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TESTING IFR AND NBU CLASSES[1]

Gábor ZSIGRI

Department of Mathematics,
Faculty of Civil Engineering,
Technical University of Budapest,
H–1521, Budapest, Hungary

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Abstract

In this paper a test is proposed for testing whether a life distribution function F is IFR or not and another test for testing that F is NBU or not. The tests are shown to be consistent. Some selected critical values and powers are tabulated using Monte Carlo methods. Finally the IFR test is compared with the natural test for testing IFR class.

Keywords: NBU classes, IFR classes, testing for ageing distributions.

1. Introduction

The classes of ageing distributions play a central role in the reliability theory. Usually we do not know a parametric form of the underlying distribution, but we know, for example, that the failure rate is increasing. This information helps us to find good tests for testing exponentiality (DOKSUM and YANDELL (1984)), to examine coherent systems and shock models, and to find out geometric properties of our distribution function (HOLLANDER and PROSCHAN (1984)). A lot of technical journals and university reports are interested in this topic for example Technometrics, Microelectron. Reliab., Biometrika, Biometrics. Several good papers about this topic can be found in the above mentioned papers references.

We shall need a lot of definitions:

DEFINITION 1. A distribution function F is a life distribution ($F \in D^+$) if $F(x) = 0$ for $x < 0$. The corresponding survival function is $\overline{F} = 1 - F$.

DEFINITION 2. Let the density function of F be f. Then the failure rate function $r(x)$ is the following:

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$$r(x) = \frac{f(x)}{\overline{F}(x)},$$

when $\overline{F}(x) > 0$.

We may interpret $r(x)dx$ as the probability that a unit alive at age x will fail in $(x, x + dx)$, where dx is small.

DEFINITION 3. F is said to be IFR (that is F has increasing the failure rate) if $r(x)$ is non-decreasing.

It is easy to see that F is IFR iff (if and only if) $-\log(\overline{F}(x))$ is convex for all $x \geq 0$.

DEFINITION 4. The distribution function F is NBU (New Better than Used) if

$$\overline{F}(x + y) \leq \overline{F}(x)\overline{F}(y)$$

for all $x, y \geq 0$.

Dividing by $\overline{F}(x)$ we get

$$P(X \geq x + y | X \geq x) \leq P(X \geq y)$$

and this inequality says that a used unit of any fixed age has stochastically smaller residual lifelength than a new one.

These two classes are widely usable so if we have a sample we may need a test for testing whether F is IFR or not and another test for testing whether F is NBU or not. Sections 2 and 3 will introduce such tests. Section 4 gives the estimated critical values and powers, and compare our IFR test and the trivial Kolmogorov type test.

2. Testing for IFR

Suppose $F \in D^+$ is continuous and we want to test

$$H_0 : \ F \in IFR \text{ vs. } H_1 : \ F \notin \ IFR.$$

Then we can use the property that F $\in$IFR iff $-\log\overline{F}(t)$ is a convex function. But (by continuity of $\overline{F}$ and thus by $-\log\overline{F}(t)$) convexity is equivalent to the following property

$$-\frac{\log\overline{F}(x) + log\overline{F}(y)}{2} \geq -\log\overline{F}(\frac{x+y}{2})$$

for all $x, y \geq 0$. In other words $F \in$ IFR iff

$$R(F) = \sup_{x,y} \overline{F}(x)\overline{F}(y) - \overline{F}^2(\frac{x+y}{2}) = 0.$$

Thus it seems reasonable to use the test statistic

$$R(F_n) = \sup_{x,y} \overline{F}_n(x)\overline{F}_n(y) - \overline{F}_n^2(\frac{x+y}{2}).$$

Since $R(F) = 0$ on H_0 and $R(F) > 0$ on H_1 we shall reject H_0 if $R(F_n)$ is large.

We shall need the following result.

PROPOSITION 1. If G is exponential and $F \in$IFR then $R(F_n)$ is stochastically smaller than $R(G_n)$. (In other words, there exists a random variable Y such that $R(F_n) \leq Y$ and Y has the same distribution as $R(G_n)$.)

PROPOSITION 2. If F is exponential then

$$\sqrt{n}R(F_N) \xrightarrow{d} h_F(B(t)),$$

where $B(t)$ is the Brownian bridge and h_F is the functional

$$h_F(f(t)) = \sup_{x,y}[\overline{F}(x)f(F(y)) + \overline{F}(y)f(F(x)) - 2f(F(\frac{x+y}{2}))\overline{F}(\frac{x+y}{2})].$$

The critical values $R(F_n) \geq c_{n,\alpha}$ satisfy

$$P(R(G_n) > c_{n,\alpha}) \leq \alpha \leq P(R(G_n) \geq c_{n,\alpha})$$

where G is exponential with parameter 1. For large enough n the critical value $c_{n,\alpha} = \frac{c_\alpha}{\sqrt{n}}$, where c_α is the upper α quantile of $h_G(B(t))$. Finally we note that if $X_1^* \leq ... X_n^*$ is the ordered sample from F then

$$R(F_n) = \sup_{1 \leq i \leq j \leq n} (\frac{n-i+1}{n})(\frac{n-j+1}{n}) - \overline{F}_n^2(\frac{X_i^* + X_j^*}{2})$$

with probability one, thus we can easily compute $R(F_n)$.

