinary Fourier Series

In an international congress report held lately (see MIKOLÁS, 1990a), a general idea due to the author has been discussed in detail, namely that the fractional integral (1-2) can be useful for summation of series not only as *function of the variable* $x$ (this way was successfully followed in a famous monograph of HARDY–RIESZ (1915), introducing the so-called 'typical means'), but there is also another, alike so fruitful possibility: the application of fractional integrals as *functions of the order* $\nu$. The most comprehensive summation process thus obtained is now called in the literature $(M)$-*summation*, and it is defined for any series of functions $\sum_{n=1}^{\infty} \varphi_n(x)$ so that the fractional integrals occurring on the right of (2-1) exist, by the formula:

$$^{(M)}\sum_{n=1}^{\infty} \varphi_n(x) = \lim_{\nu \to +0} \sum_{n=1}^{\infty} {}_{x_0}I_x^{\nu}\varphi_n. \tag{2-1}$$

Let us stress that the method (2-1) is specially fit e. g. for summing trigonometric Fourier series of the type

$$\sum_{n=-\infty}^{\infty} c_n e^{2n\pi i x}, \qquad c_n = \int_0^1 f(t)e^{-2n\pi i t}dt. \tag{2-2}$$

[2] We denote by $B_p(u)$ $(p = 0, 1, 2, \dots)$ the coefficients in the expansion $we^{uw}(e^w - 1)^{-1} = B_0(u) + B_1(u)w + B_2(u)w^2 + \dots$ $(|w| < 2\pi)$.

In this particularly important case it is suitable to split the definition (2-1) so to say into two variants:

$$^{(M\pm)}\sum_{n=-\infty}^{\infty} c_n e^{2n\pi i x} = c_0 + \lim_{\theta\to+0} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n(\pm 2n\pi i)^{-\theta} e^{2n\pi i x} \tag{2-3}$$

and to use the terms $(M_+)$-*summation* and $(M_-)$-*summation*, respectively. (Cf. MIKOLÁS, 1960a and 1960b.)

By means of (2-3), considering the closed (integral) form of the sums on the right and utilizing also some properties of the occurring kernel function $\mathcal{Z}_s$, we can obtain such results on efficiency of the $(M\pm)$-methods which highly exceed the corresponding ones in the theory of any classical summation process. Nevertheless, we want to investigate now another, similar but simpler method: the *Dirichlet* [briefly $(D)$-] *summation*. Its definition for an arbitrary series $\sum\limits_{k=0}^{\infty} A_k$ is:

$$^{(D)}\sum_{k=0}^{\infty} A_k = A_0 + \lim_{\vartheta\to+0} \sum_{k=1}^{\infty} A_k \cdot k^{-\vartheta}, \tag{2-4}$$

where we have to assume the convergence of the right-hand auxiliary series for $\vartheta > 0$ small enough.[3] We shall see that this way implies also the kernel $\mathcal{Z}_s$, but enables us to argue in the most direct manner.

We need an identity and two elementary estimates which can be deduced easily from the preliminaries about the Hurwitz zeta-function:

$$\sum_{n=1}^{\infty} n^{-\vartheta} \cos n\tau = \tag{2-5}$$

$$= \frac{1}{4}(2\pi)^{\vartheta}\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}\left[\zeta\left(1-\vartheta, \frac{\tau}{2\pi}\right) + \zeta\left(1-\vartheta, 1-\frac{\tau}{2\pi}\right)\right]$$
$$(0 < \vartheta < 1, \quad 0 < \tau \le 2\pi),$$

$$|\zeta(\Theta, x)| \le x^{-\Theta} + (1-\Theta)^{-1} + 1 \qquad (0 < \Theta < 1, \quad 0 < x \le 1), \tag{2-6}$$

$$\left|\sum_{n=1}^{N} n^{-\vartheta}\cos n\tau\right| < \left(1 + \frac{1}{1-\vartheta}\right)\pi^{1-\vartheta}\left[\tau^{\vartheta-1} + (2\pi-\tau)^{\vartheta-1}\right] \tag{2-7}$$
$$(N \ge 2, \quad 0 < \tau < 2\pi).$$

[3]The process (2-4) was firstly applied to trigonometric series by the author in (MIKOLÁS, 1960–61). For further special literary references see (ZELLER, 1958).

