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IDEAS ON THE HISTORY OF SCIENCE REVELATION OF THE CONCEPTS OF EULER AND NAVIER IN UP-TO-DATE STATICS[1]

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Abstract

Finite Element Method is one of the most important computational tools of modern statics. The main features of it are application of variational principles on one hand and discretizing of the domain, fitting at the nodes on the other hand. Both concepts are classical, their appearance in the mechanics is due to EULER and NAVIER, respectively.

Restricting ourselves to the analysis of bars and bar structures, it can be stated that EULER was engaged with the differential equation of the elastica and with the general method of solving variational problems round 250 years ago. In his genuine investigation he made use of variations being in accordance with the simple base functions of the F.E.M. Paper shows this derivation.

Navier, whose name is connected with the foundation of the theory of the elastic bars up to now, reduced the calculation of the deflection of the simply supported beam to that of the cantilever so as to investigate the sections of the structure always between two point forces, while fitting the exact solutions valid on separate intervals to each other. This idea is presented as well and the traditional results are recalled in an up-to-date symbolism.

Keywords: history of mechanics, Euler, Navier, bars, elastic behaviour, Finite Element Method, cantilever elements.

Introduction

University-level education of students attending technical schools has to convey not only practically useful knowledge of the subject matter, but increase technical and cultural intelligence as well as embed information into the intellectual behaviour. This aim is reached by discussing history of mechanics, which — from the point of view of up-to-date research — is useful by clarifying the roots of some methods that are applied right now as well as for ages, thus emphasizing their real values, too.

Biography of EULER and NAVIER furthermore their principal ideas to be treated here are presented in connection with the basic ideas of the

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Finite Element Method. At the same time we are going to make here an effort to present the Civil Engineering Mechanics in historical aspect, placing it among the historical eminencies of culture in general, especially architecture and art.

The paper deals with the outlines of the aforementioned work prepared already, as well.

Investigating Civil Engineering Mechanics as a Branch of Cultural History

First of all we are going to place mechanics in human experience. Therefore a genealogical tree of the mechanics has been prepared that depicts the topic as growing out of particular fields of the cultural history, presents the connection with special branches of learning and shows the detailed sciences into which it can be split up. The list of the categories and special domains is fairly not complete, the aim of the dividing is only a kind of giving a first idea. Consider *Fig. 1,* where horizontal lines mean proper relations.

Knowledge of the branches of the genealogy made it possible to compile chronologies in history, science and art. Then we have sketched the outlines of the history of Civil Engineering Mechanics, dealing particularly with the trends of the Hungarian science as well. We report hereby just those tables referring to the most famous scientists and the most ingenious results of our subject matter.

Both experts dealt with in details were emphasized by writing their names with capital letters. Report on life, work and importance of the others will be published elsewhere.

Based on the grouping of the events in history of culture and technology, as well as comparing simultaneous successes of human spirit we can prepare further papers as to present the treasures of science, art and literature connected to each other, in a mutual relation, as different parts of a common scale of values. This concept tries to consider the culture as a complete entity.

Further part of the paper contains biographical data about scientists of mechanics, the most famous Hungarian experts included. Information about Western scientists is indicated in a form common in travelling documents or application forms.

The matter concerning EULER and NAVIER is detailed in the next chapter.

Afterwards some revealing chapters of Civil Engineering Mechanics have been compiled, following the development trend of graphical statics,

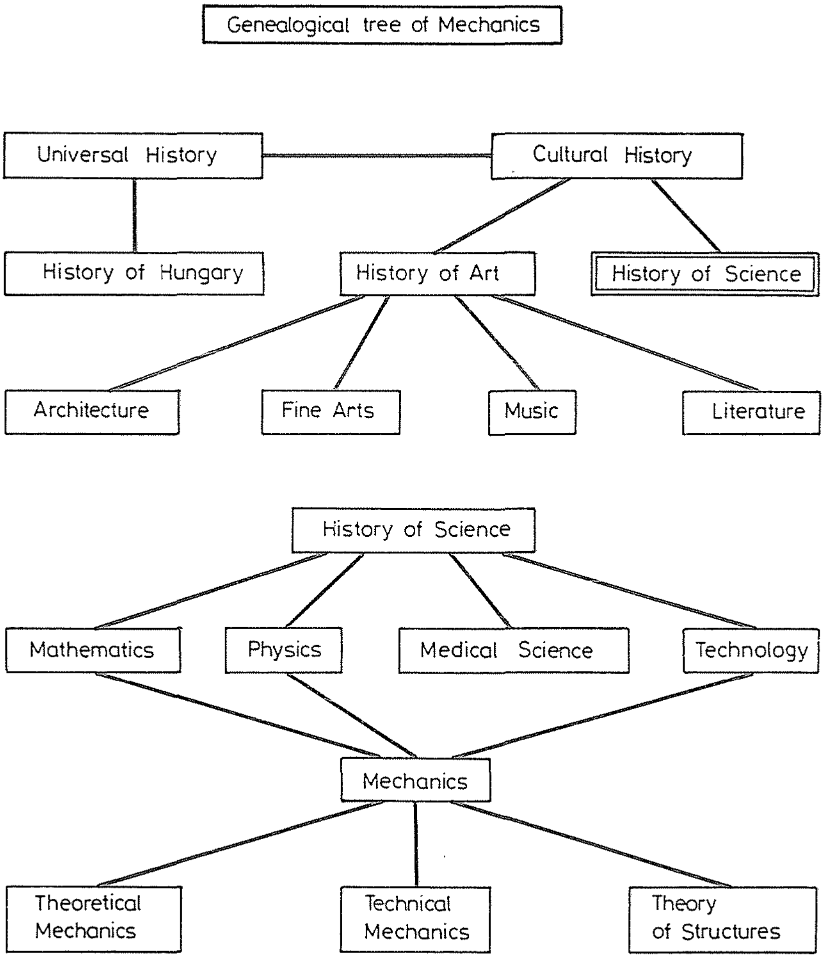

Fig. 1. Genealogical tree of Mechanics

variational calculus, and the theory of elasticity, potential theory included. Usually just the outlines of the results have been given, but sometimes derivations of interesting details have been presented, as well. Also a set of valuable works of Hungarian experts like professors KHERNDL, EGERVÁRY, SZILY, BARTA and CSONKA has been shown. Thus we were going to connect the graduate studies of mechanics with a historical view. The subject matter concerned is represented hereby once again by the ideas of EULER and NAVIER, respectively.

