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COMPUTATIONAL CONTRIBUTIONS TO A PAPER OF MIKOLÁS AND SATO ON FRANEL'S SUM

KEN-ICHI SATO

College of Engineering
Nihon University
Koriyama, Fukushima 963 Japan

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Abstract

Not long ago, the authors mentioned in the title have published a joint paper (see MIKOLÁS–SATO, 1992) about some new asymptotic results and conjectures on the so-called *Franel's sum* $Q(N)$ defined by the formula

$$Q(N) = \sum_{\nu=1}^{\Phi(N)} \left(\varrho_\nu^{(N)} - \frac{\nu}{\Phi(N)} \right)^2 . \tag{1}$$

Here $\varrho_\nu^{(N)}(\nu = 1, 2, \ldots, N)$ denote the *Farey fractions* of order N, i. e. the ascending sequence of all rationals h/k with h, k positive integers, $h \le k \le N$ and h coprime to k. The number of these fractions is $\Phi(N) = \sum_{n=1}^{N} \varphi(n)$ (φ the well-known Euler function), and we have the asymptotic equality $\Phi(N) \sim (3/\pi^2)N^2$ $(N \to \infty)$. By the representation

$$\mathcal{M}(N) = \sum_{\nu=1}^{\Phi(n)} \cos 2\pi\varrho_\nu^{(N)}, \tag{2}$$

$\Phi(N)$ is connected also with the classical Möbius summatoric function.

The importance of (1) and (2) for the analytic theory of numbers lies mainly in the fact that both of the relations

$$Q(N) = O(N^{-1+\varepsilon}), \qquad \mathcal{M}(N) = O(N^{\frac{1}{2}+\varepsilon})(\forall\varepsilon > 0) \tag{3}$$

are equivalent to the *Riemann hypothesis* on the complex zeros of $\zeta(s)$ (which is defined by analytic continuation of the series $\sum_{n=1}^{\infty} n^{-s}$, $Re\, s > 1$). These are the famous theorems of FRANEL and LITTLEWOOD, respectively. (Cf. FRANEL, 1924; LITTLEWOOD, 1912; MIKOLÁS 1949; ODLYZKO-RIELE, 1985.).

The present paper aims to develop further some numerical investigations in MIKOLÁS–SATO (1992) which cannot be approached for the time being by other mathematical tools. Our central problem will be to say somewhat more on the '*right order of magnitude*' of $Q(N)$ by computation, using also the 'test function'

$$J(N) = (\log\log\log N)/N \tag{4}$$

introduced previously, together with the well proved characterizing data:

$$\mathcal{K}(N) = Q(N)/J(N), \qquad \mathcal{R}(N) = J(N) - Q(N) \tag{5}$$

as well as with *regression analysis*. It turned out that all the essential observations formulated in the work just mentioned for the domain $3 \leq N \leq 2000$ may now be extended up to $N = 10000$, on the basis of the new enlarged 'Tables of Franel's sum' (1991–1992) of the author.

Keywords: computing methods in number theory.

Main Results

Preliminary Remarks

The numerical calculations about $Q(N)$ from $N = 2000$ to 8000 have been carried out by use of the personal computer NEC PC-9801 VX ES, RS and EPSON PC-286 BOOK.

The particularly intricate computation of $Q(N)$ from $N = 8000$ to 10000 was made using FUJITSU-FACOM M770, Model 8, Main Memory=64MB, with FACOM-OS4/MSP as Operating System (in the Computer Center of College of Engineering, Nihon University).

The programme of $Q(N)$ for $1 \leq N \leq 500$ on the personal computers was written in U-BASIC ('KIDA'S BASIC') which was constructed by Y. KIDA (Rikkyo University, Tokyo). The program of $Q(N)$ for $5000 < N < 8000$ on the personal computers was written in Turbo Pascal. The program of $Q(N)$ for $8000 < N < 10000$ at FUJITSU-FACOM-computer was written in Pascal.

Tables and figures in this paper were written by the Application Soft 'LOTUS 1-2-3'. The regression analysis has been carried out in the same manner.

We give such an *extension of Observation 2, Part(II) in* MIKOLÁS–SATO (1992), which means a great support to the probable validity of the asymptotic relation (20): $Q(N) \sim \kappa J(N)$ ($\kappa \approx 0.8$), conjectured l. c., on p. 376.

Observation(*)

We have in the 32 subintervals of equal length 250 of the interval (2000, 10000] following estimates of the form $\alpha < \mathcal{K}(N) < \beta$, *i. e.* $\alpha J(N) < Q(N) < \beta J(N)$ *(*α, β *positive constants):*

N	$\min \mathcal{K}(N)$	$\max \mathcal{K}(N)$	N	$\min \mathcal{K}(N)$	$\max \mathcal{K}(N)$
(2000,2250]	0.90039	0.92633	(6000,6250]	0.84295	0.84953
(2250,2500]	0.89616	0.92917	(6250,6500]	0.84226	0.86893
(2500,2750]	0.88732	0.91392	(6500,6750]	0.84616	0.86404
(2750,3000]	0.88500	0.93977	(6750,7000]	0.84205	0.86259
(3000,3250]	0.87638	0.90245	(7000,7250]	0.84064	0.86994
(3250,3500]	0.87569	0.90782	(7250,7500]	0.83722	0.84948
(3500,3750]	0.86965	0.88528	(7500,7750]	0.83404	0.84371
(3750,4000]	0.86830	0.88934	(7750,8000]	0.83416	0.84741
(4000,4250]	0.86711	0.88860	(8000,8250]	0.83307	0.84795
(4250,4500]	0.86141	0.88797	(8250,8500]	0.83628	0.84974
(4500,4750]	0.85699	0.87179	(8500.8750]	0.83358	0.86068
(4750,5000]	0.85682	0.87648	(8750.9000]	0.82599	0.83997
(5000,5250]	0.85354	0.86606	(9000,9250]	0.82287	0.83117
(5250,5500]	0.85166	0.86139	(9250,9500]	0.82268	0.84114
(5500,5750]	0.85452	0.86985	(9500,9750]	0.82884	0.85139
(5750,6000]	0.84781	0.86707	(9750,10000]	0.83700	0.86593