THEOREM 1. *The trigonometric Fourier series*

$$\left.\begin{aligned}&\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx)\\ \textit{with}\quad&\\ &\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt, \quad \alpha_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos ntdt,\\ &\beta_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin ntdt\end{aligned}\right\} \tag{2-8}$$

*of a bounded, $2\pi$-periodic function $f$ is $(D)$-summable at a point $x$ if and only if the limit*

$$f\,[[x]] = \lim_{\vartheta\to+0}\left[\vartheta\int_0^{\delta}\varphi(x,t)t^{\vartheta-1}\right]dt, \varphi(x,t) = \frac{1}{2}[f(x+t)+f(x-t)] \tag{2-9}$$

*($\delta > 0$, arbitrarily small) exists. The value (2-9) does not depend on $\delta$, and yields the $(D)$-sum of (2-8) at $x$, provided that it exists.*

*In particular, $f\,[[x]] = \frac{1}{2}[f(x+0)+f(x-0)]$ holds at any point $x$ where both unilateral limits of the function exist; furthermore, the $(D)$-summability is uniform in each closed continuity interval of the function (including bilateral continuity at the end-points).*

*In our case, the domain of effectiveness of the $(D)$-method is greater than that of any Cesàro method or of the Abel–Poisson summation.*

PROOF: 1° Having in mind the definition (2-4) of the $(D)$-method, let us form the auxiliary series

$$\alpha_0 + \sum_{n=1}^{\infty} n^{-\vartheta}(\alpha_n \cos nx + \beta_n \sin x) \quad (0 < \vartheta < 1, \quad 0 \le x < 2\pi), \tag{2-10}$$

$\alpha_n$, $\beta_n$ denoting the ordinary Fourier coefficients on $[0, 2\pi]$ of a bounded $(L)$-integrable function $f$ (with the period $2\pi$).

By the above, the closed integral expression of (2-10) may be written:

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)[1+2\sum_{n=1}^{\infty} n^{-\vartheta}\cos n(x-t)]dt =$$

$$=\frac{1}{\pi}\int_0^{2\pi}\varphi(x,u)du+\frac{1}{\pi}\int_0^{2\pi} f(x-\tau)\sum_{n=1}^{\infty} n^{-\vartheta}\cos n\tau d\tau =$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\varphi(x,u)du+\frac{1}{2}(2\pi)^{\vartheta-1}\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}\cdot$$

$$\cdot\int_0^{2\pi} f(x-\tau)\left[\zeta\left(1-\vartheta,\frac{\tau}{2\pi}\right)+\zeta\left(1-\vartheta,1-\frac{\tau}{2\pi}\right)\right]d\tau$$

which by

$$\int_0^{2\pi} f(x-\tau)\left[\zeta\left(1-\vartheta,\frac{\tau}{2\pi}\right)+\zeta\left(1-\vartheta,1-\frac{\tau}{2\pi}\right)\right]d\tau =$$

$$=\int_0^{2\pi}[f(x-v)+f(x+v)]\zeta\left(1-\vartheta,\frac{v}{2\pi}\right)dv$$

takes the form

$$J=\frac{1}{2\pi}\int_0^{2\pi}\varphi(x,v)Z_\vartheta(v)dv \tag{2-11}$$

with the kernel

$$Z_\vartheta(v)=1+(2\pi)^{\vartheta}\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\mathcal{Z}_\vartheta\left(\frac{v}{2\pi}\right). \tag{2-12}$$

[Cf. (1-6).]

Regarding the reverse of order of the integration and summation, we have to stress the following: 1. the existence of the integral (2-11) is assured by the boundedness of $f$ and by the fact that the sum of the series $\sum_{n=1}^{\infty} n^{-\vartheta}\cos n\tau$ as a function of $\tau$ belongs to $L(0,2\pi)$; 2. the estimate (2-7) for the partial sums $\sum_{n=1}^{N} n^{-\vartheta}\cos n\tau$ justifies the termwise integration carried out; 3. this involves simultaneously the convergence of the series (2-10) for every $\vartheta$ and $x$ in consideration.

Taking the properties of $\zeta(\Theta, u)$ into account, we see that (2-11) is a so-called *singular integral* with one single singular (exceptional) point at $v = 0$. Namely the kernel function $Z_\vartheta(v)$ (for any fixed $\vartheta > 0$) becomes infinite in order $v^{\vartheta-1}$ as $v \to +0$, but it is *continuous and monotonously decreasing* at every $v \in (0, 2\pi)$.