Table 1
Trend of the science, upto the development of the theory of structures

Archimedes	287–212 B. C.	Carl F.Gauss	1777–1855
Leonardo da Vinci	1452–1519	LOUIS M. H. NAVIER	1785–1836
Galileo Gallilei	1564–1642	Augustin Cauchy	1789–1857
Robert Hooke	1635–1703	Gabriel Lamé	1795–1870
Isaac Newton	1643–1727	Adhémar Barré	1797–1886
Pierre Varignon	1654–1722	Émile Clapeyron	1799–1864
Johann Bernoulli	1667–1748	Karl Culmann	1821–1884
LEONARD EULER	1707–1783	James C. Maxwell	1831–1879
J. le R. D'Alembert	1717–1783	Christian Otto Mohr	1835–1918
Joseph L. Lagrange	1736–1813	Joseph Boussinesq	1842–1929

Table 2
The milestones of the science, with particular respect to the theory of structures

Gravitation, axioms of mechanics, speed, acceleration	Newton
Principle of virtual displacements	Leonardo da Vinci John Bernoulli Jnr.
Critical load of a slender column	Euler
Dynamics reduced to statics Basic equations of dynamics	D'Alembert
Basic equations of elasticity	Lagrange Cauchy
Technical theory of bending	Navier
Limiting principle of boundary effects	De Saint Venant
Reciprocal theorems of displacements	Maxwell

Finally we have collected some interesting pictures and facsimile from the cultural history of Mechanics. *Pictures 1, 2* and *3* prove that the

Table 3
Basic concepts and principles in the theory of structures.
The explorers of the main ideas

Method of sections	Culmann
Concept of the stress resultant	Winkler
Bending moment diagram	Rebhann
Three-hinged arch, solution by superposition	Müller–Breslau
Bar exchanging method	Henneberg
Moment of inertia	Cauchy
Influence lines	Weyranch Mohr Müller–Breslau
Concept of the force method	Navier
Compatibility equation of the force method	Müller–Breslau
Elastic center	Kherndl
Displacement method	Bendixen Ostenfeld
Moment distribution method	Cross

differential equation of the deflection of has become a common property of the technical culture all over the world. They show the outlines of the theory in Bulgarian language, written by Cyrillic letters, then in Japanese, by Japan letters, finally in Swedish, by Latin letters. *Picture 4* shows a page of the mimeographed lecture note of the optional lecture of the professor at the T.U.B. Joseph Barta (held 1939), dealing with the double trigonometric series solution of the simply supported elastic plate, due to the famous theory of Navier.

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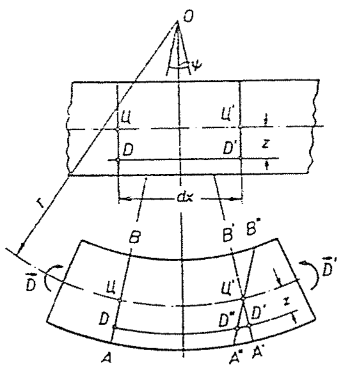
линии ще бъдат окръжности. Същото се отнася за всяка мислена надлъжна линия във вътрешността на гредата включително и оста на гредата.

Всички тези линии по повърхността и вътре в гредата ще си представяме като множество нишки, които се намират плътно една до друга, изпълвайки цялата греда. Изкривената ос на гредата се нарича еластична линия (термин, въведен от Я. Бернули). Тя лежи в цилиндрична повърхнина, която е перпендикулярна на симетрийната равнина. Частта от тази повърхнина, която попада в пространството на гредата, се нарича осов слой.



Фиг. 318

Всички отсечки от гредата, успоредни на осовия слой и на оста на гредата преди деформацията, след деформацията ще се удължат, ако попадат под слоя, или ще се скъсят, ако попадат над него. Това се вижда нагледно върху деформираната ортогонална мрежа. Отсечките от самия осов слой (включително тези от оста) запазват своите дължини при деформацията, поради това осовият слой при чистото огъване се нарича **неутрален слой**.

Да разгледаме две безкрайно близки напречни сечения, отстоящи на разстояние dx. Да означим с $d\psi$ ъгъла на тяхното релативно завъртане след деформацията и да разгледаме двете отсечки $ЦЦ'$ и DD' (фиг. 318). Първата отсечка, лежаща в неутралния слой, ще запази дължината си, а втората, лежаща във вертикална равнина под първата, ще се удължи. Да означим удължението на последната с Δdx и да прекараме права $A''B''$, успоредна на AB. От подобието на фигурите $OЦЦ'$ и $Ц'D'D''$ следва

(40.36) $$\frac{\Delta dx}{dx} = \frac{D'D''}{ЦЦ'} = \frac{z}{r}.$$

Но $\frac{\Delta dx}{dx}$ е относителното удължение на dx, следователно

(40.37) $$\varepsilon = \frac{z}{r}.$$

Ако си представим всяка нишка като цилиндрично тяло („греда"), имащо състояние чист опън, съгласно със закона на Хук (40.3) имаме $\varepsilon = \frac{\sigma}{E}$ и следователно

(40.38) $$\sigma = \frac{Ez}{r}.$$

Pic. 1

わみ角の公式は，暗記しておくべきものである。

4・2 たわみ曲線の微分方程式

図4・1(a)に示す単純ばりABに分布荷重qが作用する場合の曲げ変形について考える。はりの縦軸方向にx軸をとり，それに直交してy軸を定める。曲げの考察のときと同じように，xy平面ははりの断面の対称面とし，荷重はこの面内に作用するものとする。そうすると，たわみ曲線もこの平面内に生じる。

まず，たわみ曲線を表すたわみとたわみ角との記号について定義しておこう。たわみはvで表し，下向き，すなわちy軸の正方向のたわみを正のたわみとする。x軸より上の方向のたわみは負で表される。また，たわみ角はθで表し，変形前の軸線に対して，変形後の軸線が時針方向に回転する場合を正とする。図(a)の場合は，たわみvは，はりの縦軸上のすべての点で正であり，たわみ角θの符号は，はりの中央付近より左の部分では正，右の部分では負である。

さて，単純ばりの支点Aからの距離x，すなわち，座標xにおけるたわみv

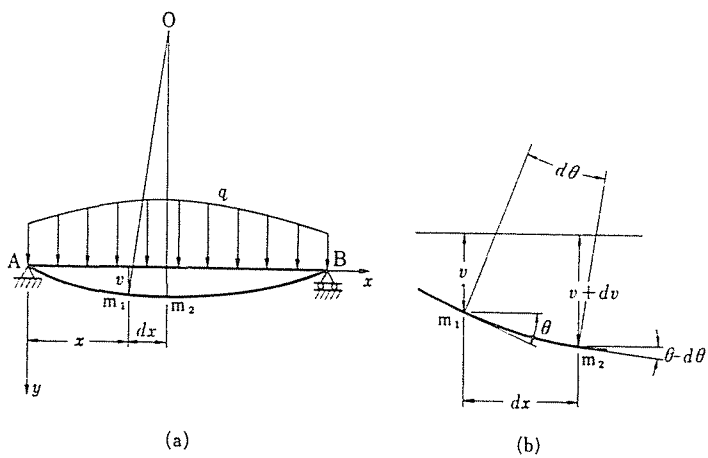


図 4・1 曲げを受けるはりのたわみ曲線と幾何学的関係

Pic. 2

Vinkeln Δm är liten så den är ungefär lika med tan för vinkeln, som är $\Delta x/R$. Den kan också uttryckas i momentet M, se (17), dvs

(93) $$\Delta m = \Delta x/R = M\Delta x/EI$$

varur

(94) $$\boxed{1/R = M/EI}$$

5.9.2 Elastiska linjen för en Euler-Bernoulli-balk

När varje balksegment i en rak balk deformeras enligt fig 46 och $\Delta x \to 0$ antar tyngdpunktslinjen en kontinuerlig, krökt kurva, fig 47. Den kallas elastiska linjen. I figuren visas en fritt upplagd balk belastad med lika stora ändmoment. Tvärkraften är då lika med noll längs balken, vilket är en förutsättning för (94). Utböjningen vinkelrätt x-axeln betecknas w. Om balken är jämnstyv, dvs EI = konstant, ger (94), eftersom $M = M_0$ = konstant, att elastiska linjen i detta fall är en cirkelkurva. Vid en annan belastning uppträder i allmänhet också tvärkrafter som ger upphov till ytterligare deformationer. Om vi bortser från dessa, erhålls ändå inte en cirkelkurva eftersom momentet och därmed M/EI ej längre är konstant längs balken.

