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POSITIVELY QUADRANT DEPENDENT BIVARIATE DISTRIBUTIONS WITH GIVEN MARGINALS

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Abstract

Several measures for dependence of two random variables are investigated in the case of given marginals and assuming positively quadrant dependence. Beyond known quantities (Spearman, Pearson correlation coefficient, etc.) new measures are introduced here and compared with the others. Approximate values of P.Q.D. bivariate distributions are calculated. A practical application in the hydrology of flood peaks is included.

Keywords: positively quadrant dependence, bivariate distributions, approximate values.

1. Investigation of Some Nonparametric Measures of Association in Case of a Positively Quadrant Dependence

There are very many possibilities to construct measures of association and a lot of them have been proposed. Among the most familiar measures we mention the following nonparametric ones:

$$r = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(H - FG)dxdy}{\sigma_1\sigma_2} \qquad \text{(correlation coefficient Pearson))} \tag{1.1}$$

$$\varrho = 12\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(H - FG)fgdxdy = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(H - FG)fgdxdy}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}[\min(F,G) - FG]fgdxdy} \qquad \text{(Spearman)} \tag{1.2}$$

$$\tau = 4\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}Hhdxdy - 1 = \frac{1}{3}\,\frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(Hh - FGfg)dxdy}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}[\min(F,G) - FG]fgdxdy} \qquad \text{(Kendall)} \tag{1.3}$$

$$\mu = 90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)^2 fg dx dy = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)^2 fg dx dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F,G) - FG]^2 fg dx dy} \quad \text{(Höeffding)} \tag{1.4}$$

$$\gamma = \sqrt{\mu} \quad \text{(Blum–Kiefer–Rosenblatt)} \tag{1.5}$$

$$q = 4H(\tilde{x}_{\frac{1}{2}}, \tilde{y}_{\frac{1}{2}}) - 1 = \frac{H(\tilde{x}_{\frac{1}{2}}, \tilde{y}_{\frac{1}{2}}) - F(\tilde{x}_{\frac{1}{2}})G(\tilde{y}_{\frac{1}{2}})}{\min[F(\tilde{x}_{\frac{1}{2}})G(\tilde{y}_{\frac{1}{2}})] - F(\tilde{x}_{\frac{1}{2}})G(\tilde{y}_{\frac{1}{2}})} \quad \text{(Blomqvist)} \tag{1.6}$$

$$K = 4 \sup_{(x,y)} |H(x,y) - F(x)G(y)| \quad \text{(Schweizer–Wolff)} \tag{1.7}$$

It is not difficult to construct other measures. For the case of a positively quadrant dependence beyond (1.1)–(1.7.) we propose the following further measures:

$$\nu = 90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)[\min(F,G) - FG] fg dx dy = \tag{1.8}$$

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)[\min(F,G) - FG] fg dx dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F,G) - FG]^2 fg dx dy}$$

$$\omega = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H - FG}{\sqrt{F(1-F)G(1-G)}} fg dx dy \tag{1.9}$$

$$\lambda^{**} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG) dx dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F,G) - FG] dx dy} = \frac{r}{r_+}. \tag{1.10}$$

where r_t is the correlation coefficient if the joint distribution of X and Y is $H(x,y) = \min(F(x), G(y))$. For different H the values of the mentioned measures depend on H in a fairly simple way. Some relations among them are contained in the following proposition.

PROPOSITION 1

$\lambda^* \geq \frac{\varrho}{3}$	(1.11)	$\tau \geq \frac{\varrho}{3}$	(1.15)
$\lambda^{**} \geq r$	(1.12)	$\mu \geq 0.625\varrho^2$	(1.16)
$\mu \leq \nu \leq \gamma = \sqrt{\mu}$	(1.13)	$\gamma \geq \frac{\sqrt{90}}{12}\varrho$	(1.17)
$\lambda^* \geq \omega$	(1.14)	$\lambda^* \geq 0.625\varrho^2$	(1.18)

PROOF:

(1.11) follows from the fact that $\min(F,G) - FG = \frac{1}{4}$; namely

$$\text{in case} \quad F \leq G, \quad \min(F,G) - FG = F(1-G) \leq F(1-F) \leq \frac{1}{4}$$

$$\text{in case} \quad F > G, \quad \min(F,G) - FG = G(1-F) \leq G(1-G) \leq \frac{1}{4}$$

$$\lambda^* = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H - FG}{\min(F,G) - FG} fg\,dx\,dy \geq 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG) fg\,dx\,dy = \frac{\varrho}{3}$$

(1.12) follows from the fact that $r_+ = \frac{1}{\sigma_1,\sigma_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F,G) - FG]dxdy \leq$ 1 (1.13) is a consequence of the inequality of Schwarz. Namely

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)[\min(F,G) - FG] fg\,dx\,dy \leq$$

$$\leq \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)^2 fg\,dx\,dy\right]\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F,G) - FG]^2 fg\,dx\,dy\right]^{\frac{1}{2}},$$

hence

$$\frac{\nu}{90} \leq \frac{\sqrt{\mu}}{\sqrt{90}} \cdot \frac{1}{\sqrt{90}}, \quad \text{i. e.} \quad \nu \leq \sqrt{\mu} = \gamma$$

further

$$\frac{\nu}{90} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)[\min(F,G) - FG] fg\,dx\,dy \geq$$

$$\geq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)^2 fg\,dx\,dy = \frac{\mu}{90}.$$

(1.14) follows from the fact that if $F \leq G$, then $1 - F \geq 1 - G$, i. e.

$$\sqrt{F(1-G)} \leq \sqrt{G(1-F)},$$
$$F(1-G) \leq \sqrt{F(1-F)G(1-G)}$$

and if $F \geq G$

$$G(1-F) \leq \sqrt{F(1-F)G(1-G)}$$

consequently

$$\lambda^* = \int\limits_{F \leq G} \int\limits_{F \leq G} \frac{H - FG}{F(1-G)} f g dx dy + \int\limits_{F > G} \int\limits_{F > G} \frac{H - FG}{G(1-F)} f g dx dy \geq$$

$$\geq \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{H - FG}{\sqrt{F(1-F)G(1-G)}} f g dx dy = \omega.$$

To see (1.15) we have to compare

$$\tau = 4 \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} H h dx dy - 4 \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} F G f g dx dy$$

and

$$\frac{\varrho}{3} = 4 \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} H f g dx dy - 4 \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} F G f g dx dy.$$

For $H > G$, the relation

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} H f g dx dy = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} F G h dx dy \leq \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} H h dx dy$$

is valid and it follows that

$$\tau \leq \frac{\varrho}{3}.$$

(1.16) is a consequence of Schwarz inequality according to which

$$\left[\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} (H - FG) f g dx dy\right]^2 \leq \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} (H - FG)^2 f g dx dy \cdot \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} 1^2 f g dx dy$$

and

$$\left(\frac{\varrho}{12}\right)^2 \leq \frac{\mu}{90},$$

hence

$$\mu \geq \frac{90}{144}\varrho^2 = 0.625\varrho^2$$

and

$$\gamma = \sqrt{\mu} \geq \frac{\sqrt{90}}{12}\varrho.$$

2. Approximate Values of a Two-dimensional *cdf* H in case of Positively Quadrant Dependence

Let H the joint *cdf* of the pair of random variables X and Y, and let the marginal *cdf*-s F and G, respectively. We suppose that

$$H \geq FG.$$

We shall compare the probability of any quadrant $X < x$, $Y < y$ under the distribution H with the corresponding probability under the distribution $H = \lambda \min(F, G) + (1 - \lambda)FG$ for suitably chosen value of λ.

First of all, we shall determine the value of λ, for which relation:

$$\varphi(\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_\lambda - H)^2 fg dx dy = \min \tag{2.1}$$

holds.

