THE LAMBERT CONFORMAL CONIC PROJECTION

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Received Aug. 22, 1981

Presented by Prof. Dr. P. Biró

Abstract

The projection equations of Lambert conformal conic projections are transformed that the same relatively simple formulas can be applied for projections with one or two standard parallels as well. This is due to the fact that projected image on both types of projections differs only by a scale factor. This factor is the central scale factor k_0, its value is $k_0 = 1$ on projections with one standard parallel; for projections with two standard parallels, if not explicitly given, its value can be get from equation (1).

Notations

a, b = major and minor semi-axes of the ellipsoid,
φ = latitude, negative south of the equator,
φ_0 = latitude of the central parallel circle,
φ_1 φ_2 = latitudes of the two standard parallels,
λ = longitude measured from Greenwich, positive eastwards,
λ_0 = longitude of the central meridian,
γ = grid convergence,
δ = arc-to-cord correction,
ε = eccentricity of ellipsoid,
θ = half of vertex angle of cone,
ϱ, ν = radii of curvature of the ellipsoid in meridian and prime vertical respectively,
ψ = isometric latitude on the ellipsoid,
x = easting measured from the central meridian, positive eastwards,
y = northing measured from the central parallel, negative souhtwards,
X_0, Y_0 = false easting and northing,
X = final easting referred to false origin,
Y = final northing referred to false origin,
C = constant,
k = point scale factor,
k_0 = central scale factor,
K = line scale factor,
L = plane distance,

$n = \sin \varphi_0,$
$R =$ radius of projected parallel circle,
$R_0 =$ radius of projected central parallel circle,
$S =$ grid distance,
$s =$ ellipsoidal distance.

The Lambert conformal conic projection is a true projection of the rotational ellipsoid (datum surface) onto the plane. The axis of a right cone with circular base contains both poles and this cone is tangent or secant to the datum surface along one or two parallel circles respectively. The one tangent or the two secant parallel circles called *standard parallels* are equidistant lines of the projection (lines without distortion). The parallel circle along which *point scale factor* reaches minimum is called *central parallel.* On tangent projections standard parallel is identical with the central parallel circle; on secant projections the central parallel lies between standard parallels in a place one can calculate according to equation (1).

In the geodetical literature tangent and secant conic projections are usually treated as two distinct cases and one can find different projection equations for them. Actually, the projection with two standard parallels only differs from the tangent one by the *central scale factor* k_0, being $k_0 = 1$ for the projection with one and $k_0 < 1$ (but very close to the unity) for the projection with two standard parallels. The grid coordinates of projection with two standard parallels one can find after all by multiplying the coordinates of the projection with one standard parallel by the factor k_0. Accordingly the relations of the projection with two standard parallels can be applied on the basis of the same principle in both cases. The factor k_0 of a certain projection system is usually given explicity ($k_0 = 1$ for projection with one standard parallel). On the other hand when the geographical latitudes of two standard parallels are known instead of k_0, the geographical latitude φ_0 of the standard parallel and (cf. [1]) factor k_0 have to be determined:

$$\varphi_0 = \sin^{-1}\left(\frac{1}{\psi_2 - \psi_1} \ln \frac{\nu_1 \cos \varphi_1}{\nu_2 \cos \varphi_2}\right) \tag{1}$$

where ν denotes the radius of curvature in prime vertical, ψ stands for the isometric latitude and the indices refer to standard parallels. The wellknown formula of the isometric latitude is

$$\psi = \ln\left[\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\left(\frac{1 - \varepsilon \sin \varphi}{1 + \varepsilon \sin \varphi}\right)^{\varepsilon/2}\right], \tag{2}$$

with ε being the first eccentricity of the ellipsoid.

For any kind of conic projection holds

$$R_0 = k_0 \nu_0 \cot \varphi_0 = C e^{-n\varphi_0} \tag{3}$$

where

$$n = \sin \varphi_0$$

and

$$C = \frac{\nu_1 \cos \varphi_1}{n e^{-n\varphi_1}} = \frac{\nu_2 \cos \varphi_2}{n e^{-n\varphi_2}}. \tag{4}$$

From equation (3) follows

$$k_0 = \frac{n C e^{-n\varphi_0}}{\nu_0 \cos \varphi_0} = \frac{R_0 \tan \varphi_0}{\nu_0}. \tag{5}$$

It is enough to determine the constants φ_0, C and R_0 only for a certain projection system.

