OPTIMAL DESIGN OF ELASTO-PLASTIC STRUCTURES UNDER VARIOUS LOADING CONDITIONS AND DISPLACEMENT CONSTRAINTS

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Received September 19, 1989

Abstract

The paper presents a generalized approach to the optimal design of linearly elastic — perfectly plastic bar structures constructed of prismatic members. The goal is to minimize the volume of the structure subject to the constraints so that at certain points in the structure the elastic displacements and the permanent plastic displacements caused by a one-parameter static load and by a high intensity short-time dynamic pressure, respectively, do not exceed the given allowable displacement and also the structure under the action of a multi-parameter static loading shakes down. The paper presents the variational and mathematical programming formulation of the problems when the above constraints separately and also simultaneously are taken into consideration and illustrates the application by numerical examples.

1. Introduction

In the analysis and design of structures various loading conditions and several design criteria have to be taken into consideration.

A certain arrangement of the loads must be carried by the structure in elastic stage such that the displacements do not exceed the allowable elastic displacements. In addition, when the structure is submitted to a multi-parameter loading then plastic deformations might be permitted, but it must be proved that during the entire loading history these deformations do not accumulate unrestrictedly, i.e. the structure shakes down.

In some cases special extreme loads (e.g. earthquake, explosion, impact) should also be taken into account. Then again plastic deformations might be allowed but they should not exceed the values which lead to local failure or to the collapse of the entire structure. In the optimal design usually the volume of the structure is the objective function and the constraints might be the design criteria described above. Considering these criteria separately several independent optimal solutions can be determined which can form the basis of the design e.g. by choosing for every point of the structure the maximum cross-sectional area obtained by the separate solutions [2, 4]. A more general solution can be obtained however if all or several of the prescribed design criteria are simultaneously taken into consideration. This calculation leads to a single optimal solution which can be used directly to the design of the structure.

In the following the variational and mathematical programming formulation of the optimal design problems described above will be presented.

2. Fundamentals

In the following linearly elastic — perfectly plastic bar structures (frames, trusses) with given shape and geometry will be considered. The structure is composed of $i = 1,2, \ldots, n$ prismatic members with given lengths l_i and with unknown cross-sectional areas A_i as design variables. It is assumed that the specific stiffness S_i and the specific elastic and plastic strengths R_i^e and R_i^p of the members can be expressed in terms of A_i in the following general forms [2, 4]:

$$\left.\begin{aligned} S_i &= \varphi E A_i^{\alpha}, \\ R_i^e &= \psi \sigma_y A_i^{\beta}, \\ R_i^p &= \varrho \sigma_y A_i^{\gamma}. \end{aligned}\right\} \tag{1}$$

Here φ, ψ, ϱ, α, β and γ are appropriately chosen constants and E and σ_y denote the Young's Modulus and the yield stress of the material, recpectively. For example in the case of beams and frames S_i, R_i^e and R_i^p denote the specific bending stiffness, the maximum elastic moment M_i^e and the fully plastic moment M_i^p of the members. When the beam or frame has rectangular cross-sections with fixed width b_i and variable height h_i eqs. (1) reduce to

$$S_i = EJ_i = \frac{E}{12 b_i^2} A_i^3, \tag{2}$$

$$R_i^e = M_i^e = \frac{\sigma_y}{6 b_i} A_i^2, \tag{3}$$

$$R_i^p = M_i^p = \frac{\sigma_y}{4 b_i} A_i^2. \tag{4}$$

When the structure has sandwich cross-sections with fixed height h_i then eqs. (1) become

$$S_i = E \frac{h_i^2}{4} A_i, \tag{5}$$

$$R_i^e = R_i^p = \sigma_y \frac{h_i}{2} A_i. \tag{6}$$

In the following three different *loading conditions* will be considered:

a) a *one-parameter static load* $F_0(x)$ with given distribution and intensity;

b) a *multi-parameter static loading* defined by the loads $F_1(x)$, $F_2(x)$, . . . , $F_p(x)$ which can act independently or simultaneously;

c) a high intensity, short-time *dynamic pressure* $F^d(x, t)$ defined by the relationships

$$\left.\begin{aligned} &F^d(x,t) = p(t)\, F_0^d(x), \\ &p(t) = p_0, \ \text{if} \ 0 \leq t \leq t_0, \\ &p(t) = 0, \ \text{if} \ t > t_0. \end{aligned}\right\} \tag{7}$$

Here x denotes the coordinate measured along the axis of the bars and t is the time.