As $H_\lambda - H = (H_\lambda - FG) - (H - FG)$ the minimum problem can be written in the following form:

$$\begin{aligned}\varphi(\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(H_\lambda - FG) - (H - FG)]^2 fg dx dy = \\ = \lambda^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F, G) - FG]^2 fg dx dy - \\ - 2\lambda \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\min(F, G) - FG][H - FG] fg dx dy + \\ + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H - FG)^2 fg dx dy = \min.\end{aligned} \tag{2.2}$$

Due to (1.4) and (1.8) the equation (2.2) has the following form:

$$\varphi(\lambda) = \frac{\lambda^2}{90} - \frac{2\lambda\nu}{90} + \frac{\mu}{90}. \tag{2.3}$$

The function $\varphi(\lambda)$ takes its minimum if

$$\varphi'(\lambda) = \frac{2\lambda - 2\nu}{90} = 0, \quad \text{i. e. if } \lambda = \nu. \tag{2.4}$$

Then

$$\varphi(\nu) = \frac{\nu^2 - 2\nu^2 + \mu}{90} = \frac{\mu - \nu^2}{90}. \tag{2.5}$$

By (1.13)

$$\nu^2 \le \mu \le \nu$$

Therefore

$$\varphi(\nu) \le \frac{\nu - \nu^2}{90} \le \frac{1}{360} \approx 0.0027. \tag{2.6}$$

It follows from (2.5) that the smaller the difference between μ and ν^2, the better the approximation of H by H_λ is. If $H = H_\lambda$, then $\mu = \lambda^2$, $\nu = \lambda$, i. e. $\varphi(\nu) = 0$.

Result (2.6) can be improved, the upper bound can be decreased. Using (1.13) put

$$\nu = \alpha\sqrt{\mu} + (1-\alpha)\mu, \quad 0 \le \alpha = \frac{\nu - \mu}{\sqrt{\mu} - \mu} \le 1.$$

Then

$$\nu^2 = \alpha^2\mu + 2\alpha(1-\alpha)\mu\sqrt{\mu} + (1-\alpha)^2\mu^2 \ge$$
$$\ge \alpha^2\mu + 2\alpha(1-\alpha)\mu^2 + (1-\alpha)^2\mu^2 = \alpha^2\mu + (1-\alpha^2)\mu^2.$$

Thus

$$\mu - \nu^2 \le \mu - \alpha^2\mu - (1-\alpha^2)\mu^2 = (1-\alpha^2)(\mu - \mu^2).$$

Hence

$$\varphi(\nu) \le (1-\alpha^2)\frac{\mu - \mu^2}{90} \le \frac{1-\alpha^2}{360}. \tag{2.7}$$

As an example consider the distribution function

$$H = FG + \beta F(1-F)G(1-G)$$

introduced by D. MORGENSTERN (1956). This is a positively quadrant dependent if $0 \leq \beta \leq 1$.

An easy calculation shows that

$$\nu = \frac{17}{56}\beta \approx 0.3\beta; \quad \mu = \frac{\beta^2}{10}; \quad \sqrt{\mu} = \frac{\beta}{\sqrt{10}}.$$

One can easily see that

$$\varphi(\nu) = O(10^{-4}).$$

REMARK 1

As $H_\lambda - FG = \lambda[\min(F,G) - FG]$ we can say that H_λ keeps the proportion between $\min(F,G)$ and FG.

Let us now introduce the following functions of the random variables X and Y:

$$\begin{aligned} U(X,Y) &= \min[F(X),G(Y)] - H(X,Y); \\ V(X,Y) &= H(X,Y) - F(X)G(Y); \\ Z(X,Y) &= \min[F(X),G(Y)] - F(X)G(Y). \end{aligned} \tag{2.8}$$

If $H = H_\lambda$ $(0 \leq \lambda \leq 1)$ then

$$U_\lambda = (1-\lambda)Z, \quad V_\lambda = \lambda Z \quad \text{and} \quad U_\lambda = \frac{1-\lambda}{\lambda}V_\lambda, \tag{2.9}$$

i. e. between the random variables U_λ, V_λ and Z_λ there is a linear functional relationship. It follows that the correlation coefficients between the pairs (U_λ, Z), (V_λ, Z), (U_λ, V_λ) all are equal to 1.