The minimal ('m') and maximal ('M') values of $\mathcal{K}(N)$ are given to five decimals with a possible error of 1 in the last decimal. These are taken at N =2003(M),2125(m); 2331(m), 2411(M); 2530(m), 2732(M); 2803(M), 2997(m); 3182(m), 3242(M); 3295(M), 3485(m); 3670(m), 3733(M); 3905(m), 3948(M); 4161(M), 4197(m); 4261(M), 4476(m); 4526(M), 4636(m); 4885(M), 4966(m); 5011(M), 5076(m); 5410(m), 5485(M); 5674(M), 5740(m); 5908(M), 5980(m); 6154(M), 6195(m); 6262(m), 6399(M); 6582(M), 6750(m); 6943(m), 6997(M); 7027(M), 7176(m); 7253(M), 7390(m); 7582(m), 7728(M); 7808(m), 7964(M); 8004(m), 8193(M); 8418(M), 8456(m); 8554(M), 8706(m); 8751(M), 8995(m); 9059(M), 9150(m); 9276(m), 9499(M); 9618(m), 9749(M); 9889(M), 10000(m).

The situation in question is well illustrated by the Figures 10 and 11 for $\mathcal{K}(N) = Q(N)/J(N)$.

Figure 1. This gives the graph of $Q(N)$ for $10 \le N \le 100$.
Figure 2. This gives the graph of $Q(N)$ for $100 \le N \le 500$.
Figure 3. This gives the graph of $Q(N)$ and $J(N) = (\log\log\log N)/N$ for $500 \le N \le 2000$.
Figure 4. This gives the graph of $Q(N)$ and $J(N)$ for $2000 \le N \le 3000$.
Figure 5. This gives the graph of $Q(N)$ and $J(N)$ for $3000 \le N \le 5000$.
Figure 6. This gives the graph of $Q(N)$ and $J(N)$ for $5000 \le N \le 7000$.

Figure 7. This gives the graph of $Q(N)$ and $J(N)$ for $7000 \leq N \leq 10000$.
Figure 8. This gives the graph of $Q(N)$, $J(N)$ and $J(N)-Q(N)$ for $500 \leq N \leq 5000$.
Figure 9. This gives the graph of $Q(N)$, $J(N)$ and $J(N)-Q(N)$ for $5000 \leq N \leq 10000$.
Figure 10. This gives the graph of $\mathcal{K}(N) = Q(N)/J(N)$ for $500 \leq N \leq 5000$.
Figure 11. This gives the graph of $\mathcal{K}(N)$ for $5000 \leq N \leq 10000$.
Figure 12. This gives the graph of $0.6638/N - Q(N)$ for $500 \leq N \leq 5000$.
Figure 13. This gives the graph of $(0.6638/N - Q(N))/N$ for $500 \leq N \leq 5000$.
Figure 14. This gives the graph of $0.6638/N - Q(N)$ for $5000 \leq N \leq 10000$.
Figure 15. This gives the graph of $(0.6638/N - Q(N))/N$ for $5000 \leq N \leq 10000$.
Figure 16. This gives the graph of $Q(N)$ and $0.6638/N$ for $8000 \leq N \leq 10000$.
Figure 17. This gives the graph of $Q(N)$, $J(N)$ and $0.6638/N$ for $3000 \leq N \leq 10000$.

Though LOTUS 1-2-3 does not support to draw just the plot of data, we can draw line graphs which pass through every point of the corresponding data.

Table 1. The values of $Q(N)$ are given to four decimals, with a possible error of 1 in the last decimal, over the range $1 \leq N \leq 300$.
Table 2. The values of $Q(N)$ are given to five decimals, with a possible error of 1 in the last decimal, over the range $301 \leq N \leq 500$.
Table 3. The values of $Q(N)$ are given to eight decimals, with a possible error of 1 in the last decimal, for $N = 500$ to 1000 step 10.
Table 4. The values of $Q(N)$, $J(N)$ are given to eight decimals, with a possible error of 1 in the last decimal, for $N = 1000$ to 5000 step 100.
Table 5. The values of $Q(N)$, $J(N)$ are given to eight decimals, with a possible error of 1 in the last decimal, for $N = 5000$ to 10000 step 100.

Though we calculated all values of $Q(N)$, etc. for $1 \leq N \leq 10000$, we cannot give all the data in this paper.