Considering the assumptions, loading conditions and the design criteria described above the optimal design of a bar structure might be specified in the following form.

With the cross-sectional areas A_i as design variables and the volume

$$V(A_i) = \sum_{i=1}^{n} l_i A_i; \quad (i = 1, 2, \ldots, n) \tag{8}$$

of the structure as *objective function* we determine the design that minimizes V subject to the following *constraints*.

a) Under the action of the static load $F_0(x)$ the structure does not undergo plastic deformations and at given points $j = 1, 2, \ldots, m$ the *elastic displacements* w_j^e do not exceed the allowable elastic displacements w_{0j}^e, i.e.

$$Q_i^s \leq R_i^e; \quad (i = 1, 2, \ldots, n), \tag{9}$$

$$w_j^e \leq w_{0j}^e; \quad (j = 1, 2, \ldots, m). \tag{10}$$

Here Q_i^s denotes the maximum internal force caused by the load $F_0(x)$ in the i-th member of the elastic structure and R_i^e is defined by eq. (1).

b) The plastic deformations caused by the multi-parameter loading do not accumulate unrestrictedly, i.e. the structure *shakes down*.

c) At given points $j = 1, 2, \ldots m$, in the structure the plastic displacements w_j^p caused by the dynamic pressure $F^d(x, t)$ do not exceed the allowable plastic displacements w_{0j}^p, i.e.

$$w_j^p \leq w_{0j}^p; \quad (j = 1, 2, \ldots, m). \tag{11}$$

Using eqs. (1) the constraint defined by eq. (9) can be written in the form

$$\left.\begin{aligned} &A_i \geq A_{i0}, \\ \text{where} \quad &A_{i0} = \left(\frac{Q_i^s}{\psi \sigma_y}\right)^{1/\beta}. \end{aligned}\right\} \tag{12}$$

In case of beams or frames with fixed width b_i and variable height h_i eq. (3) yields

$$A_{i0} = \left(\frac{6 b_i M_i^s}{\sigma_y}\right)^{1/2}. \tag{12a}$$

Here M_i^s denotes the maximum elastic bending moment caused by the load $F_0(x)$ in the i-th member of the structure.

Using the above relationships and introducing the independent "slack variables" a_i, e_j, g_j the inequalities (9)—(11) can be converted into equality constraints which have the following forms

$$(A_i - A_{i0}) - a_i^2 = 0, \tag{9a}$$

$$(w_j^e - w_{0j}^e) + e_j^2 = 0, \tag{10a}$$

$$(w_j^p - w_{0j}^p) + g_j^2 = 0. \tag{11a}$$

Next, we discuss the above design constraints in detail and present separately the variational and mathematical programming formulation of each optimal design problem. Thereafter we will formulate a more general approach when in the optimal design all the loading conditions and displacement constraints are simultaneously taken into consideration.

3. Static analysis of the elastic structure

Under the action of the static load $F_0(x)$ the structure under consideration is in elastic state and the corresponding internal force distribution is denoted by $Q^s(x, S_i)$. Then the elastic displacement w_j^e at the point j in the structure can be obtained from the following relationship

$$w_j^e = \sum_{i=1}^{n} \int_{l_i} \frac{Q^s(x, S_i) Q_j^D(x)}{S_i} dx. \tag{13}$$

Here $Q_j^D(x)$ denotes any statically admissible internal force distribution equilibrating a "dummy unit force" acting at the point j in the direction of w_j^e. Note that $Q^e(x, S_i)$ is function of the design variable A_i, $Q_j^D(x)$ is, however, independent of it. Introducing the flexibility coefficient

$$f_{ij}(S_i) = \int_{l_i} Q^s(x, S_i) Q_j^D(x)\, dx \tag{14}$$

the *elastic displacement constraint* (10a) can be expressed in the form

$$C_j^e = \left| \sum_{i=1}^{n} \frac{f_{ij}(S_i)}{S_i} \right| - w_{0j}^e + e_j^2 = 0; \quad (j = 1, 2, \ldots, m). \tag{15}$$

3.1. Variational formulation

Using variational formulation the optimal design A_i satisfying the geometric and design constraints (9a) and (15) is identified with the stationarity of the functional

$$
\begin{aligned}
J_e = \sum_{i=1}^{n} A_i l_i + \sum_{j=1}^{m} \lambda_j \left[\left| \sum_{i=1}^{n} \frac{f_{ij}(S_i)}{S_i} \right| - w_{0j}^e + e_j^2 \right] + \\
+ \sum_{i=1}^{n} \varkappa_i (A_i - A_{i0} - a_i^2).
\end{aligned} \tag{16}
$$