$$r(U_\lambda, Z) = r(V_\lambda, Z) = r(U_\lambda, V_\lambda) = 1. \tag{2.10}$$

REMARK 2

In practical problems the two-dimensional *cdf*. H is usually unknown, but in many cases we may suppose that its marginal *cdf*-s F and G are known. If we have a sample $(X_1,Y_1),(X_2,Y_2),\ldots(X_n,Y_n)$ we have the empirical two-dimensional *cdf*, $H_n(x,y)$ and by means of F and G, we have a sample for U, V and Z:

$$\begin{aligned} U^{(i)} &= \min[F(X_i),G(Y_i)] - H_n(X_i,Y_i), \\ V^{(i)} &= H_n(X_i,Y_i) - F(X_i)G(Y_i) \end{aligned}$$

and

$$Z^{(i)} = \min F(X_i)G(Y_i) - F(X_i)G(Y_i), \quad \text{(i=1,2, \ldots,n)}$$

Table 1

Year	X(cm)	Y(day)	Year	X(cm)	Y(day)
1901	29	5	1941	204	68
1902	14	3	1942	38	7
1907	108	42		51	11
1912	72	19		60	14
	34	10			
1914	128	22	1944	4	3
1915	110	35	1952	2	5
1916	73	13	1956	39	10
				37	7
1919	266	49	1958	66	25
1920	16	2	1962	170	33
1922	124	36	1964	114	19
1924	220	51	1965	198	15
1932	273	42	1967	134	41
1937	53	11	1970	309	91
1940	197	38			
	40	8			
	28	5			

From this sample we can estimate the correlation coefficients in (5.10) and if their values are close to 1 then we may expect, that the approximation of H by H_λ 'good' or even we may accept that the null $H_0 : H = H_\lambda$ holds.

Let us consider the following example taken from the flood hydrology.

EXAMPLE

For the River Tisza in the period 1900–1970 in the second quarter every year (1 Apr.– 30 June) above the level $c = 650$ cm the following flood peaks were observed.

Testing the goodness of fit shows that the excedance X have the *cdf* : $F(x) = 1 - e^{-0.01x}$ and the duration of floods Y have the

$$cdf : G(y) = 1 - e^{-0.05y}.$$

For the joint bivariate distribution of the pair (X, Y) the sample was obtained from *Table 1*.

The value of the correlation coefficient between $V = H_\nu - FG$ and $Z = \min(F, G) - FG$ is $r(V, Z) \approx 0.9$ so we may accept the validity of hypothesis H_0:

$$H = H_\nu = \nu \min[1 - e^{-0.01x}, 1 - e^{-0.05y}] + \\ + (1 - \nu)(1 - e^{-0.01x})(1 - e^{-0.05y}). \tag{2.11}$$

Now the estimated value of ν is needed. For the *cdf* H_ν the value of ν agrees with the value of $q = 4H_\nu - 1$. The estimation of the value of q is very easy from the sample

$$\hat{q} = 4\frac{14}{31} - 1 \approx 0.8.$$

For comparison of the value of H_ν and the empirical *cdf* H_n let us consider these values in the quartile-points $(\tilde{x}_{\frac{1}{4}}, \tilde{y}_{\frac{1}{4}}), (\tilde{x}_{\frac{1}{2}}, \tilde{y}_{\frac{1}{4}}), \ldots (\tilde{x}_{\frac{3}{4}}, \tilde{y}_{\frac{3}{4}})$:

Table 2

	H_ν	H_n	$(H_\nu - H_n)^2$
$(\tilde{x}_{\frac{1}{4}}, \tilde{y}_{\frac{1}{4}})$	0.2125	0.1935	0.000484
$(\tilde{x}_{\frac{1}{2}}, \tilde{y}_{\frac{1}{4}})$	0.225	0.1935	0.000992
$(\tilde{x}_{\frac{3}{4}}, \tilde{y}_{\frac{1}{4}})$	0.225	0.1935	0.000992
$(\tilde{x}_{\frac{1}{4}}, \tilde{y}_{\frac{1}{2}})$	0.2376	0.1935	0.001945
$(\tilde{x}_{\frac{1}{2}}, \tilde{y}_{\frac{1}{2}})$	0.2376	0.1935	0.001945
$(\tilde{x}_{\frac{3}{4}}, \tilde{y}_{\frac{1}{2}})$	0.450	0.4516	0.000000
$(\tilde{x}_{\frac{1}{4}}, \tilde{y}_{\frac{3}{4}})$	0.475	0.4838	0.00007
$(\tilde{x}_{\frac{1}{2}}, \tilde{y}_{\frac{3}{4}})$	0.475	0.4838	0.00007
$(\tilde{x}_{\frac{3}{4}}, \tilde{y}_{\frac{3}{4}})$	0.712	0.680	0.00102

Hence the mean-quadratical derivation between H_n and H_ν is

$$\frac{\sum_{1}^{9}(H_\nu - H_n)^2}{9} \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_\nu - H)^2 fg\,dx\,dy = 0.00074$$

In our example above the sample size ($n = 31$) is not large enough for carrying out a test exactly, but the high value of r along with the tabulation heuristically suggests the validity of our inference.

3. A Quadratic Mean Deviation between Two Positively Quadrant Dependent Distribution Function

Denote by M_{FG} the set of all bivariate distribution functions H whose marginals are F and $G, H \geq FG$.

THEOREM *If $H_1 \in M_{FG}$ and $H_2 \in M_{FG}$ then for the quadratic mean deviation of H_1 and H_2 we have the following inequality:*

$$\frac{\mu_1 + \mu_2 - 2\gamma_1\gamma_2}{90} \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_1)^2 dF dG \leq \frac{\mu_1\mu_2 - 2\nu_1\nu_2}{90}, \qquad (3.1)$$

where μ,ν and γ are the nonparametric measures of dependence defined in (1.4), (1.8) *and* (1.5).

PROOF:

$$(H_2 - H_1) - (H_{\nu_2} - H_{\nu_1}) = (H_2 - H_{\nu_2}) - (H_1 - H_{\nu_1}),$$

$$H_{\nu_2} - H_{\nu_1} = (\nu_2 - \nu_1)[\min(F, G) - FG],$$

$$90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_1)^2 dF dG -$$

$$-2.90(\nu_2 - \nu_1) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_1)[\min(F, G) - FG] dF dG + (\nu_2 - \nu_1)^2 \leq$$

$$\leq 90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_{\nu_2})^2 dF dG + 90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_1 - H_{\nu_1})^2 dF dG.$$

As

$$90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_1)[\min(F, G) - FG] dF dG =$$

$$= 90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(H_2 - FG) - (H_1 - FG)][\min(F, G) - FG) dF = \nu_2 - \nu_1,$$

we get

$$90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_1)^2 dF dG - (\nu_2 - \nu_1)^2 \leq \mu_2 - \nu_2^2 + \mu_1 - \nu_1^2$$

and

$$90 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_2 - H_1)^2 dF dG \leq \mu_2 + \mu_1 - 2\nu_1\nu_2 \qquad (3.2)$$

hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG \le \frac{\mu_2+\mu_1-2\nu_1\nu_2}{90}. \tag{3.3}$$

On the other hand

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG = 90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(H_2-FG)-(H_1-FG)]^2dFdG.$$

Now Schwarz inequality gives

$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-FG)(H_1-FG)dFdG\right]^2 \le$$

$$\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-FG)^2dFdG \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_1-FG)^2dFdG.$$

Thus

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG \ge \mu_2-2\sqrt{\mu_1}\sqrt{\mu_2}+\mu_1=(\sqrt{\mu_2}-\sqrt{\mu_1})^2. \tag{3.4}$$

By inequalities (3.3) and (3.4) we get

$$\frac{(\sqrt{\mu_2}-\sqrt{\mu_1})^2}{90} \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG \le \frac{\mu_1+\mu_2-2\nu_1\nu_2}{90},$$

i. e.