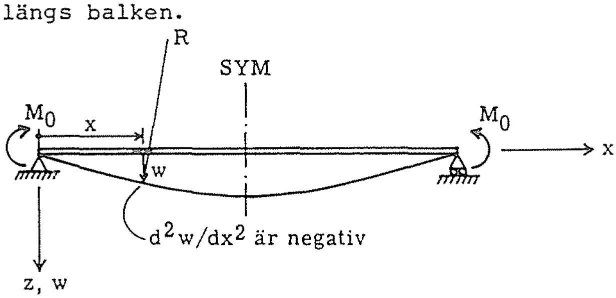


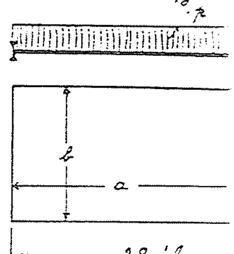
Fig 47. Elastiska linjen av balk vid ren böjning

Från matematiken hämtas följande formel för krökningsradien:

(95) $$1/R = -\frac{d^2w/dx^2}{[1 + (dw/dx)^2]^{3/2}}$$

Vid små utböjningar, som det i allmänhet är fråga om vid byggnadskonstruktioner, kan nämnaren i (95) approximeras till 1, så kombination av (94) och (95) ger

9. §. A szélén elforgathatóan megtámasztott egyenletesen megterhelt derékszögü négyszöglemez: Szimmetrikus kétszeresen végtelen sorokkal. - Derékszögü négyszöglemez esetében a (8) differenciálegyenlet megoldásának keresését össze fogom kötni a végtelen sorok alkalmazásával. A x, y tengelykereszt essék össze a négyszög két oldalával (28. ábra).



28. ábra.

A (8) differenciálegyenlet megoldására szolgáljon a

$$(37)\qquad w = \sum_h \sum_k A_{hk} \sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b}$$

felvétel, ahol h és k pozitiv egész számokat jelentenek. Ez a felvétel a $w = 0$ és $\Delta w = 0$ kerületi feltételeket a négyszögnek mind a négy oldalán kielégiti. Ha a $p(x,y)$ terhelésfüggvényt

$$p(x,y) = \sum_h \sum_k B_{hk} \sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b}$$

Fourier-sorba fejtjük, akkor e sor együtthatóit a

$$B_{hk} = \frac{4}{ab} \int_0^a \int_0^b p(x,y) \sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b}\, dx\, dy$$

kifejezés adja meg. Ebből, ha a teher egyenletes vagyis $p = \text{konst.}$, akkor

$$B_{hk} = \begin{cases} \frac{16p}{hk\pi^2}, & (h, k = 1, 3, 5, \ldots), \\ 0, & (h = 0, 2, 4, \ldots \text{ vagy } k = 0, 2, 4, \ldots). \end{cases}$$

Ennélfogva

$$(38)\qquad p = \frac{16p}{\pi^2} \sum_h \sum_k \frac{1}{hk} \sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b}, \quad (h, k = 1, 2, 3, \ldots)$$

A_{hk} együtthatók meghatározása végett (37) és (38) kifejezéseket (8)-ba helyettesitvén

$$\sum_h \sum_k A_{hk} \left(\frac{h^2\pi^2}{a^2} + \frac{k^2\pi^2}{b^2}\right)^2 \sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b} = \frac{16p}{\pi^2 N} \sum_h \sum_k \frac{1}{hk} \sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b}$$

egyenletet nyerjük. Ebből

$$A_{hk} = \begin{cases} \frac{16p}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 hk\pi^6 N}, & (h, k = 1, 3, 5, \ldots), \\ 0, & (h = 0, 2, 4, \ldots \text{ vagy } k = 0, 2, 4, \ldots) \end{cases}$$

következik, aminek a figyelembevételével az áthajlásra a

$$(39)\qquad w = \frac{16p}{\pi^6 N} \sum_h \sum_k \frac{\sin \frac{h\pi x}{a} \sin \frac{k\pi y}{b}}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 hk}, \quad (h, k = 1, 3, 5, \ldots)$$

képletet nyerjük. Ez tehát a szóban lévő kerületértékfeladat megoldása szeresen végtelen sorral kifejezve. Ebből a lemez közepének az áthajlá $x = \frac{a}{2}$, $y = \frac{b}{2}$ helyettesitéssel,

$$w_K = \frac{16p}{\pi^6 N} \sum_h \sum_k \frac{(-1)^{\frac{h+k}{2}+1}}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 hk}, \quad (h, k = 1, 3, 5, \ldots).$$

Pic. 4

Biographical Data of the Scientists Euler and Navier

Entering of the Calculus of Variations into Mechanics

NAME OF THE EXPERT:	Leonard EULER
DATE AND PLACE OF BIRTH:	April 15, 1707, Basel
DATE AND PLACE OF DEATH:	September 18, 1783, St.Petersburg
NATIONALITY:	Swiss
PROFESSION:	Mathematician, physicist

SHORT SCIENTIFIC BIOGRAPHY:

1723 –	magister
1723 –	first assistant at the Mathematical Division, Scientific Academy, St.Petersburg
1727–30	Marine lieutenant at the Tsar's Navy
1730–41	Professor at the Academy, Physics first, then Mathematics
1741–66	Member of the Prussian Academy, Berlin
1743 –	Director of the Division at the same place
1755 –	External member of the French Academy
1766–83	Working at the Academy, St.Petersburg

DATA OF ACTIVITY:

Books, papers:	Methodus inveniendi lineas curvas 1744. Introductio in analysim infinitorum 1748. Institutiones calculi integralis 1768–70. (Altogether 756 papers)
Scientific results:	Basic equations of hydrodynamics Differential equation of the calculus of variations Critical load of a slender bar Exploring the formula $\exp(\pi i) = -1$
Concepts, principles:	The existing world is the best of all worlds being. The world is generated by the ratio.

PRIVATE LIFE:

He was a good friend of the BERNOULLI brothers, also a co-worker of FRIDERICUS the GREAT. He had 12 children of two marriages. His home was burnt in 1771. He was gone blind to one eye 1735, afterwards to the other 1767.

Sources of the Theory of Structures

NAME OF THE EXPERT:	Louis Marie Henri NAVIER
DATE AND PLACE OF BIRTH:	February 15, 1785, Dijon
DATE AND PLACE OF DEATH:	August 23, 1836, Paris
NATIONALITY:	French
PROFESSION:	Civil engineer, mathematician

SHORT SCIENTIFIC BIOGRAHY:

1802 –	Entrance examination at the École Polytechnique
1804–08	Student at the same place
1808 –	Diploma in building of bridges and roads
After 1808	activity in engineering practice, member of the Corps of bridge and road constructors
1824 –	Member of the French Academy
Sept.6, 1826	Building accident of the Paris chain bridge
1829 –	Professor of Applied Mechanics at the École des Ponts et Chaussées
1830 –	Professor of Analysis and Mechanics at the École Polytechnique
After 1834	Supervisor of the bridge and road building at the Royal Ministry

DATA OF ACTIVITY:

Works:	Erection of bridges at Choisy, Argenteuil and Asnières
Books, papers:	Editorship of the book 'Traité des Ponts' by Gauthey 1813. Edition and comments to the books 'Science des Ingenieurs' and 'Architecture Hydraulique' by Belidor, 1819. Mémoire sur la flexion des verges elastiques courbes 1819. Mémoire sur les ponts suspendus 1823.
Scientific results:	Engineering theory of bending of bars

	Solution of simply supported elastic plates. A general theory of elasticity Hydrodynamics of viscous fluids (together with Stokes)
Concepts, principles:	Hypothesis of Navier (plain cross section remains plain in bending).