Here λ_j and $\varkappa_i$ denote Lagrangian multipliers. The variation of the functional J_e with respect to the variables A_i, e_j and a_i yields the following equations:

$$
\begin{aligned}
\frac{\partial J_e}{\partial A_i} = l_i + \sum_{j=1}^{m} \frac{\lambda_j}{(S_i)^2} \left| \left(\frac{\partial f_{ij}}{\partial S_i} S_i - f_{ij} \right) \frac{\partial S_i}{\partial A_i} \right| + \\
+ \varkappa_i \left(1 - \frac{\partial A_{i0}}{\partial A_i} \right) = 0; \quad (i = 1, 2, \ldots, n),
\end{aligned} \tag{17}
$$

$$
\frac{\partial J_e}{\partial e_j} = \lambda_j e_j = 0; \quad (j = 1, 2, \ldots, m), \tag{18}
$$

$$
\frac{\partial J_e}{\partial a_i} = \varkappa_i a_i = 0; \quad (i = 1, 2, \ldots, n). \tag{19}
$$

From eqs. (18) and (19) follows that along the structure either e_j or λ_j and either $\varkappa_i$ or a_i must vanish. Considering these "switching conditions" the structure can be subdivided into different regions.

In the region where $e_j = 0$ and $\varkappa_i = 0$ eqs. (15) and (17) provide $n + m$ equations for the determination of A_i and λ_j. Where, on the other hand, $e_j = 0$ and $a_i = 0$, eq. (9a) yields the solution $A_i = A_{i0}$.

In the region where $\lambda_j = 0$ according to eq. (17) $\varkappa_i \neq 0$ therefore, independently of the value of e_j, a_i must vanish. Hence for this region eq. (9a) provides again the solution $A_i = A_{i0}$.

We can conclude that the above variational formulation uniquely defines the optimal solution of our problem.

3.2. Mathematical programming formulation

In order to obtain a suitable form to the application of mathematical programming let us introduce a new design variable

$$
Y_i = \frac{1}{S_i} = \frac{A_i^{-\alpha}}{\varphi E}. \tag{20}
$$

Then the objective function (8) is expressed in the form

$$V(Y_i) = \sum_{i=1}^{n} l_i(\varphi E Y_i)^{-1/\alpha} \tag{21}$$

and the constraints (9a), (13) and (15) become

$$0 \leq Y_i \leq \frac{1}{\varphi E}\left(\frac{Q_i^s}{\psi \sigma_y}\right)^{-\frac{\alpha}{\beta}} \tag{22}$$

$$\left|\sum_{i=1}^{n} f_{ij} Y_i\right| - w_{0j}^{\varepsilon} \leq 0. \tag{23}$$

In case of *statically determinate structures* the flexibility coefficients f_{ij} and the internal forces Q_i^s are independent of Y_i therefore the constraints (22) and (23) are linear functions of Y_i and only the objective function (21) is nonlinear. This relatively simple mathematical programming problem can be solved by Wolfe's reduced gradient method [10].

In case of *statically indeterminate structures* f_{ij} and Q_i^s are functions of Y_i therefore eqs. (22) and (23) are highly nonlinear. To solve the problem the following iterative procedure can be applied.

First, we assume appropriate initial values $(Y_i)_0$ $\big($e.q. $(Y_i)_0 = \text{const.}\big)$ for the design variables, determine $(f_{ij})_0$ and $(Q_i^s)_0$ and solve the problem as it was described above.

Next, from the obtained design variables $(Y_i)_1$ we determine the coefficients $(f_{ij})_1$ and internal forces $(Q_i^s)_1$ and solve the problem again.

This procedure has to be continued until the difference between two consecutive steps is sufficiently small.