Table 1
Critical values for our IFR test

N	$\alpha = 0.2$	$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
15	.2222	.2222	.2400	.2488	.2488
20	.1975	.2025	.2100	.2275	.2475
25	.1920	.2016	.2096	.2240	.2464
30	.1788	.1822	.1955	.2100	.2322
35	.1681	.1763	.1869	.1959	.2195
40	.1593	.1650	.1743	.1875	.2100
45	.1516	.1600	.1683	.1822	.2044
50	.1464	.1524	.1616	.1724	.1976

3. Testing for NBU

Suppose again that F is continuous and we want to test

$$H_0: \ F \in NBU \text{ vs } H_1: \ F \notin NBU.$$

Introduce

$$S(F) = \sup_{x,y}[\overline{F}(x+y) - \overline{F}(x)\overline{F}(y)].$$

If $F \in H_0$ then $S(F) = 0$, otherwise $S(F) > 0$. Thus $S(F_n)$ seems to be a good test statistic.

PROPOSITION 3. If G is exponential and $F \in$NBU then $S(G_n)$ is stochastically bigger than $S(F_n)$.

PROPOSITION 4. If F is exponential then

$$\sqrt{n}S(F_n) \xrightarrow{d} k_F(B(t)).$$

where $B(t)$ is the Brownian bridge and

$$k_F(f(t)) = \sup_{x,y}[f(F(x+y)) - \overline{F}(x)f(F(y)) - \overline{F}(y)f(F(x))].$$

In this test the asymptotic critical value is again the upper α quantile of $k_G(B(t))$ where G is exponential with parameter one. Finally we note that

$$S(F_n) = \sup_{1\le i\le j\le n}[\overline{F}_n(X_i^* + X_j^*) - (\frac{n-i}{n})(\frac{n-j}{n})]$$

with probability one ($X_0^* := 0$).

Table 2
Critical values for our NBU test

N	$\alpha = 0.2$	$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
15	.2133	.2222	.2400	.2488	.2488
20	.1975	.2000	.2100	.2275	.2475
25	.1776	.1904	.2016	.2096	.2400
30	.1655	.1766	.1833	.2000	.2222
35	.1591	.1640	.1763	.1910	.2195
40	.1500	.1600	.1693	.1806	.2093
45	.1402	.1481	.1585	.1713	.1955
50	.1376	.1460	.1552	.1680	.1904

4. Critical Values and Power

Monte Carlo estimations of the critical values of $R(F_n)$ and $S(F_n)$ are in *Table 1* and *Table 2*. Each estimation is based on 10,000 trials.

Let X be a random variable uniformly distributed on $[0,1]$, and denote by F_0 the distribution function of $2X+[2X]$ (where $[x]$ is the biggest integer less than or equal to x). It is obvious that F_0 is neither in IFR nor in NBU. For this F_0 the estimated power of our tests (i.e. the probability of rejecting

$$H_0 : F \in IFR \text{ or } F \in NBU, \textit{ respectively})$$

is shown by *Table 3* and *Table 4*, respectively.

Let us compare our tests and the Kolmogorov type tests denoted by R_2 (in case of IFR) and S_2 (in case of NBU). (Kolmogorov type means that R_2 (S_2) rejects

$$H_0 : \ F \in IFR \ (F \in NBU)$$

if the Kolmogorov distance between the distribution function and the class IFR (NBU) is too big, for example greater than the critical values of the usual Kolmogorov test.) My computations show that the power of the Kolmogorov type tests is nearly zero when the sample size is less than 40. So they seem to be useless, but if we use smaller critical values the tests perform much better. Similar results can be stated as Prop. 1 and Prop. 3 that is we can estimate the critical values using the exponential distribution. The estimated power of R_2 is shown by *Table 5*.

Each power estimation was based on min. 1000 trials.

We can see that our IFR (R_1) test performs better than the Kolmogorov type (R_2) test but the difference is not too big.

Table 3
Power of our IFR test

α	$N=15$	$N=20$	$N=25$	$N=30$	$N=35$	$N=40$
.20	.68	.96	.97	.99	1	1
.15	.68	.96	.95	.99	.99	1
.10	.37	.87	.94	.97	.99	1
.05	0	.72	.78	.94	.99	.99
.01	0	.16	.33	.76	.99	.98

Table 4
Power of our NBU test

α	$N=15$	$N=20$	$N=25$	$N=30$	$N=35$	$N=40$
.20	.96	.96	.99	.99	1	1
.15	.67	.96	.98	.99	1	1
.10	.46	.90	.95	.99	.99	1
.05	0	.74	.94	.98	.98	.99
.01	0	.18	.58	.89	.90	.99

Table 5
Power of R_2 test $\alpha = 0.1$

$N=15$	$N=20$	$N=25$	$N=30$	$N=35$
.29	.39	.64	.83	.94

References

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