PRIVATE LIFE:
His uncle, the civil engineer Gauthey of Dijon has been his foster father from his age 14. Navier was an excellent teacher. His opinion was royalistic.

Basic Concept of F.E.M.
Merits of Euler and Navier with Respect to the Formulation of the Method

The fundamental idea of F.E.M. is as follows:

a) Stating either a stationarity or an extremum principle which is generally the principle of stationarity of the potential energy, utmost useful in the engineering practice.
b) Splitting up the domain into finite elements, describing the elements by several coordinate-systems (e. g. local, global, Euclidean 3D, Riemann 3D, natural, parametric etc.).
c) Defining the unknown displacement parameters of the elements, as well as the proper interpolation functions describing the displacements.
d) Corresponding to c), to select the unknown degrees of freedom.
e) Write up and solve the canonical equations.

Considering the aforementioned details, EULER proves to develop outstanding ideas in

a) analyzing the mathematical form of the extremum principle and formulating the general differential equation of the problem,
while NAVIER is involved in
b) formulating the most important structural element in engineering, i. e. the bar, making use of *local* and *global* frames, respectively, finally applying the cantilever beam as a part of a structure to be considered as a finite element.

Also he has established the basic idea of the force method useful at bars and the displacement method applied in the theory of elasticity.

Derivation of Euler's Differential Equation Using the Original Variations with Finite Elements

Next we reveal a derivation of EULER that analyses the differential equation of the problem of the bar being in simultaneous bending and compression, a problem customary in the Elementary Strength of Material. The calculation differs from the usual deduction of the EULER–LAGRANGE differential equation in the assumption of the variation itself. Instead of a complete variation of the unknown function between the prescribed boundary values, the function is varied just over two elementary intervals (*Fig. 2*). This kind of variation agrees with the use of linear finite elements (spline functions).

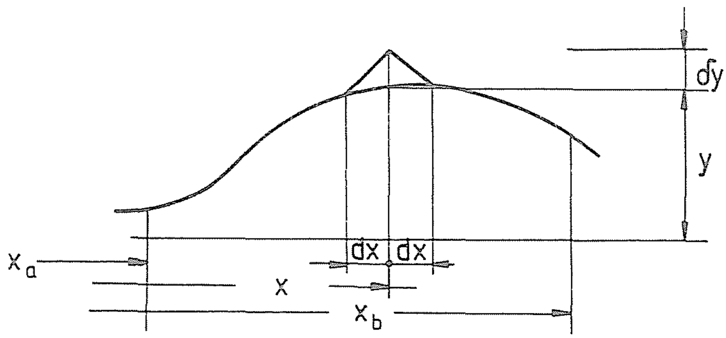


Fig. 2. Euler's spline variation

The notation is similar to our up-to-date convention rather than the original one.

To begin the analysis, let us consider the stationarity theorem of the potential energy used at a simply supported beam with straight axis. The structure is loaded by a distributed transversal load and point forces as lateral loads as well. The theorem states

$$\Pi(y) = \int_0^l \left\{ qy + \frac{P}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{EI}{2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 \right\} \mathrm{d}x = \text{stac!} \qquad (1)$$

This problem can be written in the more general mathematical form

$$J = \int_0^l f\left(x,\, y,\, \frac{\mathrm{d}y}{\mathrm{d}x},\, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) \mathrm{d}x = \min! \qquad (2)$$

By definition, the variation of the functional J reads as

$$\delta J\left(x,\, y,\, \frac{\mathrm{d}y}{\mathrm{d}x},\, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$$

$$= J\left(x,\, y+\delta y,\, \frac{\mathrm{d}y}{\mathrm{d}x}+\delta\frac{\mathrm{d}y}{\mathrm{d}x},\, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\delta\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) - J\left(x,\, y,\, \frac{\mathrm{d}y}{\mathrm{d}x},\, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right). \tag{3}$$

(1) becomes really stationary in case the expression (3) disappears at any variation δy of the function, being arbitrary, but small enough. EULER proved for the first time that the operations of variation and derivation are commutable.

$$\delta\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\delta y\,. \tag{4}$$

Also referring to the second variations and derivatives

$$\delta\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \delta\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\delta\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta y\,. \tag{5}$$

Since variation agreeing *Fig. 2* causes changes of the functional just at the neighbourhood of place x, due to three functions generated by each other,

$$\delta J = \left\{\frac{\partial f}{\partial y}\Big|_x \delta y + \frac{\partial f}{\partial y'}\Big|_x \delta y' + \frac{\partial f}{\partial y''}\Big|_x \delta y''\right\}\,\mathrm{d}x \tag{6}$$

holds. Interchanging the operations of variation and differentiation we have

$$\delta J = \left\{\frac{\partial f}{\partial y}\Big|_x \delta y + \frac{\partial f}{\partial y'}\Big|_x \frac{\mathrm{d}}{\mathrm{d}x}\delta y + \frac{\partial f}{\partial y''}\Big|_x \frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta y\right\}\,\mathrm{d}x\,. \tag{7}$$

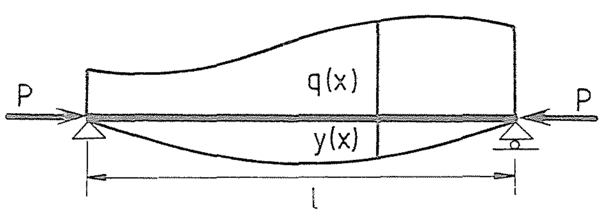


Fig. 3. Beam column

Applying the LEIBNIZ rule of differentiating products of two factors

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\Big|_x \delta y\right) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\Big|_x \delta y + \frac{\partial f}{\partial y'}\Big|_x \frac{\mathrm{d}}{\mathrm{d}x}\delta y\,. \tag{8}$$

Rearranging formula (8)

$$\frac{\partial f}{\partial y'}\Big|_x \frac{\mathrm{d}}{\mathrm{d}x}\delta y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\Big|_x \delta y\right) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\Big|_x \delta y\,. \tag{9}$$

The first term of the right side disappears provided the boundary condition is homogeneous. Then we can repeat our consideration in connection with the second variation as well:

$$\delta J = \left\{\frac{\partial f}{\partial y}\Big|_x \delta y - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\Big|_x \delta y + \frac{\partial f}{\partial y''}\Big|_x \frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta y\right\}\mathrm{d}x\,. \tag{10}$$

Rearranging the arbitrary variation δy

$$\delta J = \left\{\frac{\partial f}{\partial y}\Big|_x - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\Big|_x + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial f}{\partial y''}\Big|_x\right\}\delta y\,\mathrm{d}x\,. \tag{11}$$

This expression disappears at the whole interval $0 < x < l$ in case we have

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial f}{\partial y''} = 0 \tag{12}$$

in the parenthesis.

Performing the operations by the function contained in the integral (1), presented like the function of (2), we obtain

$$EIy^{IV} + \frac{P}{EI}y'' = q\,. \tag{13}$$

This is the well-known differential equation of the deflection line of a bar in simultaneous bending and compression. Thus we have presented one of the most important fundamental ideas of F.E.M., used already by EULER in the calculus of variations.