4. Shakedown analysis of the elasto-plastic structure

The condition of shakedown of a linearly elastic — perfectly plastic, statically q times indeterminate structure is defined by the following relationships [6, 7]

$$\left.\begin{aligned} Q_k^{\max}(S_i) + Q_k^R &\leq R_k^p, \\ Q_k^{\min}(S_i) + Q_k^R &\geq -R_k^p. \end{aligned}\right\} \quad (k = 1, 2, \ldots, s). \tag{24}$$

Here $Q_k^{\max}(S_i)$ and $Q_k^{\min}(S_i)$ denote the maximum and minimum values of the internal forces of the linearly elastic structure calculated from all the possible combinations of the multi-parameter loading $F_1(x)$, $F_2(x)$, . . . , $F_p(x)$ at the critical cross-sections $k = 1, 2, \ldots, s$ and R_k^p and Q_k^R are the plastic strengths (e.q. plastic moments) and the self-equilibrating internal residual forces of the critical cross-sections, respectively. The latter can be expressed in terms of the unknown statically indeterminate forces X_l $(l = 1, 2, \ldots, q)$ in linear forms

$$Q_k^R = \sum_{l=1}^{q} a_{kl} X_l; \quad (k = 1, 2, \ldots, s), \tag{25}$$

where a_{kl} are constant coefficients. Note that $Q_k^{\max}$, $Q_k^{\min}$ and R_k^p are functions of the design variables A_i, Q_k^R and X_l are, however, independent of them.

Inserting eq. (25) into eqs. (24) and introducing the independent "slack variables" d_k and f_k the condition of shakedown can be defined as

$$\left.\begin{aligned} C_{k1}^s &= Q_k^{\max}(S_i) + \sum_{l=1}^{q} a_{kl} X_l - R_k^p + d_k^2 = 0, \\ C_{k2}^s &= Q_k^{\min}(S_i) + \sum_{l=1}^{q} a_{kl} X_l + R_k^p - f_k^2 = 0. \end{aligned}\right\} \quad (k = 1, 2, \ldots, s) \tag{26a—b}$$

In addition, to fulfil some constructional requirements, it might be necessary to prescribe a minimum value A_0 for the cross-sectional area. This geometrical constraint is expressed as

$$(A_i - A_0) - a_i^2 = 0; \quad (i = 1, 2, \ldots, n), \tag{27}$$

where a_i is a slack variable.

4.1. *Variational formulation*

The variational formulation of the optimal design A_i satisfying the constraints (26a—b) and (27) is identified with the stationarity of the functional

$$\begin{aligned} J_s = {} & \sum_{i=1}^{n} l_i A_i + \sum_{k=1}^{s} \mu_k \left[Q_k^{\max}(S_i) + \sum_{l=1}^{q} a_{kl} X_l - R_k^p + d_k^2 \right] + \\ & + \sum_{k=1}^{s} \nu_k \left[Q_k^{\min}(S_i) + \sum_{l=1}^{q} a_{kl} X_l + R_k^p - f_k^2 \right] + \sum_{i=1}^{n} \varkappa_i (A_i - A_0 - a_i^2). \end{aligned} \tag{28}$$

Here μ_k, ν_k and $\varkappa_i$ denote Lagrangian multipliers. The variation of the functional J_s with respect to the variables A_i, X_l, d_k, f_k and a_i leads to the following equations:

$$\begin{aligned} \frac{\partial J_s}{\partial A_i} = {} & l_i + \sum_{k=1}^{s} \mu_k \frac{\partial Q_k^{\max}(S_i)}{\partial S_i} \frac{\partial S_i}{\partial A_i} - \frac{\partial R_k^p}{\partial A_i} + \\ & + \sum_{k=1}^{s} \nu_k \left[\frac{\partial Q_k^{\min}(S_i)}{\partial S_i} \frac{\partial S_i}{\partial A_i} + \frac{\partial R_k^p}{\partial A_i} \right] + \varkappa_i = 0; \quad (i = 1, 2, \ldots, n), \end{aligned} \tag{29}$$

$$\frac{\partial J}{\partial X_l} = \sum_{k=1}^{s} (\mu_k + \nu_k) a_{kl} = 0; \quad (l = 1, 2, \ldots, q), \tag{30}$$

$$\frac{\partial J_s}{\partial d_k} = \mu_k d_k = 0; \quad (k = 1, 2, \ldots, s), \tag{31}$$

$$\frac{\partial J_s}{\partial f_k} = \nu_k f_k = 0; \quad (k = 1, 2, \ldots, s). \tag{32}$$

Considering the "switching conditions" (30)—(32) different regions can be distinguished in the structure.

In the region where $d_k = 0$, $f_k \neq 0$ and $\varkappa_i = 0$ according to eq. (32) ν_k must vanish. Then, eqs. (26a), (29) and (30) provide $(s + n + q)$ equations for the determination of ν_k, X_l and A_i.