Table 1
Table of $Q(N)$ (from $N = 1$ to 300)

N	$Q(N)$	N	$Q(N)$	N	$Q(N)$	N	$Q(N)$	N	$Q(N)$	N	$Q(N)$
1	0.0000	51	0.0092	101	0.0053	151	0.0038	201	0.0031	251	0.0024
2	0.0000	52	0.0089	102	0.0051	152	0.0038	202	0.0031	252	0.0024
3	0.0139	53	0.0095	103	0.0053	153	0.0037	203	0.0030	253	0.0023
4	0.0139	54	0.0092	104	0.0052	154	0.0036	204	0.0030	254	0.0023
5	0.0272	55	0.0086	105	0.0051	155	0.0036	205	0.0029	255	0.0023
6	0.0167	56	0.0082	106	0.0050	156	0.0035	206	0.0029	256	0.0023
7	0.0261	57	0.0080	107	0.0053	157	0.0036	207	0.0028	257	0.0023
8	0.0242	58	0.0079	108	0.0052	158	0.0036	208	0.0028	258	0.0023
9	0.0237	59	0.0083	109	0.0054	159	0.0036	209	0.0027	259	0.0023
10	0.0193	60	0.0080	110	0.0053	160	0.0036	210	0.0027	260	0.0023
11	0.0244	61	0.0085	111	0.0052	161	0.0035	211	0.0028	261	0.0022
12	0.0201	62	0.0084	112	0.0051	162	0.0035	212	0.0028	262	0.0022
13	0.0250	63	0.0080	113	0.0053	163	0.0035	213	0.0027	263	0.0023
14	0.0214	64	0.0079	114	0.0053	164	0.0035	214	0.0027	264	0.0022
15	0.0178	65	0.0076	115	0.0051	165	0.0034	215	0.0027	265	0.0022
16	0.0170	66	0.0073	116	0.0050	166	0.0034	216	0.0027	266	0.0022
17	0.0198	67	0.0076	117	0.0048	167	0.0035	217	0.0027	267	0.0022
18	0.0179	68	0.0074	118	0.0047	168	0.0034	218	0.0027	268	0.0022
19	0.0209	69	0.0071	119	0.0046	169	0.0034	219	0.0027	269	0.0022
20	0.0186	70	0.0069	120	0.0045	170	0.0034	220	0.0027	270	0.0022
21	0.0160	71	0.0073	121	0.0045	171	0.0033	221	0.0027	271	0.0022
22	0.0150	72	0.0071	122	0.0045	172	0.0033	222	0.0026	272	0.0022
23	0.0168	73	0.0075	123	0.0044	173	0.0034	223	0.0026	273	0.0022
24	0.0154	74	0.0073	124	0.0043	174	0.0034	224	0.0026	274	0.0022
25	0.0153	75	0.0071	125	0.0043	175	0.0033	225	0.0026	275	0.0022
26	0.0147	76	0.0069	126	0.0042	176	0.0033	226	0.0026	276	0.0022
27	0.0142	77	0.0065	127	0.0044	177	0.0032	227	0.0026	277	0.0022
28	0.0130	78	0.0064	128	0.0043	178	0.0032	228	0.0026	278	0.0022
29	0.0143	79	0.0068	129	0.0043	179	0.0033	229	0.0026	279	0.0022
30	0.0136	80	0.0066	130	0.0042	180	0.0032	230	0.0026	280	0.0022
31	0.0153	81	0.0066	131	0.0044	181	0.0033	231	0.0025	281	0.0022
32	0.0148	82	0.0063	132	0.0043	182	0.0033	232	0.0025	282	0.0022
33	0.0134	83	0.0067	133	0.0042	183	0.0033	233	0.0026	283	0.0023
34	0.0127	84	0.0066	134	0.0041	184	0.0032	234	0.0025	284	0.0022
35	0.0116	85	0.0063	135	0.0041	185	0.0031	235	0.0025	285	0.0022
36	0.0109	86	0.0062	136	0.0040	186	0.0031	236	0.0025	286	0.0022
37	0.0118	87	0.0060	137	0.0041	187	0.0031	237	0.0025	287	0.0022
38	0.0115	88	0.0058	138	0.0041	188	0.0030	238	0.0024	288	0.0022
39	0.0110	89	0.0061	139	0.0042	189	0.0030	239	0.0025	289	0.0022
40	0.0104	90	0.0060	140	0.0041	190	0.0030	240	0.0025	290	0.0022
41	0.0111	91	0.0058	141	0.0041	191	0.0031	241	0.0025	291	0.0021
42	0.0105	92	0.0056	142	0.0040	192	0.0030	242	0.0025	292	0.0021
43	0.0114	93	0.0056	143	0.0039	193	0.0031	243	0.0025	293	0.0022
44	0.0108	94	0.0056	144	0.0039	194	0.0031	244	0.0025	294	0.0022
45	0.0101	95	0.0055	145	0.0038	195	0.0031	245	0.0024	295	0.0021
46	0.0098	96	0.0054	146	0.0038	196	0.0030	246	0.0024	296	0.0021
47	0.0105	97	0.0055	147	0.0038	197	0.0031	247	0.0024	297	0.0021
48	0.0101	98	0.0054	148	0.0037	198	0.0031	248	0.0024	298	0.0021
49	0.0101	99	0.0052	149	0.0038	199	0.0032	249	0.0024	299	0.0020
50	0.0098	100	0.0051	150	0.0037	200	0.0032	250	0.0023	300	0.0020

Table 2
Table of $Q(N)$ (from $N = 301$ to 500)