Navier's Method of Calculating the Displacement of Simply Supported Beams

Investigating the simply supported beam, the analysis of the cantilever serves as a basis. NAVIER has written up the deflection of a cantilever loaded at the free end by a point load, solving the boundary value problem of the differential equation of a bar in bending. Notations are shown in *Fig. 4*. The formulae of the solution are

$$y(x) = \frac{P}{EI}\left(\frac{x^2 z}{2} - \frac{x^3}{6}\right) = \frac{Px^2}{2EI}\left(z - \frac{x}{3}\right), \tag{14}$$

furthermore

$$y(z) = \frac{Pz^3}{3EI}, \qquad y'(z) = \frac{Pz^2}{2EI}. \tag{15}$$

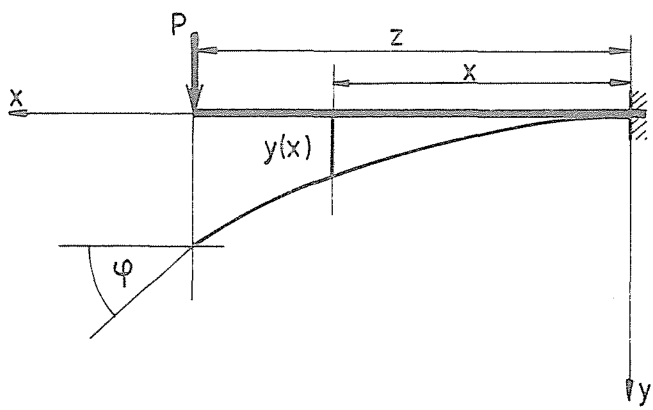


Fig. 4. Cantilever beam

Considering the cantilever as a fundamental element, while the simply supported beam as the ensemble of two different cantilevers having common clamping-in sections, NAVIER determines the influence line of the simply supported girder.

Fig. 5 shows the simply supported beam, consisting of two cantilevers. The most important idea of the analysis is that the displacement of the cross-section below the load is common, irrespective of whether it is calculated from the left or from the right. Thus it is possible, first of all, to describe the angular rotation of the cross-section below the load (*Fig. 6*).

The geometrical condition of the joining is

$$e_{1A} + e_{2A} = e_{1B} - e_{2B}, \tag{16}$$

where obviously

$$e_{1A} = P\frac{l-z}{l}\frac{z^3}{3EI} \qquad e_{2A} = z\phi, \tag{17}$$

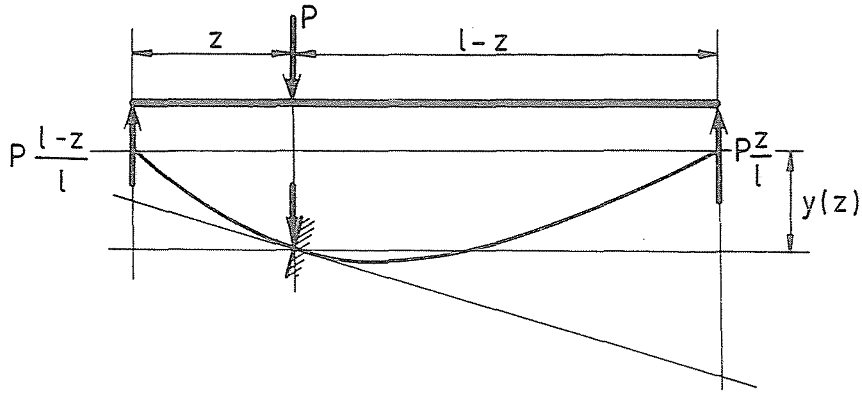


Fig. 5. Simply supported beam

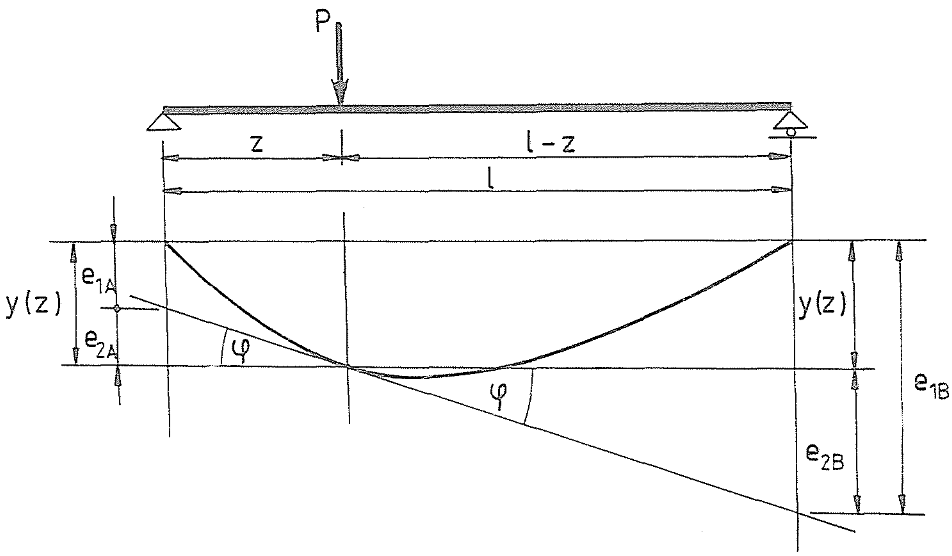


Fig. 6. Compatibility condition

furthermore

$$e_{1B} = P\frac{z}{l}\frac{(l-z)^3}{3EI}\,, \qquad e_{2B} = (l-z)\phi\,. \tag{18}$$

The formulae have been developed by assuming small displacements.

Replacing (17) and (18) in (16) and rearranging the result

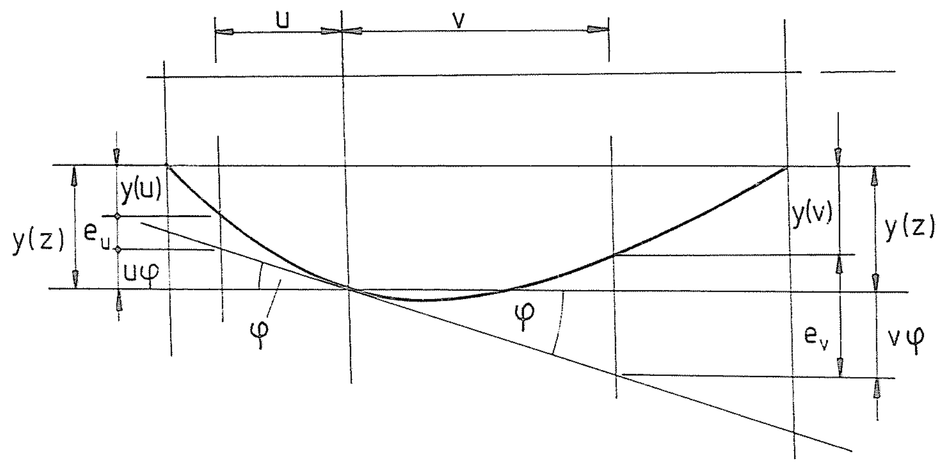


Fig. 7. Coupled cantilevers

$$\phi = \frac{P}{3EIl} z(l-z)(l-2z)\,. \tag{19}$$

Making use of the local frames presented in *Fig. 7* we can describe the displacement functions of both parts of the bar, distinguished by the load.