On the other hand, in the region where $d_k \neq 0$, $f_k = 0$ and $\varkappa_i = 0$ according to eq. (31) μ_k must vanish. Hence, eqs. (26b), (29) and (30) provide again $(s + n + q)$ equations for the determination of ν_k, X_l and A_i.

In the region where $d_k = 0$, $f_k = 0$ and $\varkappa_i = 0$ both μ_k and ν_k can be different from zero. Now eqs. (26a—b), (29) and (30) provide $(2s + n + q)$ equations from which μ_k, ν_k, X_l and A_i can be calculated.

It can be easily seen that in all the remaining parts of the structure the above switching conditions yield a constrained solution i.e. in these regions $A_i = A_0$. Hence, we can conclude that the above variational formulation uniquely defines the optimum solution of the problem.

4.2. Mathematical programming formulation

Introducing a new design variable

$$R_i^p = \varrho \sigma_y A_i^\gamma \tag{33}$$

and inserting this relationship into the objective function (8) the mathematical programming formulation of the problem under consideration can be defined as below.

Minimize

$$V(R_i^p) = \sum_{i=1}^{n} l_i \left(\frac{R_i^p}{\varrho \sigma_y} \right)^{1/\gamma} \tag{34}$$

subject to

$$\left. \begin{aligned} Q_k^{\max}(S_i) + \sum_{l=1}^{q} a_{kl} X_l \leq R_i^p \\ Q_k^{\min}(S_i) + \sum_{l=1}^{q} a_{kl} X_l \geq R_i^p \end{aligned} \right\} \quad (k = 1, 2, \ldots, s) \tag{35a}$$

$$R_i^p \geq R_0^p = \varrho \sigma_y A_0^\gamma. \tag{35b}$$

Here the last equation corresponds to the geometrical constraint defined by eq. (27). This is a nonlinear mathematical programming problem which can be solved by iteration.

Assuming appropriate initial values $(R_i^p)_0$ (e.g. $(R_i^p)_0 = \text{const.}$) we determine $(S_i)_0$ and the corresponding internal forces $(Q_k^{\max})_0$ and $(Q_k^{\min})_0$. Then we get a mathematical programming problem in which only the objective function is nonlinear. Solving this problem by the use of the reduced gradient method we determine $(X_i)_1$ and $(R_i^p)_1$. From these $(S_i)_1$ the corrected values $(Q_k^{\max})_1$ and $(Q_k^{\min})_1$ can be calculated and the above procedure can be continued until the difference between two consecutive steps is sufficiently small.

5. Dynamic analysis of the rigid-perfectly plastic structure

The maximum permanent displacements of a rigid — perfectly plastic structure subjected to a high intensity short-time dynamic pressure given by eq. (7) can be determined among others by the kinematic approximation [3, 5]. The basic idea of this approximation is that during the dynamic response the structure has stationary motion which is described by a function expressed in product form

$$w^p(x, t) = W(t)\, w^k(x). \tag{36}$$

Here $w^k(x)$ denotes any arbitrary kinematically admissible displacement field (yield mechanism) and $W(t)$ is an unknown displacement parameter function. $W(t)$ is determined by the differential equation of motion of the structure and reaches its maximum value W^p when the structure comes to standstill. Omitting the details for W^p the following expression can be obtained [5, 6]

$$W^p = \frac{1}{2} K p_0 t_0^2 \left(\frac{p_0}{p^k} - 1 \right). \tag{37}$$

Here

$$K = \frac{\int\limits_L F_0^d(x)\, w^k(x)\, dx}{\varrho \sum\limits_{i=1}^{n} A_i \int\limits_{l_i} [w^k(x)]^2\, dx}, \tag{38}$$

ϱ is the density per unit volume of the material and p^k denotes the kinematically admissible multiplier associated with the load $F_0^d(x)$ and displacement field $w^k(x)$ and is defined by the expression (6)

$$p^k = \frac{\sum\limits_{i=1}^{n} R_i^p \bar{q}_i^k}{\int\limits_L F_0^d(x)\, w^k(x)\, dx}. \tag{39}$$

Here $\bar{q}_i^k$ denotes the sum of the absolute values of the generalized strains (e.g. rotations) occuring in the perfectly plastic cross-sections (e.g. in the plastic hinges) of the bar i.

Making use of eq. (37) the approximate values of the maximum plastic displacements at the points $j = 1, 2, \ldots, m$ in the structure can be expressed in the form

$$w^{p} = W^p w_j^k = \frac{1}{2} K p_0 t_0^2 \left(\frac{p_0}{p^k} - 1 \right) w_j^k. \tag{40}$$

Note that the accuracy of the approximation might be improved by introducing several kinematically admissible displacement fields [5, 6]. Then, the maximum values of the permanent displacement obtained by the use of these displacement fields are competent in the design.