Forward solution

The following equations should be used to get grid coordinates of Lambert projection from geographical φ, λ coordinates of a certain point on the surface of the ellipsoid (Fig. 1.):

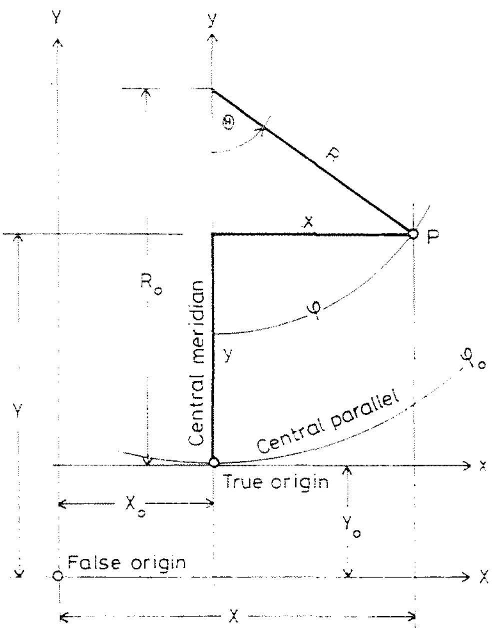

Fig. 1

$$R = C\, e^{-n\psi}, \tag{6}$$

$$\Theta = n(\lambda - \lambda_0) \text{ East of Greenwich,} \tag{7}$$

$$\Theta = n(\lambda_0 - \lambda) \text{ West of Greenwich.}$$

$$\begin{aligned} Y &= Y_0 + (R_0 - R\cos\Theta) = Y_0 + y, \\ X &= X_0 + R\sin\Theta = X_0 + x. \end{aligned} \tag{8}$$

Back solution

$$\begin{aligned} y &= Y - Y_0, \\ x &= X - X_0, \end{aligned} \tag{9}$$

$$\Theta = \tan^{-1} \frac{x}{R_0 - y}\,, \tag{10}$$

$$\begin{aligned} \lambda &= \lambda_0 + \frac{\Theta}{n} \quad \text{East of Greenwich,} \\ \lambda &= \lambda_0 - \frac{\Theta}{n} \quad \text{West of Greenwich} \end{aligned} \tag{11}$$

$$R = \sqrt{x^2 + (R_0 - y)^2} \tag{12}$$

$$\varphi = 2\tan^{-1}\left[\left(\frac{C}{R}\right)^{1/n} \left(\frac{1 + \varepsilon \sin\varphi}{1 - \varepsilon\sin\varphi}\right)^{\varepsilon/2}\right] - \frac{\pi}{2}\,. \tag{13}$$

The last equation enables us to determine φ only iteratively. As a first approximation of φ we accept

$$(\varphi) = 2\tan^{-1}\left(\frac{C}{R}\right)^{1/n} - \frac{\pi}{2} \tag{14}$$

as

$$\left(\frac{1 + \varepsilon\sin\varphi}{1 - \varepsilon\sin\varphi}\right)^{\varepsilon/2} \approx 1$$

holds approximately.

Substituting (φ) from (14) into the right side of (13) yields better approximation of φ. After the next substitution into (13) it can be decided by comparing the newly calculated value with the previous one whether to stop this process or not. Practical considerations show that three approximations are enough to get φ within 0,0001″ (seconds of arc).

Grid convergence

The absolute value and sign of *grid convergence* γ on the Lambert conic projection is the same as for Θ, so it depends only on geographical longitude. It can be achieved by means of (7) or (10) using geographical or plane coordinates respectively.

Point scale factor

Point scale factor depends only on geographical latitude, it follows than, that it assumes constant values along parallel circles. Rewriting (5) for arbitrary parallels, the result is

$$k = \frac{n\,C\,e^{-n\psi}}{\nu\cos\varphi} = \frac{n\,R}{\nu\cos\varphi}\,.$$

This equation could be used when either geographical or grid coordinates are given as well. Of course first R and φ have to be calculated using backward solution.

Line scale factor

The ratio of plane distance L to ellipsoidal distance s is the *line scale factor*

$$K = \frac{L}{s}\,.$$

Although line scale factor is in principle different than point scale factor, if distance is less than 5 km,

$$K \approx k_m$$

where k_m is the point scale factor calculated at the mid-point of the lines. Up to 15 km distance

$$K \approx \frac{k_1 + k_2}{2}$$

where k_1 and k_2 are point scale factors belonging to the terminating points of the line. Nearly up to 100 km

$$K \approx \frac{1}{6}\,(k_1 + 4k_m + k_2)$$

is valid.

Arc-to-chord correction

The arc-to-chord correction belonging to the terminating points of a line is (Fig. 2.):

$$\delta_1 = \frac{X_2 - X_1}{2\varrho_0^2 k_0^2 \sin 1''}\left(Y_1 - Y_0 + \frac{Y_2 - Y_1}{3}\right) = \frac{x_2 - x_1}{2\varrho_0^2 k_0^2 \sin 1''}\left(y_1 + \frac{y_2 - y_1}{3}\right),$$

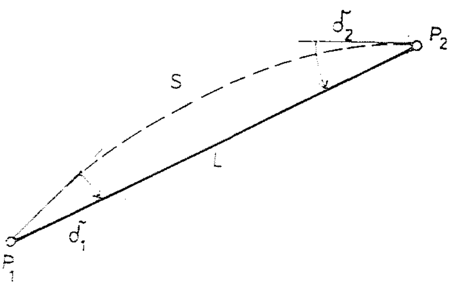

Fig. 2

where X_1, Y_1, X_2, Y_2 are coordinates of the terminating points and ϱ_0 denotes the mean radius of curvature of the ellipsoid at latitude φ_0. For practical tasks, one may ommit k_0 as $k_0 \approx 1$.

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