On the left side, starting from the fictitious clamping in:

$$y(u) = y(z) - u\phi - e_u = y(z) - u\phi - \frac{P(l-z)}{EIl}\left(\frac{u^2 y}{2} - \frac{u^3}{3}\right)\,. \tag{20}$$

Since at the left support we have

$$y(u=-z) = 0\,, \tag{21}$$

the deflection at the cross-section containing the load is

$$y(z) = \frac{P}{3EIl}(l-z)(4z-l)z^2\,. \tag{22}$$

On the right side, also starting from the fictitious clamping in,

$$y(v) = y(z) + v\phi - e_v = y(z) + v\phi - \frac{Pz}{EIl}\left[\frac{v^2(l-z)}{2} - \frac{v^3}{3}\right]\,. \tag{23}$$

Applying the reference frame of *Fig. 8* we can describe the suitable forms of the deflection function at both intervals:

If $0 \leq x \leq z$ then

$$y(x) = y(z) - (z-x)\phi - \frac{P(l-z)}{EIl}\left[\frac{(z-x)^2 z}{2} - \frac{(z-x)^3}{3}\right], \qquad (24.a)$$

while if $z \leq x \leq l$ then

$$y(x) = y(z) - (z-x)\phi - \frac{Pz}{EIl}\left[\frac{(x-z)^2(l-z)}{2} - \frac{(x-z)^3}{3}\right]. \qquad (24.b)$$

Finally, by selecting

$$P = 1, \qquad y = y(x,z) \qquad (25)$$

we obtain the deflection influence function of the cross-section z as well, due to the theorem of MAXWELL.

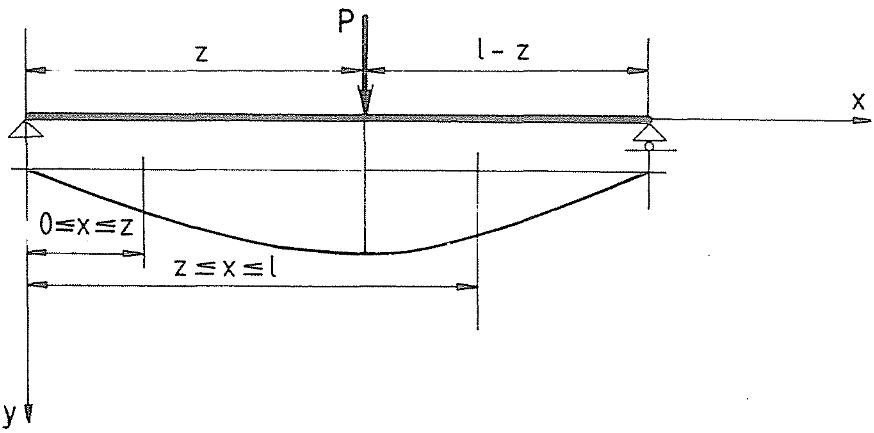


Fig. 8. Green-functions

The flexibility as well as the stiffness parameters of the simply supported bar can be obtained from the results of NAVIER, too. Thus, first of all we have to determine the angular rotation of an arbitrary cross-section due to a couple acting upon it. This latter can be treated as the entity of two equal and opposite forces, so the former results can be used.

$$M = pP, \qquad p \to 0, \qquad P \to \infty. \qquad (26)$$

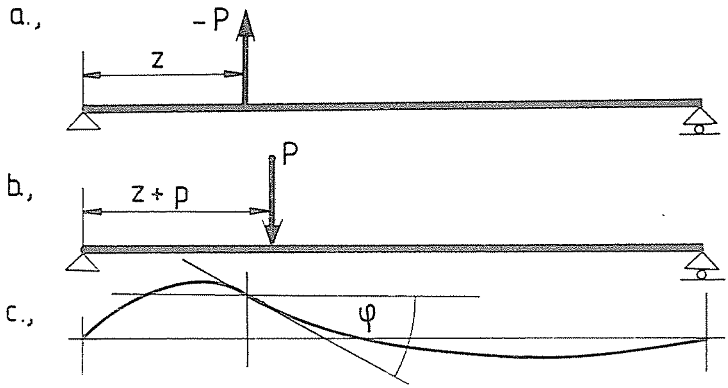


Fig. 9. Couple loading on the simply supported beam

The forces are presented in part a) and b) of *Fig. 9,* respectively. The deflection graph containing the rotation as the fundamental point of the solution is shown in *Fig. 9c.*

Due to the force showing upwards

$$\phi_1 = \frac{-P}{3EIl} z(l-z)(l-2z) , \tag{27}$$

while in the presence of the force showing downwards

$$\phi_2 = \frac{P}{3EIl}(z+p)(l-z-p)\left[l-2(z+p)\right] . \tag{28}$$

The result looked for is obtained by the limit of the sum of these latter angular rotations.

$$\phi = \lim_{\substack{p\to 0\\ P\to\infty\\ pP=M}} (\phi_1+\phi_2) = -\frac{M}{3EI}\left(3z - 3\frac{z^2}{l} - l\right) . \tag{29}$$

We also need the angular rotation caused by the point force at the right support of the simply supported beam. The calculation can be carried out by making use of a cantilever element, considering that the load of the cantilever is just the right-side reaction of the beam.

The cantilever is presented in *Fig. 10a,* while the geometry is shown in *Fig. 10b.* Hence

$$\phi_B = \phi - \frac{Pz}{l}\frac{(l-z)^2}{2EI} , \tag{30}$$

or in detail

$$\phi_B = \frac{P}{EI}\left[\frac{z(l-z)(2z-l)}{3l} - \frac{z}{l}\frac{(l-z)^2}{2}\right] = \frac{P}{6EI}\left(12z^2 - 7\frac{z^3}{l} - 5zl\right) . \tag{31}$$

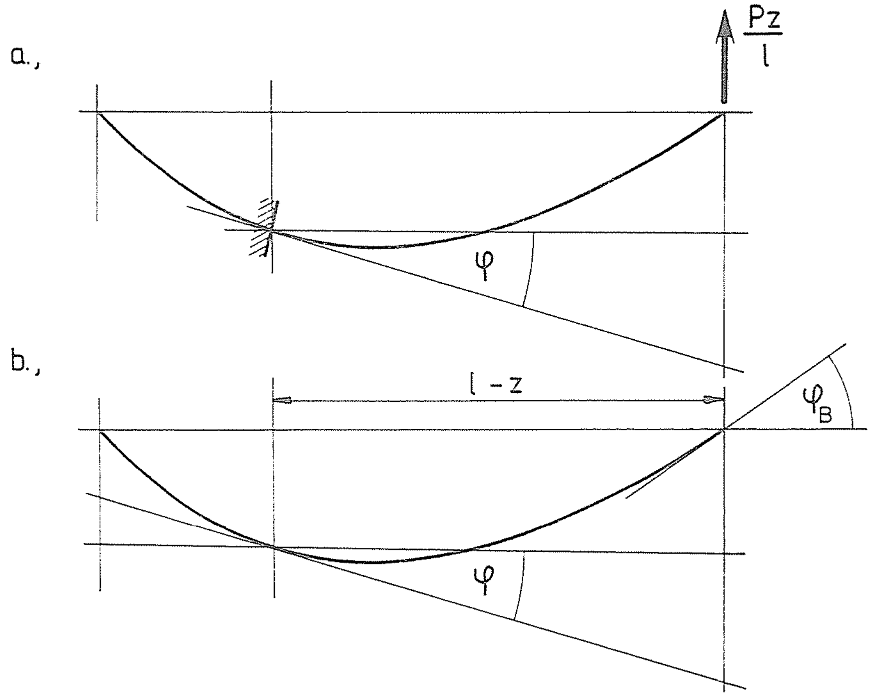


Fig. 10. Right side cantilever

This formula is suited to the determination of the right support's angular rotation due to a concentrated couple acting at the cross-section of coordinate z. Similarly to the operation (29), we obtain from the left-side force showing upwards,

$$\phi_{1B} = -\frac{P}{6EI}\left(12z^2 - \frac{7z^3}{l} - 5zl\right) = -\frac{Pz}{6EI}\left(12z - 7\frac{z^2}{l} - 5l\right) \tag{32}$$

while from the right-side force showing downwards

$$\phi_{2B} = -\frac{P}{6EI}(z+p)\left[12(z+p) - 7\frac{(z+p)^2}{l} - 5l\right] . \tag{33}$$