Inserting eqs. (38) and (39) into eq. (40) and introducing the notations

$$G = \int_L F_0^d(x) w^k(x)\, dx, \quad D_i = \int_{l_i} [w^k(x)]^2 dx \tag{41}$$

for the design constraint (11a) we get the expression

$$C_j^d = \frac{p_0 t_0^2 G |w_j^k|}{2\varrho \sum_{i=1}^{n} D_i A_i} \left(\frac{p_0 G}{\sum_{i=1}^{n} R_i^p \bar{q}_i^k} - 1 \right) - w_{0j}^p + g_j^2 = 0; \quad (j = 1, 2, \ldots, m) \tag{42}$$

and the geometrical constraint (27) has the form

$$(A_i - A_0) - a_i^2 = 0; \quad (i = 1, 2, \ldots, n). \tag{43}$$

5.1. *Variational formulation*

The variational formulation of optimal design A_i satisfying the design and geometric constraints (42) and (43) is identified with the stationarity of the functional

$$J_p = \sum_{i=1}^{n} l_i A_i + \sum_{j=1}^{m} \psi_j \left[\frac{p_0 t_0^2 G |w_j^k|}{2\varrho \sum_{i=1}^{n} D_i A_i} \left(\frac{p_0 G}{\sum_{i=1}^{n} R_i^p \bar{q}_i^k} - 1 \right) - w_{0j}^p + g_j^2 \right] + \tag{44}$$

$$+ \sum_{i=1}^{n} \varkappa_i (A_i - A_0 - a_i^2).$$

Here ψ_j and $\varkappa_i$ denote Lagrangian multipliers. The variation of the functional J_p with respect to the variables A_i, g_j and a_i yields the following equations:

$$\frac{\partial J_p}{\partial A_i} = l_i + \frac{p_0 t_0^2 G}{2\varrho} \frac{\partial}{\partial A_i}\left[\frac{1}{\sum\limits_{i=1}^{n} D_i A_i}\left(\frac{p_0 G}{\sum\limits_{i=1}^{n} R_i^p \bar{q}_i^k} - 1\right)\right]\sum_{j=1}^{m}\psi_j |w_j^k| + \tag{45}$$
$$+ \varkappa_i = 0; \quad (i = 1, 2, \ldots, n),$$

$$\frac{\partial J_p}{\partial g_j} = \psi_j g_j = 0; \quad (j = 1, 2, \ldots, m), \tag{46}$$

$$\frac{\partial J_p}{\partial a_i} = \varkappa_i a_i = 0; \quad (i = 1, 2, \ldots, n). \tag{47}$$

Similarly to the former problems we can see that for the region where $g_j = 0$ and $\varkappa_i = 0$ an unconstrained solution can be obtained for the determination of A_i and ψ_j. In the other parts of the structure we get a constrained solution, i.e. $A_i = A_0$. Hence, the above variational formulation uniquely defines the optimal solution of the problem under consideration.

5.2. *Mathematical programming formulation*

The mathematical programming formulation of the problem under consideration is as follows.
Minimize

$$V(A_i) = \sum_{i=1}^{n} l_i A_i \tag{48}$$

subject to

$$\frac{p_0 t_0^2 G |w_j^k|}{2\varrho \sum\limits_{i=1}^{n} D_i A_i}\left(\frac{p_0 G}{\sum\limits_{i=1}^{n} R_i^p \bar{q}_i^k} - 1\right) - w_{0j}^p \leq 0; \quad (j = 1, 2, \ldots, m) \tag{49}$$

$$A_0 - A_i \leq 0; \quad (i = 1, 2, \ldots, n). \tag{50}$$

This nonlinear mathematical programming problem can be solved by iteration. Assuming proper initial values $(A_i)_0$ (e.q. $(A_i)_0 = \text{const.}$) we calculate $(R_i^p)_0$. Then we get a linear mathematical programming problem from which $(A_i)_1$ and $(R_i^p)_1$ can be obtained. Then we have to repeat the above procedure until the requested accuracy is reached.

6. A unified approach to optimal design

In the above investigations we took the three loading conditions and displacement constraints separately into consideration and obtained three independent optimal solutions for the structure. To find a single solution which satisfies all the design criteria and leads to the minimum volume of the structure we have to unify the above solutions including all the design constraints in the variational formulation. This problem can be defined as follows.

