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AN INVERSE MARKOV–CHEBYSHEV INEQUALITY

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Abstract

Suppose that X is an arbitrary nonnegative random variable with three given moments $E(X)$, $E(X^2)$ and $E(X^3)$. Lower bounds will be given for the tail probabilities $P(X > a)$.

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One of the most classical and most investigated class of probability inequalities gives bounds on the expected value $Ef(X)$ given $Eg_i(X) = c_i$, $i = 1, 2, \ldots, m$ where X is a real valued random variable, $c_i \in R$, f and $g_i \in R \rightarrow R$ such that the expectations above exist. In case $g_i(x) = x^i$ this is a modification of the moment problem (see SHOHAT and TAMARKIN,(1943) Ch.III). The best known special case for $X \geq 0$ is $P(X \geq a) \leq Eg(X)/g(a)$ ($m = 1, f(x) = I_a(x)$, the indicator function of $[a, \infty)$) which is Markov's inequality for $g(x) = x$ and Chebyshev's inequality for $g(x) = x^2$. These inequalities can be found in most introductory texts (for more information on their history and recent advances see the References). On the other hand it is hard to find in the literature lower bounds on $P(X \geq a)$. LOÈVE (1977, p. 159) gives one for bounded random variables: if $P(0 \leq X \leq c) = 1$ then $P(X \geq a) \geq [Eg(X) - g(a)]/c$. If only $a > E(X) > 0$ is given then clearly the best lower bound is trivial (=0). The same holds if both $a > E(X) > 0$ and $E(X^2)$ (or Var$X$) are given. In case $a < E(X) = 1$ FELLER (1966, p. 152) provides the inequality $P(X > a) \leq (1-a)^2/E(X^2)$. To get nontrivial bounds in general case suppose that $E(X)$, $E(X^2)$ and $E(X^3)$ are given and we seek a Markov–Chebyshev type lower bound in the form

$$P(X \geq a) \geq a_0 + a_1 E(X)/a + a_2 E(X^2)/a^2 + a_3 E(X^3)/a^3. \qquad (1)$$

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THEOREM. *If $X \geq 0$ and $E(X^3)$ is finite then*

$$P(X > a) \geq \sup_{\alpha > 1} \frac{(2\alpha - 3\alpha^2)E(X)/a + (3\alpha^2 - 1)E(X^2)/a^2 + (1 - 2\alpha)E(X^3)/a^3}{\alpha^2(1-\alpha)^2} \qquad (2)$$

and if there exists a random variable X supported on $\{0, a, b\}$ for some $b > a$ with prescribed first, second and third moments then for this random variable X (2) is an equality (thus in many cases (2) cannot be improved).

REMARKS. *Observe that on the right-hand side of (2) the sum of the coefficients is 0.*

The explicit value of the best α is not simple, it is a solution of a cubic equation, and depends on $E(X^i)$, $i = 1, 2, 3$ and a. However, we need not use the best value. E.g. if we choose $\alpha = 2$ we get a simple nontrivial (but not necessarily best) bound

$$P(X > a) \geq -2E(X)/a + \frac{11}{4}E(X^2)/a^2 - \frac{3}{4}E(X^3)/a^3. \qquad (3)$$

THE PROOF OF THE THEOREM *If (1) holds for all $X \geq 0$ with finite $E(X^3)$ then it surely holds for all random variables degenerate at $x \geq 0$. Thus for the indicator function $I_a(x)$ of $[a, \infty)$ we have*

$$I_a(x) \geq p(x) = a_0 + a_1 x/a + a_2 x^2/a^2 + a_3 x^3/a^3. \qquad (4)$$

Since $p(x) \leq 0$ on $(0, a)$, bounded from above on (a, ∞), there exists an $x_0 < 0$ such that $p(x_0) = 0$. If $p(x) < 0$ on $[a, \infty)$ then (4) does not give any nontrivial bound. Therefore there must exist an $x_1 \geq a$ such that $p(x_1) = 0$. Denote by $b \in (a, \infty)$ the unique number where $p(x)$ takes its maximum in (a, ∞). To get the best possible lower bounds we may suppose that $p(b) = 1$. For simplicity put $x_0 = 0$ and $x_1 = a$. Then we have the following conditions

$$p(0) = p(a) = 0, \quad p(b) = 1, \quad p'(b) = 0.$$

Using the notation $\alpha = b/a$ we get

$$p_\alpha^*(x) = \frac{(2\alpha - 3\alpha^2)x/a + (3\alpha^2 - 1)x^2/a^2 + (1 - 2\alpha)x^3/a^3}{\alpha^2(1-\alpha)^2},$$

therefore $I_a(X) \geq p^*_\alpha(X)$ *for every* $\alpha > 1$. *Taking expectations on both sides and then supremum for* $\alpha > 1$ *on the right hand side we get (2). We get equality in (2) if* X *is supported on* $\{0, a, b\}$.

REMARK. *The restriction* $X \geq 0$ *is essential. If* X *may take any* $x \in R$ *then (4) cannot hold since on* R *the right hand side of (4) is not bounded. Therefore nontrivial lower bounds on* $P(|X| > a)$ *require at least four moments* $E(X^i)$, $i = 1, 2, 3, 4$.

Examples

1. *If* X *is a random variable having the same moments* $E(X^i)$, $i = 1, 2, 3$ *as the uniform random variable on* $(0, 1)$ *then* $E(X^i) = (i+1)^{-1}$ *and thus (3) gives* $P(X > 1/2) \geq 1/6$. *This bound is sharp and is achieved for the random variable* $P(X = 0) = 1/6$, $P(X = 1/2) = 2/3$ *and* $P(X = 1/6) = 1/6$. *Loève's bound with* $g(x) = x^3$ *and* $c = 1$ *is* $1/8$.

2. *If* X *is a random variable having the same moments as the exponential distribution with mean* $1/\lambda$, *then* $EX^2 = 2/\lambda^2$ *and* $EX^3 = 6/\lambda^3$. *Thus the lower bound (2) for* $P(X \geq \frac{1}{c\lambda})$ *is* $1/12$ *for* $c = 1$, $27/120$ *for* $c = 2$, *and* $125/688$ *for* $c = 3$. *For the exponentially distributed* X, *the corresponding exact probabilities are* $e^{-1} = .3679$, $e^{-1/2} = .6065$, *and* $e^{-1/3} = .7165$. *Loève's inequality does not cover this case.*

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