N	$Q(N)$	N	$Q(N)$	N	$Q(N)$	N	$Q(N)$
301	0.00198	351	0.00172	401	0.00157	451	0.00138
302	0.00197	352	0.00171	402	0.00156	452	0.00138
303	0.00195	353	0.00173	403	0.00155	453	0.00137
304	0.00194	354	0.00172	404	0.00154	454	0.00136
305	0.00192	355	0.00170	405	0.00153	455	0.00135
306	0.00191	356	0.00170	406	0.00151	456	0.00135
307	0.00194	357	0.00167	407	0.00150	457	0.00137
308	0.00192	358	0.00167	408	0.00149	458	0.00136
309	0.00191	359	0.00170	409	0.00151	459	0.00135
310	0.00190	360	0.00169	410	0.00150	460	0.00134
311	0.00193	361	0.00169	411	0.00149	461	0.00136
312	0.00192	362	0.00170	412	0.00148	462	0.00135
313	0.00196	363	0.00168	413	0.00147	463	0.00138
314	0.00195	364	0.00167	414	0.00146	464	0.00137
315	0.00194	365	0.00167	415	0.00145	465	0.00137
316	0.00193	366	0.00166	416	0.00145	466	0.00136
317	0.00197	367	0.00168	417	0.00145	467	0.00139
318	0.00196	368	0.00167	418	0.00144	468	0.00138
319	0.00193	369	0.00165	419	0.00145	469	0.00136
320	0.00191	370	0.00164	420	0.00145	470	0.00136
321	0.00190	371	0.00162	421	0.00147	471	0.00135
322	0.00189	372	0.00162	422	0.00146	472	0.00134
323	0.00187	373	0.00164	423	0.00145	473	0.00133
324	0.00186	374	0.00163	424	0.00145	474	0.00133
325	0.00184	375	0.00162	425	0.00143	475	0.00132
326	0.00183	376	0.00161	426	0.00142	476	0.00131
327	0.00183	377	0.00159	427	0.00141	477	0.00130
328	0.00181	378	0.00158	428	0.00141	478	0.00130
329	0.00180	379	0.00161	429	0.00140	479	0.00132
330	0.00180	380	0.00161	430	0.00139	480	0.00131
331	0.00182	381	0.00159	431	0.00141	481	0.00130
332	0.00181	382	0.00159	432	0.00140	482	0.00130
333	0.00179	383	0.00162	433	0.00142	483	0.00129
334	0.00179	384	0.00161	434	0.00142	484	0.00129
335	0.00178	385	0.00160	435	0.00142	485	0.00127
336	0.00177	386	0.00160	436	0.00141	486	0.00127
337	0.00179	387	0.00159	437	0.00140	487	0.00129
338	0.00178	388	0.00157	438	0.00140	488	0.00128
339	0.00179	389	0.00160	439	0.00142	489	0.00127
340	0.00177	390	0.00159	440	0.00142	490	0.00126
341	0.00177	391	0.00157	441	0.00141	491	0.00128
342	0.00176	392	0.00156	442	0.00141	492	0.00128
343	0.00176	393	0.00156	443	0.00144	493	0.00126
344	0.00175	394	0.00156	444	0.00143	494	0.00126
345	0.00172	395	0.00156	445	0.00141	495	0.00125
346	0.00173	396	0.00155	446	0.00140	496	0.00125
347	0.00174	397	0.00157	447	0.00138	497	0.00123
348	0.00173	398	0.00158	448	0.00138	498	0.00123
349	0.00175	399	0.00156	449	0.00140	499	0.00125
350	0.00173	400	0.00155	450	0.00140	500	0.00125

Table 3
Table of $Q(N)$, $J(N)$, etc. (from $N = 500$ to 1000 step 10)

N	$Q(N)$	$J(N)$	$K(N) = Q(N)/J(N)$	$R(N) = J(N) - Q(N)$
500	0.00124812	0.00120524	1.03557	-0.00004288
510	0.00123291	0.00118502	1.04041	-0.00004789
520	0.00117604	0.00116550	1.00904	-0.00001054
530	0.00117218	0.00114664	1.02228	-0.00002555
540	0.00112355	0.00112840	0.99570	0.00000485
550	0.00112412	0.00111076	1.01203	-0.00001336
560	0.00110477	0.00109369	1.01014	-0.00001109
570	0.00110660	0.00107715	1.02734	-0.00002944
580	0.00108632	0.00106113	1.02373	-0.00002518
590	0.00106395	0.00104561	1.01755	-0.00001835
600	0.00104618	0.00103054	1.01517	-0.00001564
610	0.00101564	0.00101593	0.99972	0.00000029
620	0.00102354	0.00100174	1.02177	-0.00002181
630	0.00098421	0.00098795	0.99621	0.00000374
640	0.00096435	0.00097456	0.98952	0.00001021
650	0.00097419	0.00096154	1.01315	-0.00001264
660	0.00098397	0.00094888	1.03698	-0.00003509
670	0.00096500	0.00093657	1.03036	-0.00002844
680	0.00095603	0.00092458	1.03402	-0.00003145
690	0.00092271	0.00091290	1.01074	-0.00000980
700	0.00089782	0.00090154	0.99588	0.00000372
710	0.00088711	0.00089046	0.99624	0.00000335
720	0.00086801	0.00087966	0.98676	0.00001165
730	0.00085684	0.00086913	0.98586	0.00001229
740	0.00084596	0.00085886	0.98497	0.00001291
750	0.00083144	0.00084884	0.97950	0.00001740
760	0.00082950	0.00083907	0.98860	0.00000956
770	0.00082445	0.00082952	0.99389	0.00000507
780	0.00080846	0.00082020	0.98569	0.00001174
790	0.00079465	0.00081109	0.97973	0.00001644
800	0.00078717	0.00080219	0.98128	0.00001502
810	0.00077115	0.00079349	0.97185	0.00002234
820	0.00075687	0.00078499	0.96417	0.00002813
830	0.00076224	0.00077667	0.98142	0.00001443
840	0.00075040	0.00076854	0.97639	0.00001814
850	0.00073946	0.00076058	0.97223	0.00002112
860	0.00074414	0.00075279	0.98850	0.00000865
870	0.00073535	0.00074517	0.98683	0.00000981
880	0.00072632	0.00073770	0.98457	0.00001138
890	0.00072912	0.00073039	0.99826	0.00000127
900	0.00070469	0.00072323	0.97437	0.00001854
910	0.00069078	0.00071621	0.96450	0.00002543
920	0.00068554	0.00070933	0.96646	0.00002379
930	0.00068172	0.00070259	0.97030	0.00002087
940	0.00067282	0.00069598	0.96672	0.00002316
950	0.00066381	0.00068950	0.96273	0.00002570
960	0.00065748	0.00068314	0.96243	0.00002566
970	0.00064703	0.00067691	0.95587	0.00002987
980	0.00064500	0.00067079	0.96156	0.00002578
990	0.00063953	0.00066478	0.96201	0.00002525
1000	0.00063871	0.00065889	0.96937	0.00002018