Actually, the circumstance will be most important for us that after subtracting an appropriate term bearing the 'singularity' at $v = 0$, the remaining part of the kernel function $Z_\vartheta$ tends uniformly to 0 in $0 \le v \le 2\pi$ as $\vartheta \to +0$. More precisely, by the definition of $\zeta(s, u)$ and using elementary properties of the gamma-function, we can write

$$\left.\begin{aligned} &\left| Z_\vartheta(v) - 2\pi\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}v^{\vartheta-1}\right| \le \\ &\le \left|1 - (2\pi)^\vartheta\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta+1)^{-1}\right| + \\ &+ (2\pi)^\vartheta\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\frac{\vartheta}{\Gamma(\vartheta+1)}; \end{aligned}\right\} \tag{2-13}$$

and both terms of the last bound tend to 0 with $\vartheta$, independently of $v$.

2° Let now split the integral (2-11) into three parts:

$$\left.\begin{aligned} J = &\frac{1}{2\pi}\int_0^{2\pi}\varphi(x,v)\left[Z_\vartheta(v) - 2\pi\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}v^{\vartheta-1}\right]dv + \\ &+\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}\int_0^{\delta}\varphi(x,v)v^{\vartheta-1}dv + \\ &+\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}\int_\delta^{2\pi}\varphi(x,v)v^{\vartheta-1}dv = J_1 + J_2 + J_3 \end{aligned}\right\} \tag{2-14}$$

$\delta$ denoting a fixed positive number $< 1$.

As far as the first term is concerned, with any given $\varepsilon > 0$ a number $\vartheta'_\varepsilon < 1$ can be associated such that

$$|J_1| \le \frac{1}{2\pi}\int_0^{2\pi}|\varphi(x,v)|\cdot\varepsilon dv = \varepsilon\cdot K \qquad (\vartheta < \vartheta'_\varepsilon), \tag{2-15}$$

where $K = \sup_{t\in[0,2\pi]} f(t)$.

On the other hand, using again the gamma-function we get:

$$\left|J_2 - \vartheta \int_0^\delta \varphi(x,v)v^{\vartheta-1}dv\right| =$$

$$= \vartheta\left|\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta+1)^{-1} - 1\right|\left|\int_0^\delta \varphi(x,v)v^{\vartheta-1}dv\right| < \tag{2-16}$$

$$< \left|\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta+1)^{-1}\right| \cdot K\vartheta\int_0^1 v^{\vartheta-1}dv =$$

$$= K\left|\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta+1)^{-1} - 1\right| < K\varepsilon$$

provided that $\vartheta < \vartheta''_\varepsilon$.

Finally, there exists a number $\vartheta'''_\varepsilon > 0$ such that

$$|J_3| < \left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta)^{-1}\cdot K(2\pi)^{\vartheta-1}\int_\delta^{2\pi}\left(\frac{v}{2\pi}\right)^{-1}dv < \tag{2-17}$$

$$< \vartheta\left(\cos\frac{\pi\vartheta}{2}\right)^{-1}\Gamma(\vartheta+1)^{-1}\cdot K\cdot 2\pi\log(2\pi/\delta) < K\varepsilon,$$

if only $\vartheta < \vartheta'''_\varepsilon$.

Summing up, (2-14)–(2-17) yield together

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(x,v)Z_\vartheta(v)dv - \vartheta\int_0^\delta \varphi(x,v)v^{\vartheta-1}dv\right| < 3K\varepsilon$$

for $\vartheta$ sufficiently small; this is equivalent to the statement that the limits

$$\lim_{\vartheta\to+0}\left[\frac{1}{2\pi}\int_0^{2\pi}\varphi(x,v)Z_\vartheta(v)dv\right], \quad \lim_{\vartheta\to+0}\left[\vartheta\int_0^\delta \varphi(x,v)v^{\vartheta-1}dv\right]$$

can exist only simultaneously, and in case of existence they are equal.