The optimal design A_i that takes the three loading cases into consideration and satisfies the three displacement constraints described above and the geometrical constraint (9-a) and (13) is identified with the stationarity of the functional

$$J_g = \sum_{i=1}^{n} l_i A_i + \sum_{j=1}^{m} \lambda_j C_j^e + \sum_{k=1}^{s} [\mu_k C_{k1}^s + \nu_k C_{k2}^s] + \\ + \sum_{j=1}^{m} \psi_j C_j^d + \sum_{i=1}^{n} \varkappa_i (A_i - A_{i0} - a_i^2). \tag{51}$$

Here λ_j, μ_k, ν_k and ψ_j are Lagrangian multipliers and C_j^e, C_{k1}^s, C_{k2}^s and C_j^d denote the functions given by eqs. (15), (26a—b) and (42).

The variation of the functional J_g with respect to the variables A_i, e_j, d_k, f_k, g_j and a_i leads to the equation

$$\frac{\partial J_g}{\partial A_i} = l_i + \sum_{j=1}^{m} \lambda_j \frac{\partial C_j^e}{\partial A_i} + \sum_{k=1}^{s} \left[\mu_k \frac{\partial C_{k1}^s}{\partial A_i} + \nu_k \frac{\partial C_{k2}^s}{\partial A_i} \right] + \\ + \sum_{j=1}^{m} \psi_j \frac{\partial C_j^d}{\partial A_i} + \varkappa_i \left(1 - \frac{\partial A_{i0}}{\partial A_i} \right) = 0; \quad (i = 1, 2, \ldots, n) \tag{52}$$

and to the switching conditions

$$\left. \begin{aligned} &\lambda_j e_j = 0, \ \mu_k d_k = 0, \ \nu_k f_k = 0, \\ &\sum_{k=1}^{s} (\mu_k + \nu_k) a_{kl} = 0, \ \psi_j g_j = 0, \ \varkappa_i a_i = 0. \end{aligned} \right\} \tag{53}$$

Following our former considerations it can be stated that the above variational formulation uniquely defines the optimal solution of the generalized problem.

Solving the generalized problem by the use of mathematical programming the same iterative procedure can be applied which was described above.

7. Dual formulation and multicriterion optimization

Consider the case where the displacements under consideration are confined merely to a specific point B in the structure. Then the functions $w_B^e(A_i)$ and $w_B^p(A_i)$ of the elastic and permanent displacements at the point B can be regarded as *objective functions* and the optimization problem might be formulated in various ways.

a) *minimum volume:* $V(A_i) = \min!$
fixed elastic displacement: $w^e_B(A_i) = w^e_{0B}$
fixed permanent displacement: $w^p_B(A_i) = w^p_{0B}$
shakedown of structure.

b) *minimum elastic displacement:* $w^e_B(A_i) = \min!$
fixed volume: $V(A_i) = V_0$
fixed permanent displacement: $w^p_B(A_i) = w^p_{0B}$
shakedown of structure.

c) *minimum permanent displacement:* $w^p_B(A_i) = \min!$
fixed volume: $V(A_i) = V_0$
fixed elastic displacement: $w^e_B(A_i) = w^e_{0B}$
shakedown of structure.

Furthermore, if we consider more functions involved in the optimization procedure as the criterion of optimal design we get the problem of *multicriterion optimization*. Then, the simplest approach is if the objective function is constructed as a set of the weighted individual criteria, as follows:

$$F(A_i) = \psi_1 V(A_i) + \psi_2 w^e_B(A_i) + \psi_3 w^p_B(A_i).$$

Here $\psi_1 \geq 0$, $\psi_2 \geq 0$ and $\psi_3 \geq 0$ are the weighting factors representing the relative importance of individual criteria. By their proper choice (including $\psi_i = 0$) the optimization problem under consideration can be formulated in various ways. The more general formulation of multicriterion optimization is described elsewhere [1, 8, 10].