Table 4
Table of $Q(N)$, $J(N)$, etc. (from $N = 1000$ to 5000 step 100)

N	$Q(N)$	$J(N)$	$K(N) = Q(N)/J(N)$	$R(N) = J(N) - Q(N)$
1000	0.00063871	0.00065889	0.9694	0.00002018
1100	0.00059501	0.00060541	0.9828	0.00001040
1200	0.00052811	0.00056023	0.9427	0.00003213
1300	0.00049221	0.00052153	0.9438	0.00002932
1400	0.00046032	0.00048800	0.9433	0.00002769
1500	0.00043205	0.00045865	0.9420	0.00002660
1600	0.00040227	0.00043274	0.9296	0.00003047
1700	0.00038839	0.00040969	0.9480	0.00002130
1800	0.00035986	0.00038904	0.9250	0.00002918
1900	0.00033760	0.00037044	0.9113	0.00003284
2000	0.00032801	0.00035359	0.9276	0.00002558
2100	0.00030778	0.00033825	0.9099	0.00003048
2200	0.00029520	0.00032423	0.9105	0.00002903
2300	0.00028095	0.00031136	0.9023	0.00003041
2400	0.00027729	0.00029950	0.9258	0.00002221
2500	0.00025867	0.00028854	0.8965	0.00002987
2600	0.00024845	0.00027837	0.8925	0.00002993
2700	0.00024084	0.00026892	0.8956	0.00002808
2800	0.00024026	0.00026011	0.9237	0.00001985
2900	0.00022879	0.00025188	0.9083	0.00002309
3000	0.00021695	0.00024416	0.8886	0.00002721
3100	0.00021036	0.00023692	0.8879	0.00002656
3200	0.00020377	0.00023010	0.8855	0.00002634
3300	0.00020093	0.00022368	0.8983	0.00002275
3400	0.00019514	0.00021762	0.8967	0.00002248
3500	0.00018655	0.00021189	0.8804	0.00002533
3600	0.00018007	0.00020646	0.8722	0.00002639
3700	0.00017608	0.00020131	0.8747	0.00002523
3800	0.00017200	0.00019641	0.8757	0.00002442
3900	0.00016695	0.00019176	0.8706	0.00002481
4000	0.00016345	0.00018733	0.8725	0.00002388
4100	0.00016239	0.00018310	0.8869	0.00002071
4200	0.00015528	0.00017907	0.8671	0.00002379
4300	0.00015463	0.00017521	0.8825	0.00002058
4400	0.00014860	0.00017152	0.8664	0.00002292
4500	0.00014539	0.00016799	0.8655	0.00002260
4600	0.00014164	0.00016460	0.8605	0.00002296
4700	0.00013882	0.00016136	0.8603	0.00002254
4800	0.00013727	0.00015824	0.8675	0.00002097
4900	0.00013503	0.00015524	0.8698	0.00002021
5000	0.00013092	0.00015236	0.8593	0.00002144

Table 5
Table of $Q(N)$, $J(N)$, etc. (from $N = 5000$ to 10000 step 100)