Thus as a result of the concentrated couple we have

$$\phi_B = \lim_{\substack{p \to 0 \\ P \to \infty \\ pP = M}} (\phi_{1B} + \phi_{2B}) = \frac{M}{6EI}\left(2z + \frac{z^2}{l} - l\right) . \tag{34}$$

Thereafter the flexibility coefficients can also be written, since they are defined as angular rotations due to unit couples acting at the ends of the beam (*Fig. 11*).

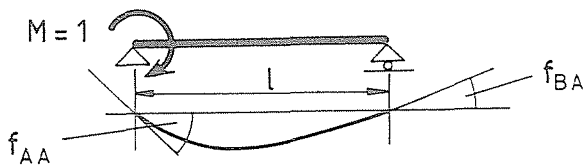


Fig. 11. Definition of the flexibility

$$f_{AA} = \phi(z=0) = \frac{l}{3EI} , \qquad f_{BA} = \phi_B(z=0) = \frac{-l}{6EI} . \tag{35}$$

Finally the flexibility and the stiffness matrices, respectively, read as

$$\mathbf{F} = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} , \qquad \mathbf{K} = 2\frac{EI}{l}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} . \tag{36}$$

The Force Method of Navier

The principle of the force method, the selection of a redundant force and the stating of a compatibility equation having geometrical content — all these are explored also by NAVIER. Certainly this idea was missing before his activity since even the model of the bar was also not existing. Performing the solution, NAVIER starts from the elastic deflection line of the beam in bending, applying the usual differential equation. However, he does not use the superposition principle, instead he applies the boundary conditions belonging to the differential equation as well as the transition conditions valid there at the reference point of the load. Thus he obtains five unknown quantities, included the redundant point force. These can be determined directly one after the other.

The problem itself is presented in *Fig. 12a.* The primary structure is shown in *Fig. 12b.* Derivation of the right side reaction Q as an influence function reads as follows:

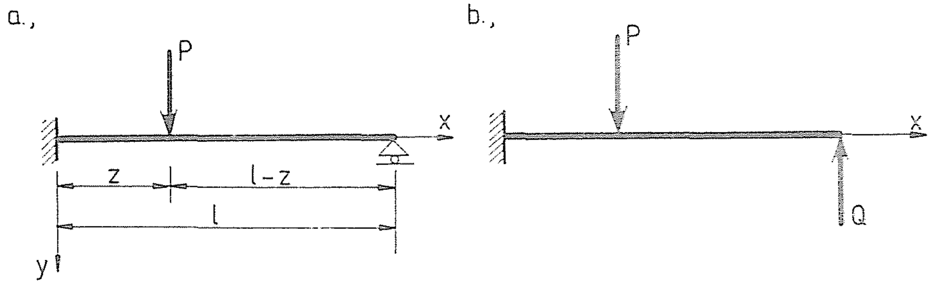


Fig. 12. Indeterminate beam

The differential equation of the deflection curve of the girder in elastic bending is

$$y'' = -\frac{M}{EI}. \tag{37}$$

Since we have different expressions for the bending moment depending on whether the cross-section is situated at the left or at the right side of the load, respectively, (37) has to be written up with respect to two different intervals

$$\begin{aligned} &x \equiv x_1 \leq z, && x \equiv x_2 \leq z, \\ &EIy_1'' = P(z - x_1) - Q(l - x_1), && EIy_2'' = -Q(l - x_2), \end{aligned} \tag{38}$$

Q being unknown. Boundary conditions generated by the geometry read as

$$y_1(0) = 0, \qquad y_1'(0) = 0, \qquad y_2(l) = 0, \tag{39}$$

while the transition conditions are

$$y_1(z) = y_2(z), \qquad y_1'(z) = y_2'(z). \tag{40}$$

Boundary conditions

$$y_2''(l) = 0, \qquad y_2'''(l) = \frac{Q}{EI} \tag{41}$$

due to statics are fulfilled automatically.

Integrating the differential equation (38) twice yields altogether four indefinite constants. The fifth unknown is the redundant Q itself. On the other hand (39) and (40) deliver just five independent conditions, so the unknown quantities can be determined.

Referring to (38) and (39)

$$C_1 = 0\,, \qquad C_2 = 0\,, \qquad C_3 l + C_4 = Q\frac{l^3}{3}\,, \tag{42}$$

while from (38) and (40) we have

$$C_3 = \frac{Pz^2}{2}\,, \qquad C_4 = -\frac{Pz^3}{6}\,. \tag{43}$$

Finally

$$Q = \frac{P}{2l^3} z^3 (3l - z)\,. \tag{44}$$

The influence line of Q is presented in *Fig. 13.*

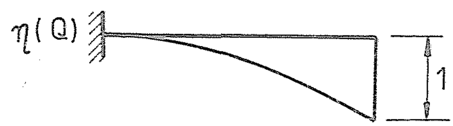


Fig. 13. Influence line of the redundant

Generalization of Navier's Cantilever Method

It is interesting to investigate the ability of the method used originally by NAVIER in order to solve simply supported beams, with respect to the application to indeterminate beams, e. g. continuous structures as well. Also we are looking for the reason why this latter problem has been solved just many years later by CLAPEYRON who was a successor of NAVIER at the French Academy.

Thus the generalization of NAVIER's method has been analyzed first in case of beams clamped at one end, and simply supported at the other one. Afterwards beams clamped at both ends were investigated, as well.

This is a simple matter of fact, just we have to extend the compatibility equations of the basic solution to both cases of more complicated boundaries by applying further geometrical conditions at the inflexion points of the deflection lines that are still unknown. The calculation results in a mixed method, containing compatibility equations(s) as canonical equations, while possessing the position of the inflexion point(s) as unknown quantity. Thus we obtain cubic algebraic equations, so the fundamental idea is not too suitable for generalizing.

The application of the cantilever-like finite elements at the calculation of bars, clamped at one end while supported at the other end is shown in *Fig. 14*. The structure consists of three finite elements. The first extends from the wall to the inflexion point of the deflection line. The second holds from this point to the action point of the point load, while the third one extends to the right support. The abscissa of the inflexion point, the shear force at the same place finally the angular rotation of the cross-section under the load are the unknowns of the problem.