3° Assuming that both of the limits $f(x+0)$ and $f(x-0)$ exist, we obtain for $0 < \eta < \delta < 1$:

$$\left|\vartheta \int_0^\delta \varphi(x,v) v^{\vartheta-1} dv - \frac{1}{2}[f(x+0)+f(x-0)]\right| \le$$

$$\le \frac{\vartheta}{2} \int_0^\delta [|f(x+v)-f(x+0)| + |f(x-v)-f(x-0)|] v^{\vartheta-1} dv +$$

$$+\frac{1}{2}|f(x+0)+f(x-0)| \left|\vartheta \int_0^\delta v^{\vartheta-1} dv - 1\right| \le$$

$$\le \frac{1}{2}\left\{\sup_{v\in[0,\eta]}[|f(x+v)-f(x-v)|] + \sup_{v\in[0,\eta]}[|f(x-v)-f(x-0)|]\right\} +$$

$$+\frac{\vartheta}{2}\int_\eta^\delta [|f(x+v)-f(x+0)| + |f(x-v)-f(x-0)|] v^{-1} dv +$$

$$+\frac{1}{2}|f(x+0)+f(x-0)|(1-\delta^\vartheta).$$

The last upper bound becomes plainly as small as we please, if first $\eta$, next (after fixing $\eta$) the number $\vartheta$ is chosen small enough. Since the bounds in (2-15)–(2-17) are independent of $x$, also the assertion on uniform summability follows.

4° In order to show that the $(D)$-method is more effective than any Cesàro or the Abel–Poisson process, we refer to the well-known fact that the divergent series $\sum_{n=1}^{\infty} n^{-(1+i\tau)}$ $(\tau \neq 0)$, by a Tauberian theorem of HARDY and LITTLEWOOD, is summable by *none* of the methods just mentioned. Nevertheless, this series is plainly summable in the (2-4) sense, because the continuity of $\zeta(s)$ for $s \neq 1$ implies

$$\lim_{\vartheta\to+0} \sum_{n=1}^{\infty} n^{-(1+i\tau)} \cdot n^{-\vartheta} = \lim_{\vartheta\to+0} \zeta(1+\vartheta+i\tau) = \zeta(1+i\tau). \tag{2-18}$$

Thus the verification of the theorem is completed.

## Connection of the Integro-Differential Operator $\mathcal{Z}_s$ with Diophantine Approximations and the Riemann Hypothesis

Let us denote by $\langle x\rangle$, as earlier, the difference $x - [x]$, i.e. the so-called 'fractional part' of a real number $x$. According to a classical theorem of

KRONECKER (1884), which is of fundamental significance in the theory of *Diophantine approximations*, the sequence $\langle nx \rangle$ $(n = 1, 2, \dots)$ lies everywhere densely on the real line in case of any fixed *irrational* $x$; furthermore, these points are at the same time *uniformly distributed* modulo 1 in H. WEYL's sense. (See e.g. WEYL, 1916.)

After a further important result of SIERPINSKI, namely that

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \langle nx \rangle = \frac{1}{2}, \quad \text{i.e.} \quad \sum_{n=1}^{N} B_1(\langle nx \rangle) = o(N) \tag{3-1}$$

for every fixed irrational $x$, since the twenties, numerous applications of Diophantine (ordinary or integral) mean estimates relating to *Bernoulli polynomials* have been found in number theory, analysis, television and radio technology. (Cf. e.g. HARDY–LITTLEWOOD, 1922a, 1922b, GÁL, 1949; GÁL–KOKSMA, 1950; MIKOLÁS, 1957, 1960c, 1990b; MORDELL, 1958; VAN DER POL, 1953).

This situation and the fact that recently the kernel function $\mathcal{Z}_s(u)$ turned out to be a natural extension of all Bernoulli functions $B_r(\langle nx \rangle)$ together (see the introduction), suggested looking for deeper connections between the 'fractional' operator $\mathcal{Z}_s$ and the theory of Diophantine approximations. In the sequel, we shall deal with such a contribution to the problem which concerns *Riemann's* famous *hypothesis* (1859): each complex zero of the function $\zeta(s)$ has the real part $1/2$.