N	$Q(N)$	$J(N)$	$K(N) = Q(N)/J(N)$	$R(N) = J(N) - Q(N)$
5000	0.00013092	0.00015236	0.8593	0.00002144
5100	0.00012889	0.00014958	0.8616	0.00002070
5200	0.00012607	0.00014691	0.8582	0.00002083
5300	0.00012342	0.00014433	0.8551	0.00002091
5400	0.00012104	0.00014185	0.8533	0.00002081
5500	0.00011938	0.00013945	0.8561	0.00002007
5600	0.00011861	0.00013713	0.8649	0.00001852
5700	0.00011630	0.00013489	0.8622	0.00001859
5800	0.00011309	0.00013273	0.8520	0.00001964
5900	0.00011272	0.00013063	0.8629	0.00001791
6000	0.00010911	0.00012860	0.8484	0.00001950
6100	0.00010728	0.00012664	0.8471	0.00001936
6200	0.00010545	0.00012474	0.8454	0.00001929
6300	0.00010416	0.00012289	0.8476	0.00001873
6400	0.00010520	0.00012110	0.8638	0.00001589
6500	0.00010197	0.00011936	0.8543	0.00001739
6600	0.00010081	0.00011767	0.8567	0.00001686
6700	0.00009843	0.00011603	0.8483	0.00001761
6800	0.00009712	0.00011444	0.8486	0.00001733
6900	0.00009601	0.00011289	0.8504	0.00001689
7000	0.00009579	0.00011139	0.8599	0.00001560
7100	0.00009325	0.00010992	0.8483	0.00001668
7200	0.00009143	0.00010849	0.8427	0.00001706
7300	0.00009008	0.00010711	0.8411	0.00001702
7400	0.00008856	0.00010575	0.8374	0.00001719
7500	0.00008786	0.00010443	0.8413	0.00001657
7600	0.00008630	0.00010315	0.8366	0.00001685
7700	0.00008547	0.00010190	0.8388	0.00001642
7800	0.00008427	0.00010067	0.8371	0.00001640
7900	0.00008357	0.00009948	0.8400	0.00001591
8000	0.00008205	0.00009832	0.8345	0.00001627
8100	0.00008125	0.00009718	0.8360	0.00001594
8200	0.00008133	0.00009607	0.8465	0.00001474
8300	0.00007984	0.00009499	0.8406	0.00001515
8400	0.00007904	0.00009393	0.8415	0.00001489
8500	0.00007889	0.00009289	0.8492	0.00001401
8600	0.00007822	0.00009188	0.8513	0.00001366
8700	0.00007602	0.00009089	0.8364	0.00001487
8800	0.00007499	0.00008992	0.8339	0.00001493
8900	0.00007425	0.00008898	0.8345	0.00001472
9000	0.00007283	0.00008805	0.8271	0.00001522
9100	0.00007182	0.00008714	0.8242	0.00001532
9200	0.00007103	0.00008626	0.8234	0.00001523
9300	0.00007051	0.00008539	0.8258	0.00001488
9400	0.00006969	0.00008453	0.8244	0.00001484
9500	0.00007040	0.00008370	0.8411	0.00001330
9600	0.00006891	0.00008288	0.8315	0.00001397
9700	0.00006909	0.00008208	0.8417	0.00001299
9800	0.00006868	0.00008129	0.8449	0.00001261
9900	0.00006877	0.00008052	0.8540	0.00001175
10000	0.00006676	0.00007977	0.8370	0.00001300

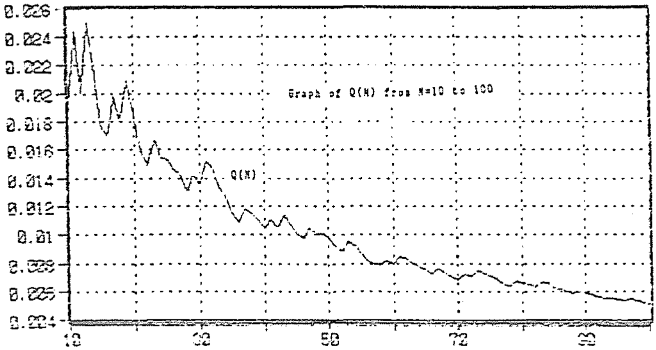


Fig. 1.

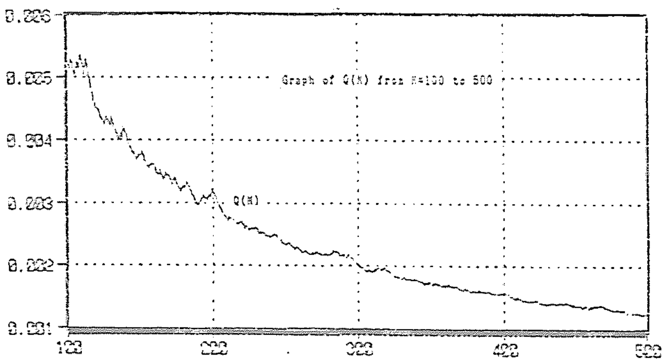


Fig. 2.

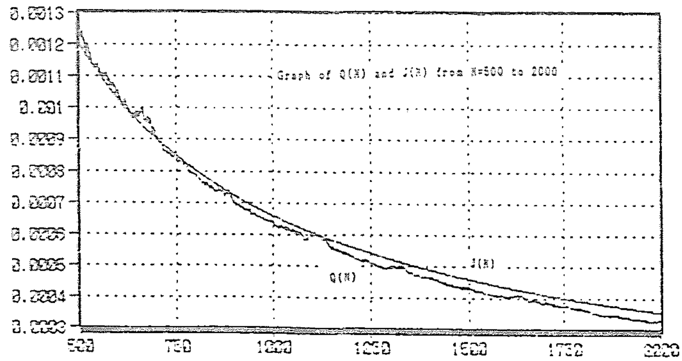
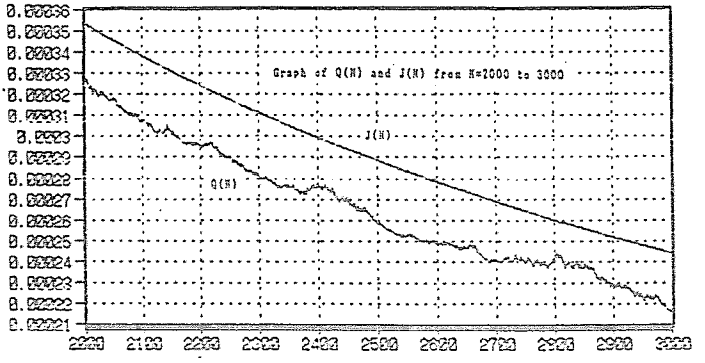


Fig.3

Fig. 3.