$$\frac{\mu_1+\mu_2-2\gamma_1\gamma_2}{90} \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG \le \frac{\mu_1+\mu_2-2\gamma_1\gamma_2}{90} \tag{3.5}$$

Since

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG =$$

$$= 90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(H_2-FG)-(H_1-FG)]^2dFdG =$$

$$= \mu_1+\mu_2-2\cdot 90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-FG)(H_1-FG)dFdG.$$

From (3.5)

$$\frac{\nu_1\nu_2}{90} \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2 - FG)(H_1 - FG)dFdG \leq \frac{\gamma_1\gamma_2}{90}$$

We note that if:

$$H_2 = \lambda_2 \min(F, G) + (1 - \lambda_2)FG$$

and

$$H_1 = \lambda_1 \min(F, G) + (1 - \lambda_1)FG,$$

then

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2 - H_1)^2 dFdG = (\lambda_2 - \lambda_1)^2.$$

In this case

$$\mu_2 = \lambda_2^2, \qquad \nu_2 = \lambda_2,$$
$$\mu_1 = \lambda_1^2, \qquad \nu_1 = \lambda_1.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\lambda_2} - H_{\lambda_1})^2 dFdG = \frac{\mu_2 + \mu_1 - 2\nu_1\nu_2}{90} = \frac{\mu_2 + \mu_1 - 2\gamma_1\gamma_2}{90}. \quad (3.6)$$

Here the following question can be posed. If we approximate the positively quadrant dependent distribution functions H_1 and H_2 by the linear combinations:

$$H_{\nu_1} = \nu_1 \min(F, G) + (1 - \nu_1)FG \quad \text{and} \quad H_{\nu_2} = \nu_2 \min(F, G) + (1 - \nu_2)FG$$

then what is the relation of the quadratic deviations

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2 - H_1)^2 dFdG \quad \text{and} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\nu_2} - H_{\nu_1})^2 dFdG.$$

We shall show that on the one hand

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2 - H_1)^2 dFdG \geq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\nu_2} - H_{\nu_1})^2 dFdG, \quad (3.7)$$

on the other hand:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\nu_2}-H_{\nu_1})^2dFdG\leq\frac{1}{180}\approx 0.005. \quad (3.8)$$

For

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(H_2-H_1)^2-(H_{\nu_2}-H_{\nu_1})]^2dFdG=$$

$$=90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG-2\cdot 90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)(H_{\nu_2}-H_{\nu_1})dFdG+$$

$$+90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\nu_2}-H_{\nu_1})^2dFdG.$$

Since

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\nu_2}-H_{\nu_1})^2dFdG=(\nu_2-\nu_1)^2$$

and

$$90\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)(H_{\nu_2}-H_{\nu_1})dFdG=(\nu_2-\nu_1)^2$$

it follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(H_2-H_1)-(H_{\nu_2}-H_{\nu_1})]^2dFdG=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG-\frac{(\nu_2-\nu_1)^2}{90}\geq 0,$$

i. e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2-H_1)^2dFdG\geq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_{\nu_2}-H_{\nu_1})^2dFdG,$$

thus (3.7) is proved.

If (2.7) is also taken into account then:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(H_2 - H_1) - (H_{\nu_2} - H_{\nu_1})]^2 dF dG =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(H_2 - H_{\nu_2}) - (H_1 - H_{\nu_1})]^2 dF dG \leq$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_2 - H_{\nu_2})^2 dF dG + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(H_1 - H_{\nu_1})^2 dF dG \leq$$

$$\leq \frac{(1-\alpha^2)(\mu_2 - \mu_2^2) + (1-\alpha_1^2)(\mu_1 - \mu_1^2)}{90} \leq \frac{2(\alpha_1^2 + \alpha_2^2)}{360},$$

where

$$\alpha_1 = \frac{\nu_1 - \mu_1}{\sqrt{\mu_1} - \mu_1}; \qquad \alpha_2 = \frac{\nu_2 - \mu_2}{\sqrt{\mu_2} - \mu_2}.$$

This proves the relation (3.8).

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