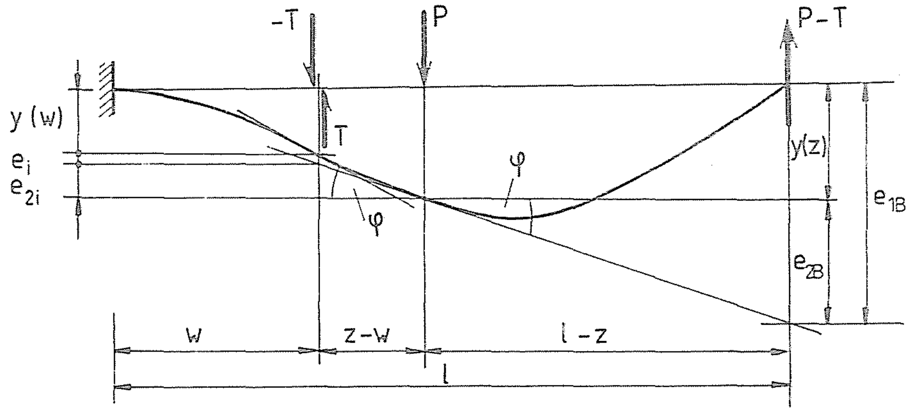


Fig. 14. Navier's solution

According to the first equation generated by the strength of materials, the deflection value at the cross-section of the load calculated from the left and calculated from the right, respectively, have to agree each other. By the notations of the figure

$$e_{1i} + e_{2i} + y(w) = e_{1B} - e_{2B} \tag{45}$$

holds. Here

$$e_{1i} = T\frac{(z-w)^3}{3EI}, \qquad e_{2i} = (z-w)\phi, \qquad y(w) = \frac{Tw^3}{3EI} \tag{46}$$

and

$$e_{1B} = (P-T)\frac{(l-z)^3}{3EI}, \qquad e_{2B} = (l-z)\phi. \tag{47}$$

Replacing (46) and (47) in (45), respectively, we obtain a compatibility equation as follows

$$(l-w)\phi = P\frac{(l-z)^3}{3EI} - \frac{T}{3EI}\left\{l^3 + 3\left[lz(z-l) + zw(w-z)\right] - 2w^3\right\}. \tag{48}$$

Another geometrical equation can be written by considering *Fig. 15a.* Thus the relationship between absolute and relative angular rotations reads as

$$y'(w) - \vartheta_i = \phi\,, \tag{49}$$

hence by applying MOHR'S theorem

$$\frac{Tw^2}{2EI} - \frac{T(z-w)^2}{2EI} = \phi\,, \tag{50}$$

therefore

$$\frac{T}{2EI}z(2w-z) = \phi\,. \tag{51}$$

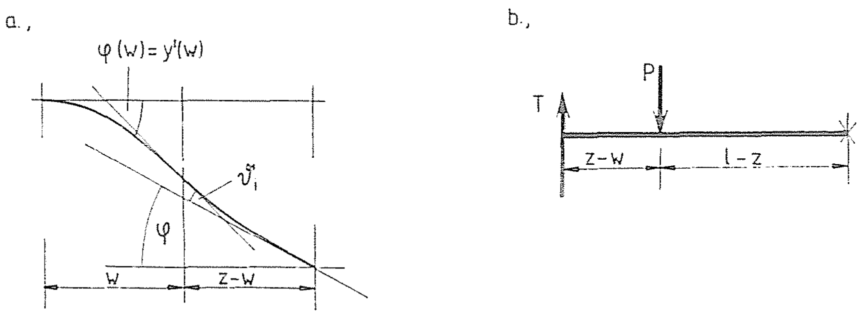


Fig. 15. Geometrical relationship. Equilibrium

We apply thereafter the equilibrium condition suited to *Fig. 15b,* that is the moment-equilibrium equation with respect to the right support. Thus

$$T = P\frac{l-z}{l-w}\,. \tag{52}$$

Referring to (50) and (52)

$$\phi = \frac{P}{2EI}\frac{z(l-z)(2w-z)}{l-w}\,. \tag{53}$$

Finally we replace (52) and (53) to (48) as to obtain

$$\frac{z(l-z)(2w-l)}{2}$$

$$= \frac{(l-z)^3}{3} - \frac{l-z}{3(l-w)}\left\{l^3 + 3\left[lz(z-l) + zw(w-y)\right] - 2w^3\right\}. \qquad (54)$$

The static indeterminacy of the structure is released by this latter compatibility equation. It has to be pointed out that the unknown quantity is neither a stress resultant nor a displacement, it is rather the coordinate w of the inflexion point.

Beam Clamped in at Both Ends

The previous investigation has been extended to the clamped in beam presented in *Fig. 16a,* as well. Both abscissae of the inflexion points of the deflection line are unknown (*Fig. 16b*). Geometrical relationships are presented in *Fig. 16/c,* while the idea of the first geometrical equation agrees to that dealt with previously. The second geometrical equation described there must be replaced here by two separate relationships, while the equilibrium condition can be obtained treating the situation presented in *Fig. 17.*

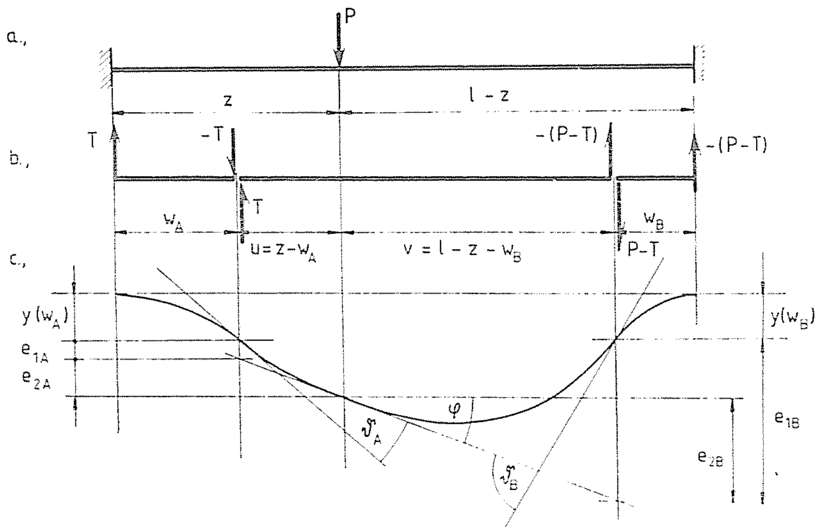


Fig. 16. Beam built in at both ends

Disregarding tedious details we obtain finally the following compatibility equations

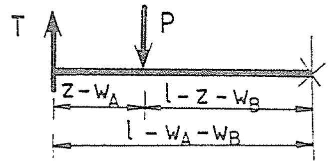


Fig. 17. Equilibrium relationship

$$(z - w_A)(l - z)(2w_B - l - z) = (l - z - w_B)(2w_A - z)\,, \qquad (55)$$

$$(l - z - w_B)z(2w_A - z) = \frac{1}{3}\left[\frac{z - w_A}{l - w_A - w_B}(l - z - w_B)^3 - \frac{l - z - w_B}{l - w_A - w_B}(z - w_A)^3 + \frac{z - w_A}{l - w_A - w_B}w_B^3 - \frac{l - z - w_B}{l - w_A - w_B}w_A^3\right]\,. \qquad (56)$$

Considering the nonlinear equations (55) and (56) we can state that the method of NAVIER using finite cantilever elements is not too suitable for solving problems related to bar structures. The technical mechanics of the early 19th century was not yet in trim for investigation of complicated questions like this.

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