8. Numerical example

Consider a frame composed of 3 prismatic bars which have rectangular cross-sections with fixed breadth $b = 10$ cm and unknown height h (Fig. 1). The areas A_1, A_2 and A_3 of the cross-sections are the design variables. The frame is composed of linearly elastic-perfectly plastic material with Young's Modulus $E = 2\times10^4$ kN/cm², yield stress $\sigma_y = 20$ kN/cm² and density $\varrho = 8000$ kg/m³ and subjected to two concentrated forces $F_1 = 20$ kN and $F_2 = 60$ kN. Following our former investigations three different loading cases will be taken into consideration.

a) *Static analysis* (ST). Under the action of the static forces $F_1 = 20$ kN and $F_2 = 60$ kN the structure must be in elastic stage and the horizontal and vertical elastic displacements at the points 1 and 4 should not exceed the allowable elastic displacements $w^e_{01} = 7$ cm and $w^e_{04} = 12$ cm, respectively.

b) *Shakedown analysis* (SH). Under the action of the static variable forces $0 \leq F_1 \leq 20$ kN and $0 \leq F_2 \leq 60$ kN which can act separately or in combination, the structure has to shake down.

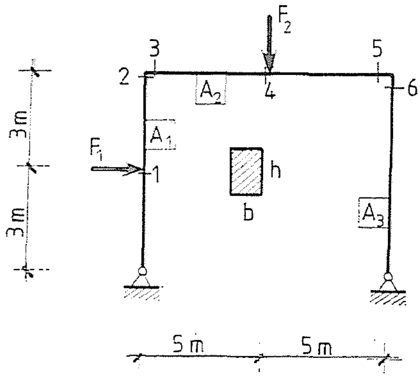

Fig. 1

c) *Dynamic analysis* (DY). Under the action of the dynamic forces $F_1^d = p_0 \times 20$ kN and $F_2^d = p_0 \times 60$ kN the horizontal and vertical plastic displacements at the points 1 and 4 should not exceed the allowable plastic displacements $w_{01}^p = 14$ cm and $w_{04}^p = 24$ cm, respectively. In accordance with eq. (7) the intensity and the duration of the dynamic pressure is $p_0 = 1.3$ and $t_0 = 0.5$ s, respectively. The possible yield mechanisms of the structure are shown in Fig. 2. These 4 mechanisms, however, depending on whether at the

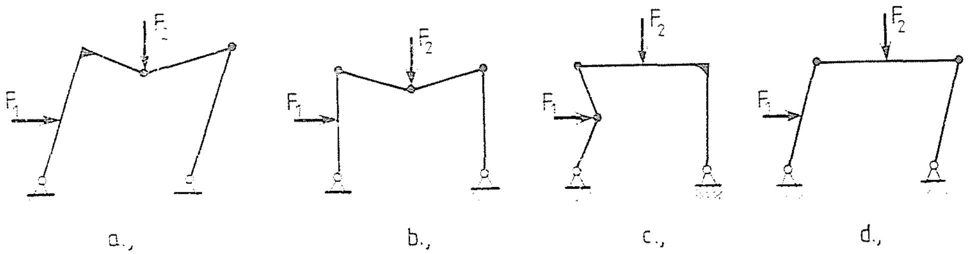

Fig. 2

corners the plastic hinges develop in the columns or in the beam, lead to 12 different solutions for the plastic displacements. To improve the accuracy of the approximate solution in the dynamic analysis all these displacements were taken into consideration and their maximum value was competent in the solution.

The aim of our investigation is to determine the design variables A_1, A_2 and A_3 that minimize the volume

$$V = 6A_1 + 10A_2 + 6A_3$$

of the frame subject to the constraints described above.

First we solved the problem by taking the three loading conditions independently into consideration. The results of these calculations are shown in the first three rows of the Table 1.

Table 1

Case	Analysis	A_1 [cm^2]	A_2 [cm^2]	A_3 [cm^2]	V [m^3]
1	ST	164.6	288.4	384.9	0.6181
2	SH	100.5	168.3	148.6	0.3177
3	DY	82.1	176.8	176.8	0.3321
4	ST + SH	164.6	288.4	384.9	0.6181
5	SH + DY	101.0	175.8	168.9	0.3377
6	ST + DY	164.6	288.4	384.9	0.6181
7	ST+SH+DY	164.6	288.4	384.9	0.6181

Then we determined the optimal solutions taking simultaneously the combinations of two loading cases and all the three loading cases, respectively into account. The last four rows of Table 1 contain the results of these investigations.

It is interesting to note that in the determination of the maximum plastic displacements caused by the dynamic load in case 3 the yield mechanism was competent that had plastic hinges at the points 2, 4 and 5, while, in case 5 in the competent yield mechanism the plastic hinges were located at the points 4 and 5. In cases 6 and 7 the constraint prescribed for the plastic displacements was inactive.

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