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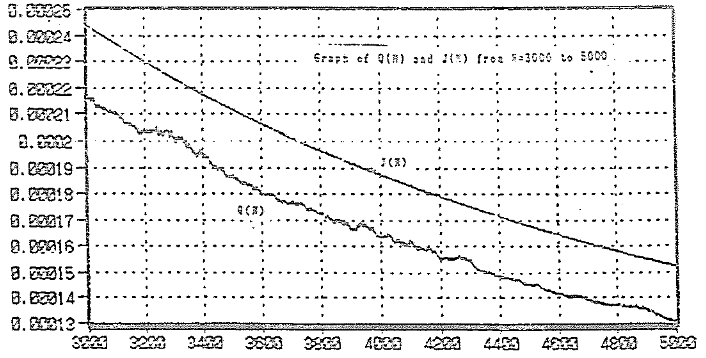

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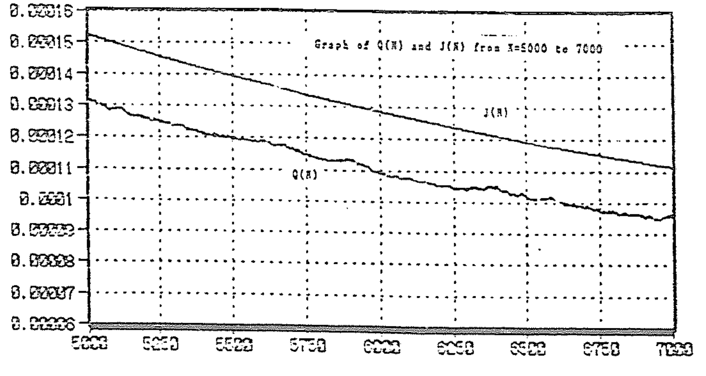

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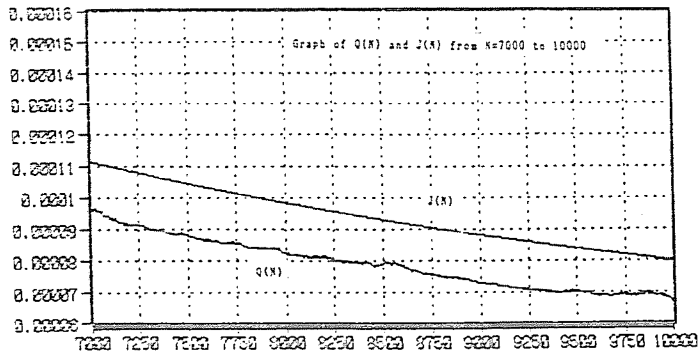

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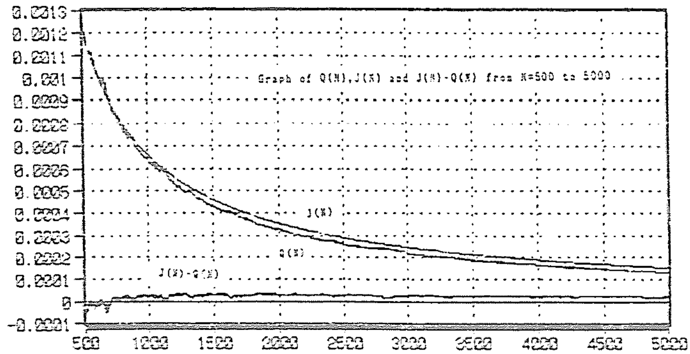

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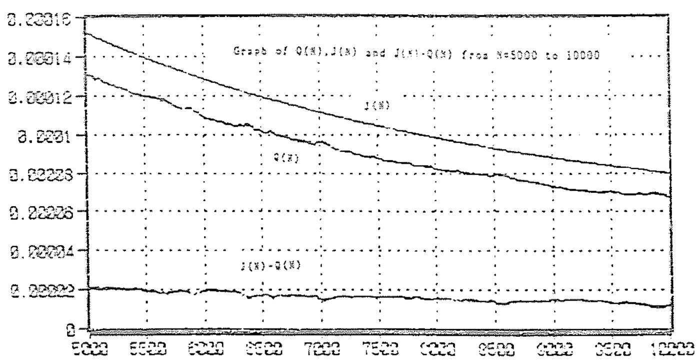

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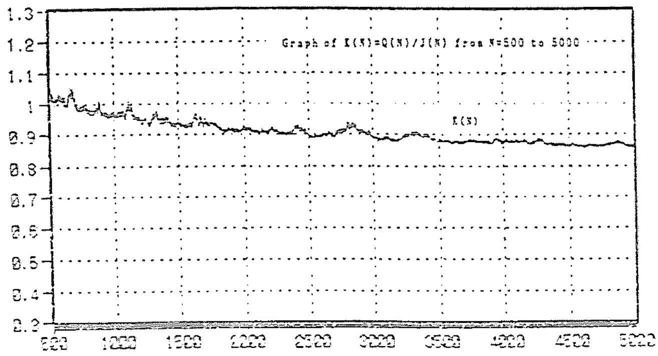
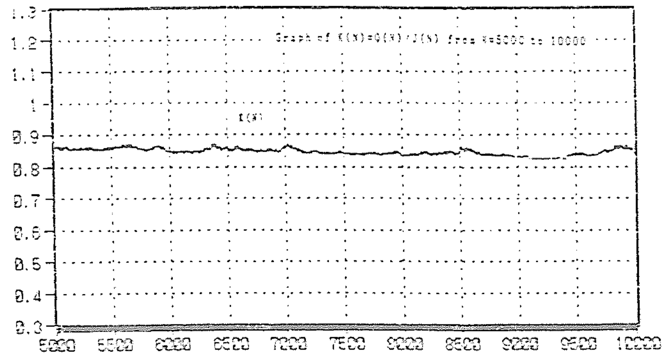


Fig. 10.