First of all, we recall a few concepts and theorems from the analytic theory of numbers. Let $\mathcal{M}(N) = \sum_{n=1}^{N} \mu(n)$ denote the well-known summatoric Möbius function, $\Phi(N) = \sum_{n=1}^{N} \varphi(n)$ the summatoric pendant of Euler's function. Then $\Phi(N)$ gives simultaneously the number of all fractions (rational numbers) $h/k$ with $0 < h \le k \le N$, $(h, k) = 1$, $k = 1, 2, \dots, N$ in ascending order, i.e. of the so-called *Farey series of order N*. The usual notation for the $\nu$-th term of this sequence is $\varrho_\nu^N$ $(\nu = 1, 2, \dots, \Phi(N))$.

A classical theorem of LITTLEWOOD (1912) which has been later strongly generalized by MIKOLÁS (1949, 1950, 1951a, 1951b) asserts that the validity of the estimate

$$\mathcal{M}(N) = \sum_{\nu=1}^{\Phi(N)} \cos 2\pi \varrho_\nu^{(N)} = O\left(N^{\frac{1}{2}+\varepsilon}\right), \quad \forall \varepsilon > 0 \tag{3-2}$$

is equivalent to the Riemann hypothesis. On the other hand, we have the nice theorem of FRANEL (1924) saying that Riemann's hypothesis is true if and only if

$$\mathcal{Q}(N) = \sum_{\nu=1}^{\Phi(N)} \left( \varrho_\nu^{(N)} - \frac{\nu}{\Phi(N)} \right)^2 = O\left(N^{-1+\varepsilon}\right), \quad \forall \varepsilon > 0. \tag{3-3}$$

We remark at once that the proof of Franel's theorem is based on an important expedient of the theory of Diophantine approximations, a formula due to Landau:

$$\int_0^1 \left( \langle au \rangle - \frac{1}{2} \right) \left( \langle bu \rangle - \frac{1}{2} \right) du = \frac{1}{12} \frac{(a,b)}{\{a,b\}} = \frac{(a,b)^2}{12ab}, \tag{3-4}$$

where $a$, $b$ are natural numbers and $(a,b)$, $\{a,b\}$ denote the greatest common divisor and the least common multiple of this couple, resp.

For our purposes, it is also essential that *Franel's sum* (3-3) has an alternative representation (cf. e.g. LANDAU, 1927, pp. 172–173):

$$\mathcal{Q}(N) = \frac{1}{\Phi(N)} \left\{ \int_0^1 \left[ \sum_{n=1}^{N} \left( \langle nx \rangle - \frac{1}{2} \right) \mathcal{M}\left(\frac{N}{n}\right) \right]^2 dx - \frac{1}{12} \right\}, \tag{3-5}$$

which indicates by the occurrence of $B_1(\langle nx \rangle)$ on the right explicitly the 'Diophantine approximatic' background of $\mathcal{Q}(N)$. So we are led to the idea: a strong generalization of the square-integral in (3-5) with the kernel function $\mathcal{Z}_s(u)$ instead of $B_1(\langle u \rangle)$, i.e. the study of

$$\mathcal{H}_s(N) = \int_0^1 \left[ \sum_{n=1}^{N} \mathcal{Z}_s(nx) \mathcal{M}\left(\frac{N}{n}\right) \right]^2 dx \tag{3-6}$$

could yield maybe a corresponding extension of Franel's result (3-3). The conjecture is correct, since

THEOREM 2. *The Riemann hypothesis is true if and only if in the case of any fixed $\varepsilon > 0$ for $s > 1/2$ we have the relation*

$$\mathcal{H}_s(N) = O\left(N^{1+\varepsilon}\right). \tag{3-7}$$

PROOF: 1° Suppose that Riemann's hypothesis holds. Then, by the above-mentioned theorem of Littlewood, to any fixed $\varepsilon > 0$ there exists a $C = C(\varepsilon)$ positive constant for which

$$|\mathcal{M}(N)| < C(\varepsilon) N^{\frac{1}{2}+\frac{\varepsilon}{2}} \quad (N = 1, 2, \dots). \tag{3-8}$$

On the other hand, an extension of (3-4) according to the author yields (cf. MIKOLÁS, 1957, p. 46; and 1960c, p. 159):

$$\int_0^1 \mathcal{Z}_s(au)\mathcal{Z}_s(bu)du = \frac{2\zeta(2s)}{(2\pi)^{2s}}\left(\frac{(a,b)}{\{a,b\}}\right)^2 \quad \left(s > \frac{1}{2}\right). \tag{3-9}$$

So, on the basis of (3-6) and (3-8), (3-9), we can write with $K_s = 2\zeta(2s)(2\pi)^{-2s}$:

$$|\mathcal{H}_s(N)| = \left|\sum_{a,b=1}^{N} \mathcal{M}\left(\frac{N}{a}\right)\mathcal{M}\left(\frac{N}{b}\right)\int_0^1 \mathcal{Z}_s(au)\mathcal{Z}_s(bu)du\right| \leq$$

$$\leq K_s \sum_{a,b=1}^{N}\left|\mathcal{M}\left(\frac{N}{a}\right)\right|\left|\mathcal{M}\left(\frac{N}{b}\right)\right|\frac{(a,b)^s}{\{a,b\}^s} \leq$$

$$\leq K_s C(\varepsilon)^2 \sum_{a,b=1}^{N}\left(\frac{N}{a}\right)^{\frac{1+\varepsilon}{2}}\left(\frac{N}{b}\right)^{\frac{1+\varepsilon}{2}} \cdot \frac{(a,b)^{2s}}{(ab)^s} =$$

$$= K_s C(\varepsilon)^2 N^{1+\varepsilon}\sum_{a,b=1}^{N}\frac{(a,b)^{2s}}{(ab)^{s+\frac{1+\varepsilon}{2}}};$$

and hence, using the notations $(a,b) = c$; $a = \alpha c$, $b = \beta c$:

$$\frac{|\mathcal{H}_s(N)|}{K_s C(\varepsilon)^2 N^{1+\varepsilon}} \leq \sum_{\substack{\alpha,\beta,c=1\\(\alpha,\beta)=1}}^{\infty}\frac{c^{2s}}{(\alpha c\cdot\beta c)^{s+\frac{1+\varepsilon}{2}}} \leq \sum_{\alpha,\beta,c=1}^{\infty}\frac{1}{\alpha^{s+\frac{1+\varepsilon}{2}}\cdot\beta^{s+\frac{1+\varepsilon}{2}}\cdot c^{1+\varepsilon}}.$$

Since the triple series in the last term is plainly convergent, if $s + \frac{1+\varepsilon}{2} > 1+\varepsilon > 1$, for every $s > 1/2$ we obtain (3-7).

$2^\circ$ Conversely, assume that in the case of each $s > 1/2$, to any given $\varepsilon > 0$ a number $N_0 = N_0(\varepsilon)$ and a constant $\Lambda = \Lambda(\varepsilon)$ can be found for which we have the inequality

$$|\mathcal{H}_s(N)| < \Lambda(\varepsilon)N^{1+\varepsilon} \quad (N \geq N_0(\varepsilon)).$$

Then putting $s = 1$, we get specially that for any fixed $\varepsilon > 0$, and at suitable choice of certain constants $N_0 = N_0(\varepsilon)$, $\Lambda = \Lambda(\varepsilon)$, it holds [cf. (3-4) and the positivity of the integrand]:

$$|\mathcal{H}_1(N)| = \int_0^1\left[\sum_{n=1}^{N}\left(\langle nu\rangle - \frac{1}{2}\right)\mathcal{M}\left(\frac{N}{n}\right)\right]^2 du =$$

$$= \frac{1}{12}\sum_{a,b=1}^{N}\mathcal{M}\left(\frac{N}{a}\right)\mathcal{M}\left(\frac{N}{b}\right)\frac{(a,b)^2}{ab} < \Lambda(\varepsilon)N^{1+\varepsilon} \quad (N \geq N_0(\varepsilon)).$$

Hence it follows by (3-5) and $\Phi(N) \sim \frac{3}{\pi^2}N^2$ $(N \to \infty)$:

$$\mathcal{Q}(N) = \frac{1}{\Phi(N)}\left(\mathcal{H}_1(N) - \frac{1}{12}\right) = O\left(N^{-1+\varepsilon}\right). \tag{3-10}$$

But a well-known inequality for $|\mathcal{M}(N)|$ yields

$$\mathcal{M}(N) = O(N\sqrt{\mathcal{Q}(N)}),$$

so that the application of (3-10) leads to

$$\mathcal{M}(N) = O\left(N^{\frac{1}{2}+\frac{\varepsilon}{2}}\right).$$

Taking still into account Littlewood's theorem (3-2), we can conclude the validity of Riemann's hypothesis.

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