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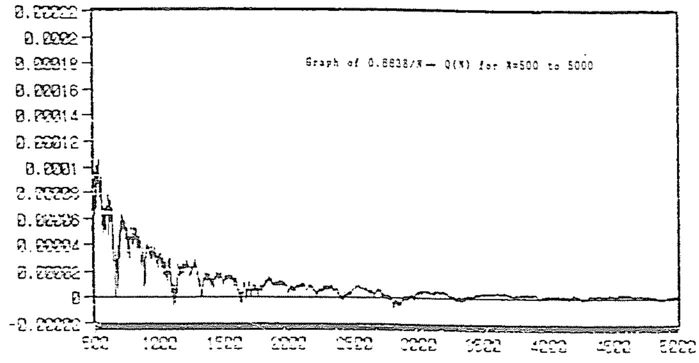

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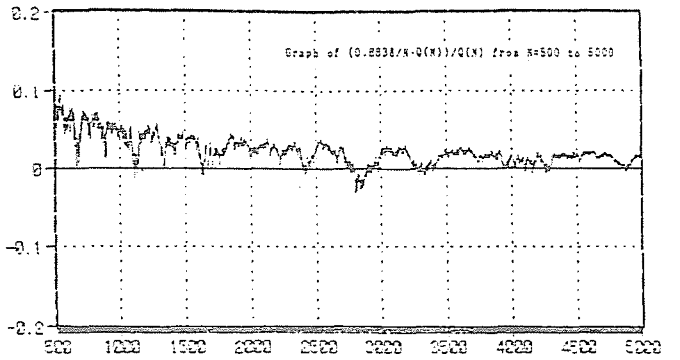


Fig. 13.

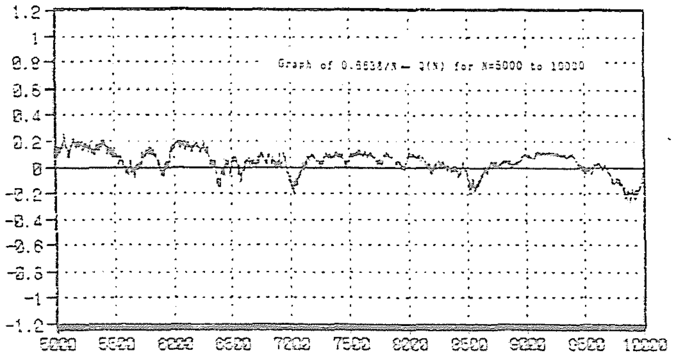
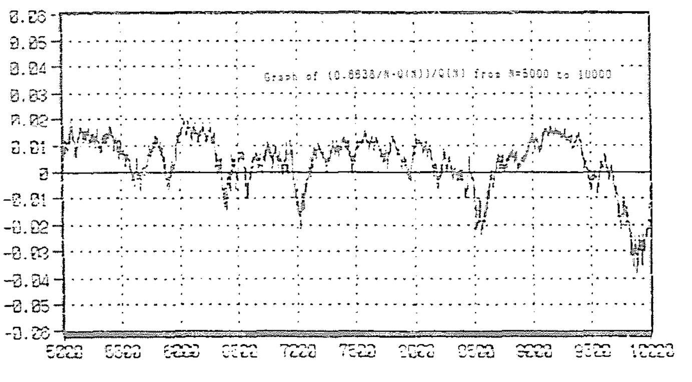


Fig. 14.



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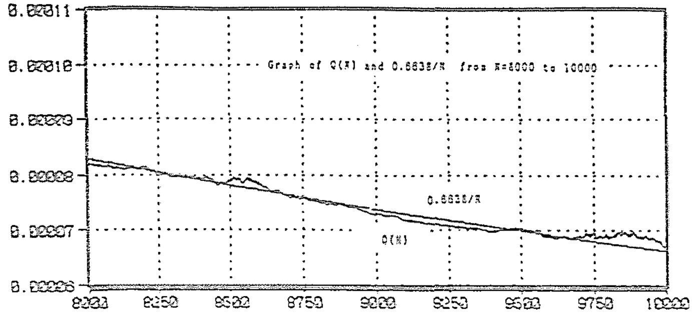

Fig. 16.

Result of Regression analysis about $Q(N)$ for $8000 \le N \le 10000$

Regression line(curve)	:	$y = \alpha^{*}t, t = 1/x$
Sample size	:	$n = 2001$
Degree of freedom	:	$p = 2000$
Regression coefficient of t	:	$\alpha = 0.6638287$
Standard error of α	:	0.0001767
Standard error of y	:	$s = 0.0000008$
Coefficient of determination	:	$R^2 = 0.9626483$

$y = 0.6638/x$

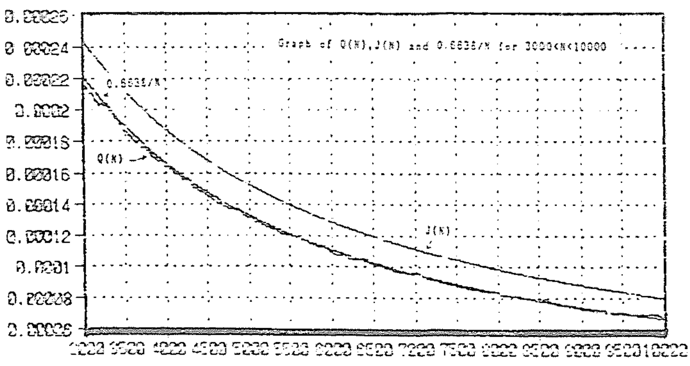

Fig. 17.

Acknowledgements

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